J. Math. Computer Sci., 30 (2023), 48–66

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Decision making on the mappings' ideal solution of a fuzzy non-linear matrix system of Kannan-type

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Abstract

Since proving many fixed point theorems in a given space requires either growing the space itself or growing the self-mapping that works on it, both of these options are good. The operators' ideal generated by a weighted binomial matrix in the Nakano sequence space of extended s-fuzzy functions is constructed. Some structures for it based on geometry and topology are presented. It has been proven that the Kannan contraction operator has a unique fixed point in this class. Lastly, sufficient conditions such that a fuzzy non-linear matrix system of Kannan-type has a unique solution in this ideal class are investigated and a numerical example to explain our results are given.

Keywords: Binomial matrix, Nakano sequence space, extended s-fuzzy functions, multiplication mapping, Kannan contraction mapping.

2020 MSC: 46B10, 46C05, 46E15.

1. Notations

(1) $\mathcal{N}$: The set of nonnegative integers.

(2) $\mathfrak{R}$: The set of real numbers.

(3) c_0: The space of convergent to zero sequences of reals.

(4) ℓ_∞: The space of bounded sequences of reals.

(5) ℓ_q: The space of q-absolutely summable sequences of reals.

(6) $\mathbb{D}(\mathcal{G}, \mathcal{V})$: The space of all bounded linear operators from an infinite-dimensional Banach space $\mathcal{G}$ into an infinite-dimensional Banach space $\mathcal{V}$.

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doi: 10.22436/jmcs.030.01.06

Received: 2022-08-17 *Revised:* 2022-09-23 *Accepted:* 2022-09-27

(7) $\mathbb{D}(\mathcal{G})$: The space of all bounded linear mappings from an infinite-dimensional Banach space $\mathcal{G}$ into itself.

(8) $\mathbb{F}(\mathcal{G}, \mathcal{V})$: The space of all finite rank linear mappings from $\mathcal{G}$ into $\mathcal{V}$.

(9) $\mathcal{A}(\mathcal{G}, \mathcal{V})$: The space of approximable linear mappings from $\mathcal{G}$ into $\mathcal{V}$.

(10) $\mathcal{K}(\mathcal{G}, \mathcal{V})$: The space of compact linear mappings from $\mathcal{G}$ into $\mathcal{V}$.

(11) $\mathbb{D}$: The ideal of bounded mappings between each two infinite-dimensional Banach spaces.

(12) ω^F: The class of all sequence spaces of fuzzy reals.

(13) $\mathcal{E}^F$: The linear space of sequences of fuzzy functions.

(14) $[r]$: The integral part of the real number r.

(15) $\overline{e_r} = (\overline{0}, \overline{0}, \dots, \overline{1}, \overline{0}, \overline{0}, \dots)$, where $\overline{1}$ displays at the r^{th} place.

(16) $\overline{\theta} = (\overline{0}, \overline{0}, \overline{0}, \dots)$.

(17) $\mathcal{F}$: The space of finite sequences of fuzzy numbers.

(18) $\uparrow$: The space of every monotonic increasing sequences of positive reals.

(19) $\downarrow$: The space of every monotonic decreasing sequences of positive reals.

(20) $\downarrow^F$: The space of every monotonic decreasing sequences of fuzzy functions.

(21) $\mathfrak{R}^{+\mathbb{N}}$: The space of every sequences of positive reals.

(22) $\mathcal{G}_0$ and $\mathcal{V}_0$: Arbitrary Banach spaces.

(23) I_q: The unit mapping on the q-dimensional Hilbert space ℓ_2^q.

(24) $\alpha_r(T)$: The r-th approximation number of the bounded linear operator T.

(25) $d_r(T)$: The r-th Kolmogorov number of the bounded linear operator T.

2. Introduction

Probability theory, fuzzy set theory, soft sets, and rough sets all help explore uncertainty. These theories have limitations, though. [1, 11, 13, 16, 19, 21, 38] have more details and examples. Let Φ be the set of all closed and bounded intervals on $\mathfrak{R}$. If $h = [h_1, h_2]$ and $j = [j_1, j_2]$ in Φ, define a metric ρ on Φ by

$$\rho(h, j) = \max\{|h_1 - j_1|, |h_2 - j_2|\}.$$

Matloka [20] showed that ρ is a metric on Φ and (Φ, ρ) is a complete metric space. The λ-level set of a fuzzy real number h, $0 < \lambda < 1$, denoted by h^λ, is defined as

$$h^\lambda = \{p \in \mathfrak{R} : h(p) \geqslant \lambda\}.$$

The set of all upper semi-continuous, normal, convex fuzzy number, and h^λ is compact, is denoted by $\mathfrak{R}([0,1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0,1])$, if we define $t \in \mathfrak{R}([0,1])$ by

$$\bar{t}(y) = \begin{cases} 1, & y = t, \\ 0, & y \neq t. \end{cases}$$

The additive identity and multiplicative identity in $\mathfrak{R}[0,1]$ are denoted by $\bar{0}$ and $\bar{1}$, respectively. Assume $h, j \in \mathfrak{R}[0,1]$ and the λ-level sets are $[h]^\lambda = [h_1^\lambda, h_2^\lambda]$, $[j]^\lambda = [j_1^\lambda, j_2^\lambda]$, $\lambda \in [0,1]$. If $\bar{\rho} : \mathfrak{R}[0,1] \times \mathfrak{R}[0,1] \to \mathfrak{R}^+ \cup \{0\}$ is defined by $\bar{\rho}(h,j) = \sup_{0 \leqslant \lambda \leqslant 1} \rho(h^\lambda, j^\lambda)$, then $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space. For more details on fuzzy functions and their properties, see [3, 12, 17, 24, 25]. If $(w_l) \in \mathfrak{R}^{+\mathcal{N}}$, we have defined the space $\left(E^F_{u,v}(q,w)\right)_\tau$ under τ as:

$$\left(E^F_{u,v}(q,w)\right)_\tau = \left\{\bar{h} = (\overline{h_m}) \in \omega^F : \tau(\delta\bar{h}) < \infty, \text{ for some } \varepsilon > 0\right\},$$

where $\tau(\bar{h}) = \sum_{m=0}^{\infty} \left(\frac{\bar{\rho}\left(\sum_{z=0}^{m} A(m,z) q_{m,z} \overline{h_z}, \bar{0}\right)}{(u+v)^m}\right)^{w_m}$, $q_{l,z} \in (0,\infty)$, for all $l, z \in \mathcal{N}$, and $A(l,z) = \left[{l \atop z}\right] u^z v^{l-z}$. Clearly, if $(w_l) \in \ell_\infty \cap \mathfrak{R}^{+\mathcal{N}}$, then

$$\left(E^F_{u,v}(q,w)\right)_\tau = \left\{\bar{h} = (\overline{h_b}) \in \omega^F : \tau(\delta\bar{h}) < \infty, \text{ for all } \delta > 0\right\}.$$

Notations 2.1 ([8]).

$$\begin{aligned}
\overline{\mathbb{D}^s}_{\mathcal{E}^F} &:= \left\{\overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})\right\}, \text{ where } \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V}) := \left\{V \in \mathbb{D}(\mathcal{G},\mathcal{V}) : ((\overline{s_j(V)})_{j=0}^\infty \in \mathcal{E}^F\right\},\\
\overline{\mathbb{D}^\alpha}_{\mathcal{E}^F} &:= \left\{\overline{\mathbb{D}^\alpha}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})\right\}, \text{ where } \overline{\mathbb{D}^\alpha}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V}) := \left\{V \in \mathbb{D}(\mathcal{G},\mathcal{V}) : ((\overline{\alpha_j(V)})_{j=0}^\infty \in \mathcal{E}^F\right\},\\
\overline{\mathbb{D}^d}_{\mathcal{E}^F} &:= \left\{\overline{\mathbb{D}^d}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})\right\}, \text{ where } \overline{\mathbb{D}^d}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V}) := \left\{V \in \mathbb{D}(\mathcal{G},\mathcal{V}) : ((\overline{d_j(V)})_{j=0}^\infty \in \mathcal{E}^F\right\},\\
\left(\overline{\mathbb{D}^s}_{\mathcal{E}^F}\right)^\Upsilon &:= \left\{\left(\overline{\mathbb{D}^s}_{\mathcal{E}^F}\right)^\Upsilon(\mathcal{G},\mathcal{V})\right\}, \text{ where}\\
\left(\overline{\mathbb{D}^s}_{\mathcal{E}^F}\right)^\Upsilon(\mathcal{G},\mathcal{V}) &:= \left\{V \in \mathbb{D}(\mathcal{G},\mathcal{V}) : ((\overline{\gamma_b(V)})_{b=0}^\infty \in \mathcal{E}^F \text{ and } \|V - \bar{\rho}(\overline{\gamma_b(V)}, \bar{0})I\| = 0 \text{ , for all } b \in \mathcal{N}\right\}.
\end{aligned}$$

Only this one try was ever made as [4, 22, 23, 26, 28] shows that much attention has been paid to the s-number operators' ideal in functional analysis. Supposing that $G \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}(q,w)\right)_\tau}(\mathcal{G},\mathcal{V})$, $\Pi : \mathcal{N}^2 \to \mathfrak{R}$, $f : \mathcal{N} \times \mathfrak{R}[0,1] \to \mathfrak{R}[0,1]$, and $P \in \mathbb{D}(\mathcal{G},\mathcal{V})$. Consider the fuzzy non-linear Matrix systems [32]:

$$\overline{s_z(G)} = \overline{s_z(P)} + \sum_{m=0}^{\infty} \Pi(z,m) f(m, \overline{s_m(G)}) \tag{2.1}$$

and $W : \overline{\mathbb{D}^s}_{\left(E^F_{u,v}(q,w)\right)_\tau}(\mathcal{G},\mathcal{V}) \to \overline{\mathbb{D}^s}_{\left(E^F_{u,v}(q,w)\right)_\tau}(\mathcal{G},\mathcal{V})$ is defined as

$$W(G) = \left(\overline{s_z(P)} + \sum_{m=0}^{\infty} \Pi(z,m) f(m, \overline{s_m(G)})\right) I. \tag{2.2}$$

Since the Banach Fixed Point Theorem book [9] was published, mathematicians have studied potential extensions and applications. Nonlinear analysis relies on the Banach contraction principle [10, 31, 37]. Kannan [18] offered a group of mappings with the same actions at specified locations as contractions. This collection is somewhat discontinuous. Tripathy et al. [33–36] looked into a fixed point theorem in a generalized fuzzy metric space, fixed point and periodic point theorems in fuzzy metric space, Banach's and Kannan's fixed point results in fuzzy 2-metric spaces, and some fixed point theorems in generalized M-fuzzy metric space. An explanation of Kannan operators in modular vector spaces was formerly attempted in Reference [15]. Bakery and Mohamed [6] offered the idea of a pre-quasi norm on the Nakano sequence space with a variable exponent that fell somewhere in the range $(0,1]$. They talked about the conditions that must be met to generate pre-quasi Banach and closed space when it is endowed with a specified pre-quasi norm, as well as the Fatou property of various pre-quasi norms on it. They also determined a fixed point for Kannan pre-quasi norm contraction mappings on it, in addition to the ideal of pre-quasi Banach mappings derived from s-numbers in this sequence space. Both of these ideals were established. In addition, several fixed point findings of Kannan non-expansive mappings on generalized Cesàro backward difference sequence space of a non-absolute type were discovered in [7]. Given that proving several fixed point theorems in a given space necessitates either extending the space itself or expanding the self-mapping that acts on it, both of these methods are possible. The aim of this paper to develop a new operators' ideal class $\overline{\mathbb{D}^s}_{\left(E^F_{u,v}(q,w)\right)_\tau}$. We have provided geometric and topological structures for it. In this class, the existence of a unique fixed point for the Kannan contraction operator is confirmed. In conclusion, we describe an application of solutions to the fuzzy non-linear matrix system and provide a numerical example of our findings.

3. Definitions and preliminaries

Lemma 3.1 ([2]). *If $w_b > 0$ and $x_b, z_b \in \mathfrak{R}$, for all $b \in \mathcal{N}$, and $\hbar = \max\{1, \sup_b w_b\}$, then*

$$|x_b + z_b|^{w_b} \leqslant 2^{\hbar-1}\left(|x_b|^{w_b} + |z_b|^{w_b}\right). \tag{3.1}$$

Definition 3.2 ([29]). A s-number is a function $s : \mathbb{D}(\mathcal{G}, \mathcal{V}) \to \mathfrak{R}^{+\mathcal{N}}$ that gives all $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$ a $(s_d(V))_{d=0}^{\infty}$ that satisfies the following conditions:

(1) $\|V\| = s_0(V) \geqslant s_1(V) \geqslant s_2(V) \geqslant \cdots \geqslant 0$, for all $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$;

(2) $s_d(VYW) \leqslant \|V\| s_d(Y) \|W\|$, if for every $W \in \mathbb{D}(\mathcal{G}_0, \mathcal{G})$, $Y \in \mathbb{D}(\mathcal{G}, \mathcal{V})$, and $V \in \mathbb{D}(\mathcal{V}, \mathcal{V}_0)$;

(3) $s_{l+d-1}(V_1 + V_2) \leqslant s_l(V_1) + s_d(V_2)$, when for every $V_1, V_2 \in \mathbb{D}(\mathcal{G}, \mathcal{V})$ and l, $d \in \mathcal{N}$;

(4) if $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$;

(5) if $\operatorname{rank}(V) \leqslant d$, then $s_d(V) = 0$, for all $V \in \mathbb{D}(\mathcal{G}, \mathcal{V})$;

(6) $s_{l \geqslant a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$.

Definition 3.3 ([5]). A sub class $\mathcal{U}$ of $\mathbb{D}$ is said to be a mappings' ideal if every $\mathcal{U}(\mathcal{G}, \mathcal{V}) = \mathcal{U} \cap \mathbb{D}(\mathcal{G}, \mathcal{V})$ satisfies the following settings.

(i) $I_\Gamma \in \mathcal{U}$, where Γ indicates Banach space of one dimension.

(ii) The space $\mathcal{U}(\mathcal{G}, \mathcal{V})$ is linear over $\mathfrak{R}$.

(iii) If $W \in \mathbb{D}(\mathcal{G}_0, \mathcal{G})$, $X \in \mathcal{U}(\mathcal{G}, \mathcal{V})$, and $Y \in \mathbb{D}(\mathcal{V}, \mathcal{V}_0)$, then $YXW \in \mathcal{U}(\mathcal{G}, \mathcal{V}_0)$.

Definition 3.4 ([14]). A function $H \in [0, \infty)^{\mathcal{U}}$ is said to be a pre-quasi norm on the ideal $\mathcal{U}$, if the following conditions are verified:

(1) Suppose $V \in \mathcal{U}(\mathcal{G}, \mathcal{V})$, $H(V) \geqslant 0$ and $H(V) = 0$, if and only if, $V = 0$;

(2) we have $Q \geqslant 1$, such that $H(\alpha V) \leqslant D|\alpha| H(V)$, for all $V \in \mathcal{U}(\mathcal{G}, \mathcal{V})$ and $\alpha \in \mathfrak{R}$;

(3) there are $P \geqslant 1$, such that $H(V_1 + V_2) \leqslant P[H(V_1) + H(V_2)]$, for every $V_1, V_2 \in \mathcal{U}(\mathcal{G}, \mathcal{V})$;

(4) there are $\sigma \geqslant 1$, such that if $V \in \mathbb{D}(\mathcal{G}_0, \mathcal{G})$, $X \in \mathcal{U}(\mathcal{G}; \mathcal{V})$, and $Y \in \mathbb{D}(\mathcal{V}, \mathcal{V}_0)$, then $H(YXV) \leqslant \sigma \|Y\| H(X) \|V\|$.

Theorem 3.5 ([14]). *Each quasi norm on the ideal $\mathcal{U}$ is a pre-quasi norm.*

Lemma 3.6 ([28]). *If $W \in \mathbb{D}(\mathcal{G}, \mathcal{V})$ and $W \notin \mathcal{A}(\mathcal{G}, \mathcal{V})$, then there are $P \in \mathbb{D}(\mathcal{G})$ and $A \in \mathbb{D}(\mathcal{V})$ such that $AWPe_j = e_j$, for all $j \in \mathcal{N}$.*

Theorem 3.7 ([28])**.** *If $\mathcal{E}^F$ is an infinite-dimensional Banach space, then*

$$\mathbb{F}(\mathcal{E}^F) \subsetneqq \mathcal{A}(\mathcal{E}^F) \subsetneqq \mathcal{K}(\mathcal{E}^F) \subsetneqq \mathbb{D}(\mathcal{E}^F).$$

Definition 3.8 ([28])**.** A Banach space $\mathcal{E}^F$ is said to be simple, when $\mathbb{D}(\mathcal{E}^F)$ has a unique non-trivial closed ideal.

Definition 3.9 ([6])**.** A mapping $W : \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V}) \to \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$ is said to be a Kannan Ξ-contraction, when there is $\zeta \in [0, \frac{1}{2})$ with $\Xi(WV - WT) \leqslant \zeta(\Xi(WV - V) + \Xi(WT - T))$, for all $V, T \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$.

4. Properties of the operators' ideal class $\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}$

Some geometric and topological properties of the operators' ideal class $\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}$ are investigated in this section.

Definition 4.1. The space $\mathcal{E}^F$ is said to be a private sequence space of fuzzy functions ($\mathfrak{pssff}$), when the next conditions are established:

(a1) If $b \in \mathcal{N}$, then $\overline{e_b} \in \mathcal{E}^F$;

(a2) If $\bar{f} = (\overline{f_b}) \in \omega^F$, $|\bar{g}| = (|\overline{g_b}|) \in \mathcal{E}^F$ and $|\overline{f_b}| \leqslant |\overline{g_b}|$ so that $b \in \mathcal{N}$, then $|\bar{f}| \in \mathcal{E}^F$;

(a3) $\left(\left|\overline{h_{[\frac{b}{2}]}}\right|\right)_{b=0}^{\infty} \in \mathcal{E}^F$, whenever $\left(\left|\overline{h_b}\right|\right)_{b=0}^{\infty} \in \mathcal{E}^F$.

Theorem 4.2. *Presume the linear sequence space $\mathcal{E}^F$ is a $\mathfrak{pssff}$, then $\overline{\mathbb{D}^s}_{\mathcal{E}^F}$ is an operators' ideal.*

Proof.
(i) If $V \in \mathbb{F}(\mathcal{G},\mathcal{V})$ and $\text{rank}(V) = n$ with $n \in \mathcal{N}$, as $\overline{e_i} \in \mathcal{E}^F$, for all $i \in \mathcal{N}$, and $\mathcal{E}^F$ is a linear space, then

$$(\overline{s_i(V)})_{i=0}^{\infty} = (\overline{s_0(V)}, \overline{s_1(V)}, \dots, \overline{s_{n-1}(V)}, \bar{0}, \bar{0}, \bar{0}, \dots) = \sum_{i=0}^{n-1} \overline{s_i(V)}\overline{e_i} \in \mathcal{E}^F,$$

hence $V \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$ so $\mathbb{F}(\mathcal{G},\mathcal{V}) \subseteq \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$.
(ii) Presume $V_1, V_2 \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$, and $\beta_1, \beta_2 \in \mathfrak{R}$ then from Definition 4.1 condition (iii), we have $(\overline{s_{[\frac{i}{2}]}(V_1)})_{i=0}^{\infty} \in \mathcal{E}^F$ and $(\overline{s_{[\frac{i}{2}]}(V_1)})_{i=0}^{\infty} \in \mathcal{E}^F$. As $i \geqslant 2[\frac{i}{2}]$, from the definition of $\bar{s}$-numbers and $\overline{s_i(P)}$ is a decreasing sequence, we get

$$\overline{s_i(\beta_1 V_1 + \beta_2 V_2)} \leqslant \overline{s_{2[\frac{i}{2}]}(\beta_1 V_1 + \beta_2 V_2)} \leqslant \overline{s_{[\frac{i}{2}]}(\beta_1 V_1) + s_{[\frac{i}{2}]}(\beta_2 V_2)} = |\beta_1|\overline{s_{[\frac{i}{2}]}(V_1)} + |\beta_2|\overline{s_{[\frac{i}{2}]}(V_2)},$$

for every $i \in \mathcal{N}$. By Definition 4.1 condition (ii) and $\mathcal{E}^F$ is a linear space, then $(\overline{s_i(\beta_1 V_1 + \beta_2 V_2)})_{i=0}^{\infty} \in \mathcal{E}^F$, then $\beta_1 V_1 + \beta_2 V_2 \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$.
(iii) Suppose $P \in \mathbb{D}(\mathcal{G}_0,\mathcal{G})$, $T \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$, and $R \in \mathbb{D}(\mathcal{V},\mathcal{V}_0)$, then $(\overline{s_i(T)})_{i=0}^{\infty} \in \mathcal{E}^F$ and as $\overline{s_i(RTP)} \leqslant \|R\|\overline{s_i(T)}\ \|P\|$, from Definition 4.1 conditions (i) and (ii) we obtain $(\overline{s_i(RTP)})_{i=0}^{\infty} \in \mathcal{E}^F$, hence $RTP \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G}_0,\mathcal{V}_0)$. □

Definition 4.3. A subspace of the $\mathfrak{pssff}$ is called a pre-modular $\mathfrak{pssff}$ when we have a function $\tau : \mathcal{E}^F \to [0,\infty)$ that satisfies the next setups:

(i) if $\bar{h} \in \mathcal{E}^F$, then $\bar{h} = \bar{\theta} \iff \tau(|\bar{h}|) = 0$, and $\tau(\bar{h}) \geqslant 0$;

(ii) suppose $\bar{h} \in \mathcal{E}^F$ and $\varepsilon \in \mathfrak{R}$, then we have $E_0 \geqslant 1$ with $\tau(\varepsilon\bar{h}) \leqslant |\varepsilon| E_0 \tau(\bar{h})$;

(iii) there are $G_0 \geqslant 1$, such that $\tau(\bar{f} + \bar{g}) \leqslant G_0(\tau(\bar{f}) + \tau(\bar{g}))$, for all $\bar{f}, \bar{g} \in \mathcal{E}^F$;

(iv) assume $|\overline{f_b}| \leqslant |\overline{g_b}|$, for every $b \in \mathcal{N}$, then $\tau(|\overline{f_b}|) \leqslant \tau(|\overline{g_b}|)$;

(v) we have $D_0 \geqslant 1$, such that $\tau(|\overline{f}|) \leqslant \tau(|\overline{f_{[.]}}|) \leqslant D_0\tau(|\overline{f}|)$;

(vi) the closure of $\mathcal{F} = \mathcal{E}^F_\tau$;

(vii) there are $\varepsilon > 0$, such that $\tau(\overline{v}, \overline{0}, \overline{0}; \overline{0}, \dots) \geqslant \varepsilon|v|\tau(\overline{1}, \overline{0}, \overline{0}, \overline{0}, \dots)$.

Definition 4.4. The $\mathfrak{pssff}$ $\mathcal{E}^F_\tau$ is said to be a pre-quasi normed $\mathfrak{pssff}$, if τ verifies the settings (i)-(iii) of Definition 4.3. The space $\mathcal{E}^F_\tau$ is called a pre-quasi Banach $\mathfrak{pssff}$, whenever $\mathcal{E}^F$ is complete equipped with τ.

Theorem 4.5. *If $\mathcal{E}^F_\tau$ is a pre-modular $\mathfrak{pssff}$, then it is pre-quasi normed $\mathfrak{pssff}$.*

Theorem 4.6. *If*

(b1) $u + v > 1$;

(b2) $(w_p)_{p\in\mathcal{N}} \in \ell_\infty \cap \uparrow$;

(b3) $(A(a,k)q_{a,k})_{k=0}^\infty \in\downarrow$ *or,* $(A(a,k)q_{a,k})_{k=0}^\infty \in\uparrow \cap \ell_\infty$ *and there exists* $C \geqslant 1$, *such that*

$$A(a, 2k+1)q_{a,2k+1} \leqslant CA(a,k)q_k;$$

(b4) *and* $(A(a,k)q_{a,k})_{a=0}^\infty \in\downarrow$,

then $(E^F_{u,v}(q,w))_\tau$ *is a pre-quasi Banach* $\mathfrak{pssff}$.

Proof. We show that $(E^F_{u,v}(q,w))_\tau$ is a pre-modular $\mathfrak{pssff}$.
(i). Evidently, $\tau(|\overline{h}|) = 0 \Leftrightarrow \overline{h} = \overline{\theta}$ and $\tau(\overline{h}) \geqslant 0$.
(a1) and (iii). If $\overline{f}, \overline{g} \in (E^F_{u,v}(q,w))_\tau$, then

$$\begin{aligned}\tau(\overline{f}+\overline{g}) &= \sum_{l=0}^\infty \left(\frac{\overline{\rho}\left(\sum_{z=0}^l A(l,z)q_{l,z}\left(\overline{f_z}+\overline{g_z}\right),\overline{0}\right)}{(u+v)^l}\right)^{w_l} \\ &\leqslant 2^{\hbar-1}\left(\sum_{l=0}^\infty \left(\frac{\overline{\rho}\left(\sum_{z=0}^l A(l,z)q_{l,z}\overline{f_z},\overline{0}\right)}{(u+v)^l}\right)^{w_l} + \sum_{l=0}^\infty \left(\frac{\overline{\rho}\left(\sum_{z=0}^l A(l,z)q_{l,z}\overline{g_z},\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right) \\ &= 2^{\hbar-1}(\tau(\overline{f})+\tau(\overline{g})) < \infty.\end{aligned}$$

Therefore, $\overline{f}+\overline{g} \in (E^F_{u,v}(q,w))_\tau$.
(ii). Presume $\lambda \in \mathfrak{R}$, $\overline{f} \in (E^F_{u,v}(q,w))_\tau$, and as $(w_l) \in\uparrow \cap \ell_\infty$, hence

$$\begin{aligned}\tau(\lambda\overline{f}) = \sum_{m=0}^\infty \left(\frac{\overline{\rho}\left(\sum_{z=0}^m A(m,z)q_{m,z}\lambda\overline{f_z},\overline{0}\right)}{(u+v)^m}\right)^{w_m} &\leqslant \sup_m |\lambda|^{w_m} \sum_{m=0}^\infty \left(\frac{\overline{\rho}\left(\sum_{z=0}^m A(m,z)q_{m,z}\overline{f_z},\overline{0}\right)}{(u+v)^m}\right)^{w_m} \\ &\leqslant E_0|\lambda|\tau(\overline{f}) < \infty,\end{aligned}$$

where $E_0 = \max\left\{1, \sup_l |\lambda|^{w_l-1}\right\} \geqslant 1$. So $\lambda\overline{f} \in (E^F_{u,v}(q,w))_\tau$. Assume $(w_l) \in\uparrow \cap \ell_\infty$, then

$$\begin{aligned}\sum_{m=0}^\infty \left(\frac{\overline{\rho}\left(\sum_{z=0}^m A(m,z)q_{m,z}\overline{(e_b)_z},\overline{0}\right)}{(u+v)^m}\right)^{w_m} &= \sum_{m=b}^\infty \left(\frac{A(m,b)q_{m,b}}{(u+v)^m}\right)^{w_m} \\ &\leqslant \sup_{m=b}^\infty (A(m,b)q_{m,b})^{w_m} \sum_{m=b}^\infty \left(\frac{1}{(u+v)^m}\right)^{w_m} < \infty.\end{aligned}$$

So $\overline{e_b} \in (E^F_{u,v}(q,w))_\tau$, for every $b \in \mathcal{N}$.

(a2) and (iv). If $|\overline{f_m}| \leqslant |\overline{g_m}|$, for each $m \in \mathcal{N}$ and $|\overline{g}| \in (E^F_{u,v}(q,w))_\tau$, then

$$\tau(|\overline{f}|) = \sum_{m=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{m} A(m,z)q_{m,z}|\overline{f_z}|,\overline{0}\right)}{(u+v)^m}\right)^{w_m} \leqslant \sum_{m=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{m} A(m,z)q_{m,z}|\overline{g_z}|,\overline{0}\right)}{(u+v)^m}\right)^{w_m} = \tau(|\overline{g}|) < \infty.$$

Therefore, $|\overline{f}| \in (E^F_{u,v}(q,w))_\tau$.

(a3) and (v). Suppose $(|\overline{f_z}|) \in (E^F_{u,v}(q,w))_\tau$ with $(w_l) \in \uparrow \cap \ell_\infty$ and $(A(l,z)q_{l,z})_{z=0}^{\infty} \in \downarrow$, hence

$$\begin{aligned}
\tau(|\overline{f_{[\frac{z}{2}]}}|) &= \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}|\overline{f_{[\frac{z}{2}]}}|,\overline{0}\right)}{(u+v)^l}\right)^{w_l} \\
&= \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{2l} A(l,z)q_{l,z}|\overline{f_{[\frac{z}{2}]}}|,\overline{0}\right)}{(u+v)^{2l}}\right)^{w_{2l}} + \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{2l+1} A(l,z)q_{l,z}|\overline{f_{[\frac{z}{2}]}}|,\overline{0}\right)}{(u+v)^{2l+1}}\right)^{w_{2l+1}} \\
&\leqslant \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{2l} A(l,z)q_{l,z}|\overline{f_{[\frac{z}{2}]}}|,\overline{0}\right)}{(u+v)^{l}}\right)^{w_{l}} + \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{2l+1} A(l,z)q_{l,z}|\overline{f_{[\frac{z}{2}]}}|,\overline{0}\right)}{(u+v)^{l}}\right)^{w_{l}} \\
&\leqslant \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(A(l,2l)q_{l,2l}|\overline{f_l}| + \sum_{z=0}^{l}\left(A(l,2z)q_{2z} + A(l,2z+1)q_{l,2z+1}\right)|\overline{f_z}|,\overline{0}\right)}{(u+v)^{l}}\right)^{w_{l}} \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l}\left(A(l,2z)q_{2z} + A(l,2z+1)q_{l,2z+1}\right)|\overline{f_z}|,\overline{0}\right)}{(u+v)^{l}}\right)^{w_{l}} \\
&\leqslant 2^{\hbar-1}\left(\sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}|\overline{f_z}|,\overline{0}\right)}{(u+v)^{l}}\right)^{w_{l}} + \sum_{l=0}^{\infty}\left(\frac{2\overline{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}|\overline{f_z}|,\overline{0}\right)}{(u+v)^{l}}\right)^{w_{l}}\right) \\
&\quad + \sum_{l=0}^{\infty}\left(\frac{2\overline{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}|\overline{f_z}|,\overline{0}\right)}{(u+v)^{l}}\right)^{w_{l}} \leqslant D_0\tau(|\overline{f}|) < \infty,
\end{aligned}$$

where $D_0 \geqslant (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar}) \geqslant 1$. So $(|\overline{f_{[\frac{z}{2}]}}|) \in (E^F_{u,v}(q,w))_\tau$.

(vi). Clearly, the closure of $\mathcal{F} = E^F_{u,v}(q,w)$.

(vii). There are $0 < \delta \leqslant \sup_l |\lambda|^{w_l - 1}$, such that $\tau(\overline{\lambda},\overline{0},\overline{0},\overline{0},\dots) \geqslant \delta|\lambda|\tau(\overline{1},\overline{0},\overline{0},\overline{0},\dots)$, for every $\lambda \neq 0$ and $\delta > 0$ if $\lambda = 0$. Assume Theorem 4.5 is established, then $(E^F_{u,v}(q,w))_\tau$ is a pre-quasi normed $\mathfrak{pssff}$. Second, to prove that $(E^F_{u,v}(q,w))_\tau$ is a Banach space, if $\overline{h^i} = (\overline{h^i_k})_{k=0}^{\infty}$ is a Cauchy sequence in $(E^F_{u,v}(q,w))_\tau$. Therefore, for every $\gamma \in (0,1)$ one gets $i_0 \in \mathcal{N}$, for every $i,j \geqslant i_0$, then

$$\tau(\overline{h^i} - \overline{h^j}) = \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\left(\overline{h^i_z} - \overline{f^j_z}\right),\overline{0}\right)}{(u+v)^l}\right)^{w_l} < \gamma^{\hbar}.$$

So $\overline{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\left(\overline{h^i_z} - \overline{h^j_z}\right),\overline{0}\right) < \gamma$. As $(\mathfrak{R}[0,1],\overline{\rho})$ is a complete metric space. Hence $(\overline{h^j_k})$ is a Cauchy sequence in $\mathfrak{R}[0,1]$, for fixed $k \in \mathcal{N}$. So it is convergent to $\overline{h^0_k} \in \mathfrak{R}[0,1]$. Then $\tau(\overline{h^i} - \overline{h^0}) < \gamma^{\hbar}$, for all $i \geqslant i_0$. Clearly, from condition (iii) we have $\overline{h^0} \in (E^F_{u,v}(q,w))_\tau$. □

Suppose Theorems 4.2 and 4.6 are satisfied, then we have the next Theorem.

Theorem 4.7. *Suppose the conditions of Theorem 4.6 are confirmed, then $\overline{\mathbb{D}^s}_{E^F_{u,v}(q,w)}$ is an operators' ideal.*

Theorem 4.8. *If s-type $\mathcal{E}^F_\tau := \left\{\overline{h} = (\overline{s_j(H)}) \in \mathfrak{R}^{\mathcal{N}} : \ H \in \mathbb{D}(\mathcal{G},\mathcal{V}) \text{ and } \tau(\overline{h}) < \infty\right\}$ and $\overline{\mathbb{D}^s}_{\mathcal{E}_\tau}$ is an operators' ideal, then*

a. *s-type $\mathcal{E}^F_\tau \supset \mathcal{F}$;*

b. *if $\left(\overline{s_j(H_1)}\right)_{j=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$ and $\left(\overline{s_j(H_2)}\right)_{j=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$, then $\left(\overline{s_j(H_1+H_2)}\right)_{j=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$;*

c. *if $\varepsilon \in \mathfrak{R}$ and $\left(\overline{s_j(H)}\right)_{j=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$, then $|\varepsilon| \left(\overline{s_j(H)}\right)_{j=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$; and*

d. *presume $\left(\overline{s_j(U)}\right)_{j=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$ and $\overline{s_j(T)} \leqslant \overline{s_j(U)}$, for every $j \in \mathcal{N}$, where $T, U \in \mathbb{D}(\mathcal{G},\mathcal{V})$, then $\left(\overline{s_j(T)}\right)_{j=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$, i.e., $\mathcal{E}^F_\tau$ is a solid space.*

Proof. Let $\overline{\mathbb{D}^s}_{\mathcal{E}_\tau}$ be a mappings' ideal.

a. We have $\mathbb{F}(\mathcal{G},\mathcal{V}) \subset \overline{\mathbb{D}^s}_{\mathcal{E}^F_\tau}(\mathcal{G},\mathcal{V})$. Hence for all $X \in \mathbb{F}(\mathcal{G},\mathcal{V})$, we have $\left(\overline{s_r(X)}\right)_{r=0}^\infty \in \mathcal{F}$. Hence $\left(\overline{s_r(X)}\right)_{r=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$. So $\mathcal{F} \subset$ s-type $\mathcal{E}^F_\tau$.

b and c. The space $\overline{\mathbb{D}^s}_{\mathcal{E}^F_\tau}(\mathcal{G},\mathcal{V})$ is linear over $\mathfrak{R}$. Hence for every $\lambda \in \mathfrak{R}$ and $X_1, X_2 \in \overline{\mathbb{D}^s}_{\mathcal{E}^F_\tau}(\mathcal{G},\mathcal{V})$, we have $X_1 + X_2 \in \overline{\mathbb{D}^s}_{\mathcal{E}^F_\tau}(\mathcal{G},\mathcal{V})$ and $\lambda X_1 \in \overline{\mathbb{D}^s}_{\mathcal{E}^F_\tau}(\mathcal{G},\mathcal{V})$. So

$$\left(\overline{s_r(X_1)}\right)_{r=0}^\infty \in \text{s-type } \mathcal{E}^F_\tau \text{ and } \left(\overline{s_r(X_2)}\right)_{r=0}^\infty \in \text{s-type } \mathcal{E}^F_\tau \Rightarrow \left(\overline{s_r(X_1+X_2)}\right)_{r=0}^\infty \in \text{s-type } \mathcal{E}^F_\tau$$

and

$$\lambda \in \mathfrak{R} \text{ and } \left(\overline{s_r(X_1)}\right)_{r=0}^\infty \in \text{s-type } \mathcal{E}^F_\tau \Rightarrow |\lambda| \left(\overline{s_r(X_1)}\right)_{r=0}^\infty \in \text{s-type } \mathcal{E}^F_\tau.$$

d. Let $A \in \mathbb{D}(\mathcal{G}_0,\mathcal{G})$, $B \in \overline{\mathbb{D}^s}_{\mathcal{E}^F_\tau}(\mathcal{G},\mathcal{V})$ and $D \in \mathbb{D}(\mathcal{V},\mathcal{V}_0)$, then $DBA \in \overline{\mathbb{D}^s}_{\mathcal{E}^F_\tau}(\mathcal{G}_0,\mathcal{V}_0)$. Therefore, since $\left(\overline{s_r(B)}\right)_{r=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$, then $\left(\overline{s_r(DBA)}\right)_{r=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$. As $\overline{s_r(DBA)} \leqslant \|D\|\overline{s_r(B)}\,\|A\|$. By condition c, let $\left(\|D\|\,\|A\|\overline{s_r(B)}\right)_{r=0}^\infty \in \mathcal{E}^F_\tau$, then $\left(\overline{s_r(DBA)}\right)_{r=0}^\infty \in$ s-type $\mathcal{E}^F_\tau$. Hence s-type $\mathcal{E}^F_\tau$ is a solid space. □

Some properties of s-type $(E^F_{u,v}(q,w))_\tau$ are presented in the next theorem according to Theorems 4.8 and 4.7.

Theorem 4.9.

a. *s-type $(E^F_{u,v}(q,w))_\tau \supset \mathcal{F}$.*

b. *If $\left(\overline{s_n(X_1)}\right)_{n=0}^\infty \in$ s-type $(E^F_{u,v}(q,w))_\tau$ and $\left(\overline{s_n(X_2)}\right)_{n=0}^\infty \in$ s-type $(E^F_{u,v}(q,w))_\tau$, then*

$$\left(\overline{s_n(X_1+X_2)}\right)_{n=0}^\infty \in \text{ s-type}(E^F_{u,v}(q,w))_\tau.$$

c. *Presume $\lambda \in \mathfrak{R}$ and $\left(\overline{s_n(X)}\right)_{n=0}^\infty \in$ s-type $(E^F_{u,v}(q,w))_\tau$, then $|\lambda| \left(\overline{s_n(X)}\right)_{n=0}^\infty \in$ s-type $(E^F_{u,v}(q,w))_\tau$.*

d. *s-type $(E^F_{u,v}(q,w))_\tau$ is a solid space.*

Under the conditions of Theorem 4.6, some properties of the ideal constructed by $(E^F_{u,v}(q,w))_\tau$ and extended s-numbers are discussed.

Theorem 4.10. *The conditions of Theorem 4.6 are sufficient only for the closure of $\mathbb{F}(\mathcal{G},\mathcal{V}) = \overline{\mathbb{D}^\alpha}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$.*

Proof. Clearly, the closure of $\mathbb{F}(\mathcal{G},\mathcal{V}) \subseteq \overline{\mathbb{D}^{\alpha}}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ from the linearity of the space $(E^F_{u,v}(q,w))_\tau$ and $\overline{e_m} \in (E^F_{u,v}(q,w))_\tau$, for all $m \in \mathcal{N}$. To show that $\overline{\mathbb{D}^{\alpha}}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}) \subseteq$ the closure of $\mathbb{F}(\mathcal{G},\mathcal{V})$, if $H \in \overline{\mathbb{D}^{\alpha}}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$, hence $(\overline{\alpha_l(H)})_{m=0}^{\infty} \in (E^F_{u,v}(q,w))_\tau$. As $\tau(\overline{\alpha_m(H)})_{m=0}^{\infty} < \infty$ and if $\gamma \in (0,1)$, so we have $l_0 \in \mathcal{N}-\{0\}$ so that $\tau((\overline{\alpha_m(H)})_{m=l_0}^{\infty}) < \frac{\gamma}{2^{\hbar+3}\delta j}$, for some $j \geqslant 1$, and $\delta = \max\left\{1, \sum_{l=l_0}^{\infty}\left(\frac{1}{(u+v)^l}\right)^{w_l}\right\}$. Since $\overline{\alpha_l(H)} \in \downarrow^F$, then

$$\begin{aligned}\sum_{l=l_0+1}^{2l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\alpha_{2l_0}(H)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} &\leqslant \sum_{l=l_0+1}^{2l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\alpha_{z}(H)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} \\ &\leqslant \sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\alpha_{z}(H)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} < \frac{\gamma}{2^{\hbar+3}\delta j}.\end{aligned} \tag{4.1}$$

Hence $U \in \mathbb{F}_{2l_0}(\mathcal{G},\mathcal{V})$ with $\text{rank}(U) \leqslant 2l_0$ and

$$\begin{aligned}\sum_{l=2l_0+1}^{3l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l} &\leqslant \sum_{l=l_0+1}^{2l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l} \\ &< \frac{\gamma}{2^{\hbar+3}\delta j}.\end{aligned} \tag{4.2}$$

Since $(w_l) \in \uparrow \cap \ell_\infty$, we get

$$\sup_{l=l_0}^{\infty} \bar{\rho}^{w_l}\left(\sum_{z=0}^{l_0} A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right) < \frac{\gamma}{2^{2\hbar+2}\delta}.$$

Therefore,

$$\sum_{l=0}^{l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l} < \frac{\gamma}{2^{\hbar+3}\delta j}.$$

From inequalities (2.1)-(4.2), one obtains

$$\begin{aligned}
&d(H,U)\\
&= \tau\left(\overline{\alpha_l(H-U)}\right)_{l=0}^{\infty}\\
&= \sum_{l=0}^{3l_0-1}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\alpha_z(H-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} + \sum_{l=3l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\alpha_z(H-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}\\
&\leqslant \sum_{l=0}^{3l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l}\\
&\quad + \sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l+2l_0} A(l+2l_0,z)q_{l+2l_0,z}\overline{\alpha_z(H-U)},\bar{0}\right)}{(u+v)^{l+2l_0}}\right)^{w_{l+2l_0}}\\
&\leqslant \sum_{l=0}^{3l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l} + \sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l+2l_0} A(l+2l_0,z)q_{l+2l_0,z}\overline{\alpha_z(H-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}
\end{aligned}$$

$$\leqslant 3\sum_{l=0}^{l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$+\sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{2l_0-1}A(l+2l_0,z)q_{l+2l_0,z}\overline{\alpha_z(H-U)}+\sum_{z=2l_0}^{l+2l_0}A(l+2l_0,z)q_{l+2l_0,z}\overline{\alpha_z(H-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$\leqslant 3\sum_{l=0}^{l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{2l_0-1}A(l+2l_0,z)q_{l+2l_0,z}\overline{\alpha_z(H-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=2l_0}^{l+2l_0}A(l+2l_0,z)q_{l+2l_0,z}\overline{\alpha_z(H-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$\leqslant 3\sum_{l=0}^{l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{2l_0-1}A(l+2l_0,z)q_{l+2l_0,z}\overline{\alpha_z(Z-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l}A(l+2l_0,z+2l_0)q_{z+2l_0}\overline{\alpha_{z+2l_0}(H-U)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$\leqslant 3\sum_{l=0}^{l_0}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l}$$
$$+2^{2\hbar-1}\sup_{l=l_0}^{\infty}\bar{\rho}^{w_l}\left(\sum_{z=0}^{l_0}A(l,z)q_{l,z}\overline{\|H-U\|},\bar{0}\right)\sum_{l=l_0}^{\infty}\left(\frac{1}{(u+v)^l}\right)^{w_l}$$
$$+2^{\hbar-1}\sum_{l=l_0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\overline{\alpha_z(H)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}<\gamma.$$

Next, one gets a negative example as $I_2\in\overline{\mathbb{D}^{\alpha}}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$, where $A(l,z)q_{l,z}=1$, for every $l,z\in\mathcal{N}$ and $w=(0,-1,2,2,2,\ldots)$. However, $(w_l)\notin\uparrow$. One obtains a negative answer of Rhoades [30] open problem about the linearity of s-type $(E^F_{u,v}(q,w))_\tau$ spaces. □

Throughout the article we will use the notation $\Xi(H)=\tau\Big(\overline{(s_b(H))_{b=0}^{\infty}}\Big)$, where $H\in\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}$.

Theorem 4.11. *The class* $\left(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau},\Xi\right)$ *is a pre-quasi Banach ideal.*

Proof. It is clear that Ξ is a pre-quasi norm on $\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}$, since τ is a pre-quasi norm on $(E^F_{u,v}(q,w))_\tau$. Let $(X_m)_{m\in\mathcal{N}}$ be a Cauchy sequence in $\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ and since $\mathbb{D}(\mathcal{G},\mathcal{V})\supseteq\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$, we have

$$\Xi(H_j-H_m)=\sum_{l=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\overline{s_z(H_j-H_m)},\bar{0}\right)}{(u+v)^l}\right)^{w_l}\geqslant\left(q_{0,0}\,\|H_j-H_m\|\right)^{w_0}.$$

Therefore, $(H_m)_{m\in\mathcal{N}}$ is a Cauchy sequence in $\mathbb{D}(\mathcal{G},\mathcal{V})$. As $\mathbb{D}(\mathcal{G},\mathcal{V})$ is a Banach space, then $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ such that $\lim_{m\to\infty} \|H_m - H\| = 0$. As $(\overline{s_l(H_m)})_{l=0}^{\infty} \in (E^F_{u,v}(q,w))_\tau$, for all $m \in \mathcal{N}$. In view of Definition 4.3 conditions (ii), (iii), and (v), we have

$$\begin{aligned}\Xi(H) &= \sum_{l=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{s_z(H)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} \\ &\leqslant 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{s_{[\frac{z}{2}]}(H-H_m)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} \\ &\quad + 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{s_{[\frac{z}{2}]}(H_m)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} \\ &\leqslant 2^{\hbar-1}\sum_{l=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{\|H-H_m\|},\bar{0}\right)}{(u+v)^l}\right)^{w_l} \\ &\quad + 2^{\hbar-1}D_0\sum_{l=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{l} A(l,z)q_{l,z}\overline{s_z(H_m)},\bar{0}\right)}{(u+v)^l}\right)^{w_l} < \infty.\end{aligned}$$

Then $(\overline{s_b(H)})_{b=0}^{\infty} \in (E^F_{u,v}(q,w))_\tau$, so $H \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$. □

Theorem 4.12. *If* $1 < w_b^{(1)} < w_b^{(2)}$ *and* $0 < q_{b,z}^{(2)} \leqslant q_{b,z}^{(1)}$*, for all* $b, z \in \mathcal{N}$*, then*

$$\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(1)}_{b,z}),(w^{(1)}_b))\right)_\tau}(\mathcal{G},\mathcal{V}) \subsetneqq \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(2)}_{b,z}),(w^{(2)}_b))\right)_\tau}(\mathcal{G},\mathcal{V}) \subsetneqq \mathbb{D}(\mathcal{G},\mathcal{V}).$$

Proof. Presume $H \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(1)}_{b,z}),(w^{(1)}_b))\right)_\tau}(\mathcal{G},\mathcal{V})$, then $(\overline{s_b(H)}) \in \left(E^F_{u,v}((q^{(1)}_{b,z}),(w^{(1)}_b))\right)_\tau$. Hence

$$\sum_{b=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{b} A(b,z)q^{(2)}_{b,z}\overline{s_z(H)},\bar{0}\right)}{(u+v)^b}\right)^{w_b^{(2)}} < \sum_{b=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{b} A(b,z)q^{(1)}_{b,z}\overline{s_z(H)},\bar{0}\right)}{(u+v)^b}\right)^{w_b^{(1)}} < \infty.$$

So $H \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(2)}_{b,z}),(w^{(2)}_b))\right)_\tau}(\mathcal{G},\mathcal{V})$. Take $(\overline{s_b(H)})_{b=0}^{\infty}$ with $\bar{\rho}\left(\sum_{z=0}^{b} A(b,z)q^{(1)}_{b,z}\overline{s_z(H)},\bar{0}\right) = \frac{(u+v)^b}{\sqrt[w_b^{(1)}]{b+1}}$, then $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ with

$$\sum_{b=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{b} A(b,z)q^{(1)}_{b,z}\overline{s_z(H)},\bar{0}\right)}{(u+v)^b}\right)^{w_b^{(1)}} = \sum_{b=0}^{\infty}\frac{1}{b+1} = \infty$$

and

$$\begin{aligned}\sum_{b=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{b} A(b,z)q^{(2)}_{b,z}\overline{s_z(H)},\bar{0}\right)}{(u+v)^b}\right)^{w_b^{(2)}} &\leqslant \sum_{b=0}^{\infty}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{b} A(b,z)q^{(1)}_{b,z}\overline{s_z(H)},\bar{0}\right)}{(u+v)^b}\right)^{w_b^{(2)}} \\ &= \sum_{b=0}^{\infty}\left(\frac{1}{b+1}\right)^{\frac{w_b^{(2)}}{w_b^{(1)}}} < \infty.\end{aligned}$$

Hence $H \notin \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(1)}_{b,z}),(w^{(1)}_b))\right)_\tau}(\mathcal{G},\mathcal{V})$ and $H \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(2)}_{b,z}),(w^{(2)}_b))\right)_\tau}(\mathcal{G},\mathcal{V})$. Clearly,

$$\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(2)}_{b,z}),(w^{(2)}_b))\right)_\tau}(\mathcal{G},\mathcal{V}) \subset \mathbb{D}(\mathcal{G},\mathcal{V}).$$

Fix $(\overline{s_b(H)})_{b=0}^{\infty}$ with

$$\bar{\rho}\left(\sum_{z=0}^{b} A(b,z)q^{(2)}_{b,z}\overline{s_z(H)},\bar{0}\right) = \frac{(u+v)^b}{\sqrt[w^{(2)}_b]{b+1}}.$$

So $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ and $H \notin \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q^{(2)}_{b,z}),(w^{(2)}_b))\right)_\tau}(\mathcal{G},\mathcal{V})$. □

Recall that, by Dvoretzky's Theorem [27] we have $\mathcal{G}/Y_j$ and $M_j \subseteq \mathcal{V}$ mapped onto ℓ_2^j through isomorphisms V_j and X_j such that $\|V_j\|\|V_j^{-1}\| \leqslant 2$ and $\|X_j\|\|X_j^{-1}\| \leqslant 2$, for all $j \in \mathcal{N}$. If T_j is the quotient operator from $\mathcal{G}$ onto $\mathcal{G}/Y_j$, I_j is the identity operator on ℓ_2^j, and J_j is the natural embedding operator from M_j into $\mathcal{V}$. Let m_j be the Bernstein numbers [26].

Theorem 4.13. *Suppose* $\left(\frac{\sum_{z=0}^{l} A(l,z)q_{l,z}}{(u+v)^l}\right)_{l=0}^{\infty} \notin \ell_{((w_l))}$, *then* $\overline{\mathbb{D}^\alpha}_{(E^F_{u,v}(q,w))_\tau}$ *is minimum.*

Proof. Presume $\overline{\mathbb{D}^\alpha}_{E^F_{u,v}(q,w)}(\mathcal{G},\mathcal{V}) = \mathbb{D}(\mathcal{G},\mathcal{V})$, hence we have $\gamma > 0$, such that $\Xi(H) \leqslant \gamma\|H\|$, for every $H \in \mathbb{D}(\mathcal{G},\mathcal{V})$ and $\Xi(H) = \sum_{b=0}^{\infty}\left(\frac{\bar{\rho}(\sum_{z=0}^{b} A(b,z)q_{b,z}\overline{\alpha_z(H)},\bar{0})}{(u+v)^b}\right)^{w_b}$. Then

$$\begin{aligned}
1 = m_z(I_j) = m_z(X_jX_j^{-1}I_jV_jV_j^{-1}) &\leqslant \|X_j\|m_z(X_j^{-1}I_jV_j)\|V_j^{-1}\| \\
&= \|X_j\|m_z(J_jX_j^{-1}I_jV_j)\|V_j^{-1}\| \\
&\leqslant \|X_j\|d_z(J_jX_j^{-1}I_jV_j)\|V_j^{-1}\| \\
&= \|X_j\|d_z(J_jX_j^{-1}I_jV_jT_j)\|V_j^{-1}\| \leqslant \|X_j\|\alpha_z(J_jX_j^{-1}I_jV_jT_j)\|V_j^{-1}\|.
\end{aligned}$$

If $0 \leqslant m \leqslant j$, hence

$$\sum_{z=0}^{m} A(m,z)q_{m,z} \leqslant \bar{\rho}\left(\sum_{z=0}^{m}\|X_j\|A(m,z)q_{m,z}\overline{\alpha_z(J_jX_j^{-1}I_jV_jT_j)\|V_j^{-1}\|},\bar{0}\right) \Rightarrow$$

$$\left(\frac{\sum_{z=0}^{m} A(m,z)q_{m,z}}{(u+v)^m}\right)^{w_m} \leqslant (\|X_j\|\|V_j^{-1}\|)^{w_m}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{m} A(m,z)q_{m,z}\overline{\alpha_z(J_jX_j^{-1}I_jV_jT_j)},\bar{0}\right)}{(u+v)^m}\right)^{w_m}.$$

So for some $\lambda \geqslant 1$, we obtain

$$\sum_{m=0}^{j}\left(\frac{\sum_{z=0}^{m} A(m,z)q_{m,z}}{(u+v)^m}\right)^{w_m} \leqslant \lambda\|X_j\|\|V_j^{-1}\|\sum_{m=0}^{j}\left(\frac{\bar{\rho}\left(\sum_{z=0}^{m} A(m,z)q_{m,z}\overline{\alpha_z(J_jX_j^{-1}I_jV_jT_j)},\bar{0}\right)}{(u+v)^m}\right)^{w_m} \Rightarrow$$

$$\sum_{m=0}^{j}\left(\frac{\sum_{z=0}^{m} A(m,z)q_{m,z}}{(u+v)^m}\right)^{w_m} \leqslant \lambda\|X_j\|\|V_j^{-1}\|\Xi(J_jX_j^{-1}I_jV_jT_j) \leqslant \lambda\gamma\|X_j\|\|V_j^{-1}\|\|J_jX_j^{-1}I_jV_jT_j\| \leqslant 4\lambda\gamma.$$

By taking $j \to \infty$, hence we have a contradiction. So $\mathcal{G}$ and $\mathcal{V}$ both cannot be infinite-dimensional, whenever $\overline{\mathbb{D}^\alpha}_{E^F_{u,v}(q,w)}(\mathcal{G},\mathcal{V}) = \mathbb{D}(\mathcal{G},\mathcal{V})$. □

Theorem 4.14. *Presume* $\left(\frac{\sum_{z=0}^{l} A(l,z)q_{l,z}}{(u+v)^l}\right)_{l=0}^{\infty} \notin \ell_{((w_l))}$, *then* $\overline{\mathbb{D}^d}_{E^F_{u,v}(q,w)}$ *is minimum.*

Theorem 4.15. *If $1 < w_l^{(1)} < w_l^{(2)}$ and $0 < q_{l,z}^{(2)} \leqslant q_{l,z}^{(1)}$, for every $l, z \in \mathcal{N}$, hence*

$$\mathbb{D}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V}), \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big) = \mathcal{A}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V}), \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big).$$

Proof. Presume $X \in \mathbb{D}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V}), \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big)$ and

$$X \notin \mathcal{A}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V}), \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big).$$

According to Lemma 3.6, then there are $Y \in \mathbb{D}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big)$ and $Z \in \mathbb{D}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big)$ with $ZXYI_b = I_b$. So for every $b \in \mathcal{N}$, one gets

$$\begin{aligned}\|I_b\|_{\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})} &= \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l} A(l,z) q_{l,z}^{(1)} \overline{s_z(I_b)}, \overline{0}\right)}{(u+v)^l}\right)^{w_l^{(1)}} \\ &\leqslant \|ZXY\| \|I_b\|_{\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V})} \\ &\leqslant \sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l} A(l,z) q_{l,z}^{(2)} \overline{s_z(I_b)}, \overline{0}\right)}{(u+v)^l}\right)^{w_l^{(2)}}.\end{aligned}$$

So, there is contradictions with Theorem 4.12. Hence

$$X \in \mathcal{A}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V}), \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big).$$

□

Corollary 4.16. *If $1 < w_l^{(1)} < w_l^{(2)}$ and $0 < q_{l,z}^{(2)} \leqslant q_{l,z}^{(1)}$, for every $l, z \in \mathcal{N}$, hence*

$$\mathbb{D}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V}), \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big) = \mathcal{K}\Big(\overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(2)}),(w_l^{(2)}))\right)_\tau}(\mathcal{G},\mathcal{V}), \overline{\mathbb{D}^s}_{\left(E^F_{u,v}((q_{l,z}^{(1)}),(w_l^{(1)}))\right)_\tau}(\mathcal{G},\mathcal{V})\Big).$$

Proof. The proof follows, since $\mathcal{A} \subset \mathcal{K}$. □

Theorem 4.17. *The class $\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}$ is simple.*

Proof. Let the closed ideal $\mathcal{K}(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}))$ include a mapping $H \notin \mathcal{A}(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}))$. According to Lemma 3.6, hence we have $P, A \in \mathbb{D}(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}))$ such that $AHPI_j = I_j$. Therefore, $I_{\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})} \in \mathcal{K}(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}))$. Hence $\mathbb{D}(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})) = \mathcal{K}(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}))$. So $\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}$ is a simple Banach space. □

Theorem 4.18. *If $\inf_l \left(\frac{\sum_{z=0}^{l} A(l,z) q_z}{(u+v)^l}\right)^{w_l} > 0$, then $\left(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}\right)^{\Upsilon}(\mathcal{G},\mathcal{V}) = \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$.*

Proof. Suppose $H \in \left(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}\right)^\Upsilon(\mathcal{G},\mathcal{V})$, then $(\overline{\gamma_m(H)})_{m=0}^\infty \in (E^F_{u,v}(q,w))_\tau$ and $\|H - \overline{\rho}(\overline{\gamma_m(H)},\overline{0})I\| = 0$, for every $m \in \mathcal{N}$. We have $H = \overline{\rho}(\overline{\gamma_m(H)},\overline{0})I$, for all $m \in \mathcal{N}$, hence

$$\overline{\rho}(\overline{s_m(H)},\overline{0}) = \overline{\rho}(\overline{s_m(\overline{\rho}(\overline{\gamma_m(H)},\overline{0})I)},\overline{0}) = \overline{\rho}(\overline{\gamma_m(H)},\overline{0}),$$

for all $m \in \mathcal{N}$. Hence $(\overline{s_m(H)})_{m=0}^\infty \in (E^F_{u,v}(q,w))_\tau$. So $H \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$.

Next, assume $H \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$. Therefore, $(\overline{s_m(H)})_{m=0}^\infty \in (E^F_{u,v}(q,w))_\tau$. So

$$\sum_{m=0}^\infty \left(\frac{\overline{\rho}\left(\sum_{z=0}^m A(m,z)q_{m,z}\overline{s_z(H)},\overline{0}\right)}{(u+v)^m}\right)^{w_m} \geqslant \inf_m \left(\frac{\sum_{z=0}^m A(m,z)q_{m,z}}{(u+v)^m}\right)^{w_m} \sum_{m=0}^\infty \left[\overline{\rho}\left(\overline{s_m(H)},\overline{0}\right)\right]^{w_m}.$$

Then $\lim_{m\to\infty}\overline{s_m(H)} = \overline{0}$. If $\|H - \overline{\rho}(\overline{s_m(H)},\overline{0})I\|^{-1}$ exists, for all $m \in \mathcal{N}$. So $\|H - \overline{\rho}(\overline{s_m(H)},\overline{0})I\|^{-1}$ exists and bounded, for every $m \in \mathcal{N}$. Then $\lim_{m\to\infty}\|H - \overline{\rho}(\overline{s_m(H)},\overline{0})I\|^{-1} = \|H\|^{-1}$ exists and bounded. Since $\left(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau},\Xi\right)$ is a pre-quasi ideal, hence

$$I = HH^{-1} \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}) \Rightarrow (\overline{s_m(I)})_{m=0}^\infty \in E^F_{u,v}(q,w) \Rightarrow \lim_{m\to\infty}\overline{s_m(I)} = \overline{0}.$$

Therefore, we have a contradiction as $\lim_{m\to\infty}\overline{s_m(I)} = \overline{1}$. So $\|H - \overline{\rho}(\overline{s_m(H)},\overline{0})I\| = 0$, for all $m \in \mathcal{N}$. Hence $\|H - \overline{\rho}(\overline{\gamma_m(H)},\overline{0})I\| = 0$, for all $m \in \mathcal{N}$. That implies $H \in \left(\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}\right)^\Upsilon(\mathcal{G},\mathcal{V})$. □

5. Fixed points of Kannan contraction type

Suppose the conditions of Theorem 4.6 are established. In this section, we have offered the existence of a fixed point of Kannan contraction mapping acting on $\overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}$. A numerical example is presented to investigate our results.

In this part, we will use $\Xi(V) = \tau\left((\overline{s_b(V)})_{b=0}^\infty\right) = \left[\sum_{l=0}^\infty\left(\frac{\overline{\rho}(\sum_{z=0}^l A(l,z)q_{l,z}\overline{s_z(V)},\overline{0})}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}}$, for all $V \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$.

Definition 5.1. A function Ξ on $\overline{\mathbb{D}^s}_{\mathcal{E}^F}$ holds the Fatou property, if for all $\{V_b\}_{b\in\mathcal{N}} \subseteq \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$ with $\lim_{b\to\infty}\Xi(V_b - V) = 0$ and all $T \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$, we have $\Xi(T - V) \leqslant \sup_b \inf_{j\geqslant b}\Xi(T - V_j)$.

Theorem 5.2. *The function Ξ does not verify the Fatou property.*

Proof. If $\{W_m\}_{m\in\mathcal{N}} \subseteq \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ so that $\lim_{m\to\infty}\Xi(W_m - W) = 0$. Then $W \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$. As for every $V \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$, then

$$\begin{aligned}
\Xi(V - W) &= \left[\sum_{l=0}^\infty\left(\frac{\overline{\rho}\left(\sum_{z=0}^l A(l,z)q_{l,z}\overline{s_z(V-W)},\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}} \\
&\leqslant \left[\sum_{l=0}^\infty\left(\frac{\overline{\rho}\left(\sum_{z=0}^l A(l,z)q_{l,z}\overline{s_{[\frac{z}{2}]}(V-W_i)},\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}} \\
&\quad + \left[\sum_{l=0}^\infty\left(\frac{\overline{\rho}\left(\sum_{z=0}^l A(l,z)q_{l,z}\overline{s_{[\frac{z}{2}]}(W-W_i)},\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}} \\
&\leqslant (2^{2\hbar-1} + 2^{\hbar-1} + 2^\hbar)^{\frac{1}{\hbar}} \sup_m \inf_{i\geqslant m}\left[\sum_{l=0}^\infty\left(\frac{\overline{\rho}\left(\sum_{z=0}^l A(l,z)q_{l,z}\overline{s_z(V-W_i)},\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}}.
\end{aligned}$$

Therefore, Ξ does not satisfy the Fatou property. □

Definition 5.3. Presume $G : \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V}) \to \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$ and $B \in \overline{\mathbb{D}^s}_{\mathcal{E}^F}(\mathcal{G},\mathcal{V})$. The operator G is called Ξ-sequentially continuous at B, if and only if, when $\lim_{m\to\infty}\Xi(W_m - B) = 0$, then $\lim_{m\to\infty}\Xi(GW_m - GB) = 0$.

Theorem 5.4. *Consider* $G : \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V}) \to \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$*. The mapping* $A \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ *is the only fixed point of* G *when the following conditions are verified:*

(i) G *is Kannan* Ξ*-contraction;*

(ii) G *is* Ξ*-sequentially continuous at* $A \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$*;*

(iii) *there is* $B \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ *with* $\{G^m B\}$ *has* $\{G^{m_i}B\}$ *converges to* A.

Proof. Presume A is not a fixed point of G, then $GA \neq A$. By the parts (ii) and (iii), we have

$$\lim_{m_i\to\infty}\Xi(G^{m_i}B - A) = 0 \text{ and } \lim_{m_i\to\infty}\Xi(G^{m_i+1}B - GA) = 0.$$

AS G is Kannan Ξ-contraction operator, then

$$\begin{aligned}0 < \Xi(GA - A) &= \Xi\left((GA - G^{m_i+1}B) + (G^{m_i}B - A) + (G^{m_i+1}B - G^{m_i}B)\right)\\ &\leqslant (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar})^{\frac{1}{\hbar}}\Xi\left(G^{m_i+1}B - GA\right) + (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar})^{\frac{2}{\hbar}}\Xi\left(G^{m_i}B - A\right)\\ &\quad + (2^{2\hbar-1} + 2^{\hbar-1} + 2^{\hbar})^{\frac{2}{\hbar}}\zeta\left(\frac{\zeta}{1-\zeta}\right)^{m_i-1}\Xi(GB - B).\end{aligned}$$

By letting $m_i \to \infty$, which is a contradiction, so A is a fixed point of G. To show the uniqueness of the fixed point A, let we have two different fixed points $A, D \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ of G. So

$$\Xi(A - D) \leqslant \Xi(GA - GD) \leqslant \zeta\Big(\Xi(GA - A) + \Xi(GD - D)\Big) = 0.$$

Then $A = D$. □

Example 5.5. Suppose that

$$M : \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z+4)A(l,z)}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_\tau}(\mathcal{G},\mathcal{V}) \to \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z+4)A(l,z)}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_\tau}(\mathcal{G},\mathcal{V})$$

and

$$M(H) = \begin{cases}\frac{H}{6}, & \Xi(H) \in [0,1),\\ \frac{H}{7}, & \Xi(H) \in [1,\infty).\end{cases}$$

For every $H_1, H_2 \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z+4)A(l,z)}\right)_{l=0}^\infty, \left(\frac{2l+3}{l+2}\right)_{l=0}^\infty\right)\right)_\tau}$, if $\Xi(H_1), \Xi(H_2) \in [0,1)$, one gets

$$\Xi(MH_1 - MH_2) = \Xi(\frac{H_1}{6} - \frac{H_2}{6}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(\Xi(\frac{5H_1}{6}) + \Xi(\frac{5H_2}{6})\Big) = \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)\Big).$$

Assume $\Xi(H_1), \Xi(H_2) \in [1,\infty)$, one obtains

$$\Xi(MH_1 - MH_2) = \Xi(\frac{H_1}{7} - \frac{H_2}{7}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{216}}\Big(\Xi(\frac{6H_1}{7}) + \Xi(\frac{6H_2}{7})\Big) = \frac{\sqrt{2}}{\sqrt[4]{216}}\Big(\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)\Big).$$

If $\Xi(H_1) \in [0,1)$ and $\Xi(H_2) \in [1,\infty)$, hence

$$\Xi(MH_1 - MH_2) = \Xi(\frac{H_1}{6} - \frac{H_2}{7}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Xi(\frac{5H_1}{6}) + \frac{\sqrt{2}}{\sqrt[4]{216}}\Xi(\frac{6H_2}{7}) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(\Xi(MH_1 - H_1) + \Xi(MH_2 - H_2)\Big).$$

So M is Kannan Ξ-contraction and $M^m(H) = \begin{cases} \frac{H}{6^m}, & \Xi(H) \in [0,1), \\ \frac{H}{7^m}, & \Xi(H) \in [1,\infty). \end{cases}$

It is clear that M is Ξ-sequentially continuous at the zero operator Θ and $\{M^m H\}$ has a $\{M^{m_j} H\}$ converging to Θ. According to Theorem 5.4, Θ is the only fixed point of M. If

$$\{H^{(a)}\} \subseteq \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z+4)A(l,z)}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_\tau}$$

with $\lim_{a\to\infty} \Xi(H^{(a)} - H^{(0)}) = 0$, where $H^{(0)} \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z+4)A(l,z)}\right)_{l=0}^{\infty}, \left(\frac{2l+3}{l+2}\right)_{l=0}^{\infty}\right)\right)_\tau}$ such that $\Xi(H^{(0)}) = 1$. As Ξ is continuous, then

$$\lim_{a\to\infty} \Xi(MH^{(a)} - MH^{(0)}) = \lim_{a\to\infty} \Xi\Big(\frac{H^{(0)}}{6} - \frac{H^{(0)}}{7}\Big) = \Xi\Big(\frac{H^{(0)}}{42}\Big) > 0.$$

Hence M is not Ξ-sequentially continuous at $H^{(0)}$. This explains M is not continuous at $H^{(0)}$.

6. Application on fuzzy non-linear dynamical system

The solution of nonlinear matrix equations (2.1) at $D \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ under the conditions of Theorem 4.6 are explained in this part, where $\Xi(G) = \left[\sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l} A(l,z) q_{l,z} \overline{s_z(G)}, \overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}}$, for every $G \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$.

Theorem 6.1. *The fuzzy non-linear Matrix systems (2.1) includes a unique solution* $D \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ *when the following settings are confirmed:*

(1) $T \in \mathbb{D}(\mathcal{G},\mathcal{V})$ *and for all* $z \in \mathcal{N}$ *we have a positive real* κ *with* $\sup_z \kappa^{\frac{w_z}{\hbar}} \in [0, 0.5)$ *so that*

$$\begin{aligned} &\left|\sum_{m\in\mathcal{N}} \Pi(z,m)\left(f(m,\overline{s_m(G)}) - f(m,\overline{s_m(T)})\right)\right| \\ &\quad \overline{\leqslant}\kappa \left[\left|\overline{s_z(P)} - \overline{s_z(G)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f(m,\overline{s_m(G)})\right| + \left|\overline{s_z(P)} - \overline{s_z(T)} + \sum_{m\in\mathcal{N}} \Pi(z,m) f(m,\overline{s_m(T)})\right|\right]; \end{aligned}$$

(2) W *is* Ξ*-sequentially continuous at a point* $D \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$*;*

(3) *we have* $B \in \overline{\mathbb{D}^s}_{(E^F_{u,v}(q,w))_\tau}(\mathcal{G},\mathcal{V})$ *with* $\{W^a B\}$ *has a* $\{W^{a_i} B\}$ *converging to* D.

Proof. We have

$$\begin{aligned} &\Xi(WG - WT) \\ &= \left[\sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l} A(l,z) q_{l,z} (\overline{s_z(G)} - \overline{s_z(T)}), \overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}} \end{aligned}$$

$$= \left[\sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\sum_{m\in\mathcal{N}}\Pi(z,m)\left(f(m,\overline{s_m(G)})-f(m,\overline{s_m(T)})\right),\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}}$$

$$\leqslant \sup_z \kappa^{\frac{t_z}{\hbar}}\left[\sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\left(\overline{s_z(P)}-\overline{s_z(G)}+\sum_{m\in\mathcal{N}}\Pi(z,m)f(m,\overline{s_m(G)})\right),\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}}$$

$$+\sup_z \kappa^{\frac{t_z}{\hbar}}\left[\sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l}A(l,z)q_{l,z}\left(\overline{s_z(P)}-\overline{s_z(T)}+\sum_{m\in\mathcal{N}}\Pi(z,m)f(m,\overline{s_m(T)})\right),\overline{0}\right)}{(u+v)^l}\right)^{w_l}\right]^{\frac{1}{\hbar}}$$

$$= \sup_z \kappa^{\frac{t_z}{\hbar}}\left(\Xi(WG-G)+\Xi(WT-T)\right).$$

In view of Theorem 5.4, one obtains a unique solution of equation (2.1) at $D \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}(q,w)\right)_\tau}(\mathcal{G},\mathcal{V})$. □

Example 6.2. Assume that $\overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z)!}\right),\left(\frac{2l+3}{l+2}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V})$, where

$$\Xi(G) = \sqrt{\sum_{l=0}^{\infty}\left(\frac{\overline{\rho}\left(\sum_{z=0}^{l}\frac{A(l,z)}{(l+z)!}\overline{s_z(G)},\overline{0}\right)}{(u+v)^l}\right)^{\frac{2l+3}{l+2}}},$$

for all $G \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z)!}\right),\left(\frac{2l+3}{l+2}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V})$. Consider the fuzzy non-linear Matrix systems:

$$\overline{s_z(G)} = \overline{e^{-(2z+3)}} + \sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\overline{s_{z-2}(G)}|}{\sinh^d|\overline{s_{z-1}(G)}|+\overline{\sin mz}+\overline{1}}, \tag{6.1}$$

for every $z \geqslant 2$, $b, d > 0$, and if $W : \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z)!}\right),\left(\frac{2l+3}{l+2}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V}) \to \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z)!}\right),\left(\frac{2l+3}{l+2}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V})$ is defined as

$$W(G) = \left(\overline{e^{-(2z+3)}} + \sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\overline{s_{z-2}(G)}|}{\sinh^d|\overline{s_{z-1}(G)}|+\overline{\sin mz}+\overline{1}}\right)I. \tag{6.2}$$

Suppose W is Ξ-sequentially continuous at a point $D \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z)!}\right),\left(\frac{2l+3}{l+2}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V})$ and we have $B \in \overline{\mathbb{D}^s}_{\left(E^F_{u,v}\left(\left(\frac{1}{(l+z)!}\right),\left(\frac{2l+3}{l+2}\right)\right)\right)_\tau}(\mathcal{G},\mathcal{V})$ such that $\{W^a B\}$ has a $\{W^{a_i} B\}$ converging to D. Obviously,

$$\left|\sum_{m=0}^{\infty}\frac{\cosh(3m-z)\cos^b|\overline{s_{z-2}(G)}|}{\sinh^d|\overline{s_{z-1}(G)}|+\overline{\sin mz}+\overline{1}}\Big(\tan(2m+1)-\tan(2m+1)\Big)\right|$$

$$\leqslant \frac{1}{25}\left|\overline{e^{-(2z+3)}}-\overline{s_z(G)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\overline{s_{z-2}(G)}|}{\sinh^d|\overline{s_{z-1}(G)}|+\overline{\sin mz}+\overline{1}}\right|$$

$$+\frac{1}{25}\left|\overline{e^{-(2z+3)}}-\overline{s_z(T)}+\sum_{m=0}^{\infty}\frac{\tan(2m+1)\cosh(3m-z)\cos^b|\overline{s_{z-2}(T)}|}{\sinh^d|\overline{s_{z-1}(T)}|+\overline{\sin mz}+\overline{1}}\right|.$$

From Theorem 6.1, the dynamical system (6.1) includes a unique solution D.

7. Conclusion

We presented in this article some topological and geometric properties of $\overline{\mathbb{D}^s}_{\left(E^F_{u,v}(q,w)\right)_\tau}$. The existence of a fixed point in the Kannan contraction mapping on this class is explored. To put our findings to the test, we introduce a numerical experiment. In addition, an effective implementation of the fuzzy non-linear dynamical system is discussed. The ideal spectrum of mappings, the fixed points of any contraction mappings on this new fuzzy functions class, and a new general class of solutions for many stochastic nonlinear dynamical systems are investigated.

Acknowledgements

This research was funded by the University of Jeddah, Jeddah, Saudi Arabia under the grant No. UJ-20-111-DR. The authors acknowledge with thanks and gratitude the University of Jeddah for its moral, financial, and technical support.

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