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Solitary and periodic wave solutions of the loaded Boussinesq and the loaded modified Boussinesq equation

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Abstract

In this article, we establish new traveling wave solutions for the loaded Boussinesq equation and the loaded modified Boussinesq equation by the functional variable method. The performance of this method is reliable and effective and gives the exact solitary wave solutions and periodic wave solutions of the loaded Boussinesq equation and its modifications. The traveling wave solutions obtained via this method are expressed by hyperbolic functions and the trigonometric functions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features, including bell-shaped solitary wave solutions, singular soliton solutions and solitary wave solutions of kink type. This method presents a wider applicability for handling nonlinear wave equations.

Keywords: Loaded modified Boussinesq equation, hyperbolic functions, trigonometric functions, periodic wave solutions, soliton solutions, functional variable method.

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1. Introduction

Boussinesq equation(BE) has wide range of usage in wide variety of physical systems, chemistry, hydrodynamics, mechanics, biology, solid state physics, long wave and engineering [13, 26]. In 1872, Josseph Boussinesq formulated model equation for the propagation of long waves on the surface of water with a small amplitude which is expressed in the following basic form

$$u_{tt} - \alpha u_{xx} - \beta(u^2)_{xx} + \gamma u_{xxxx} = 0.$$

This equation is usually distinguished as good ($\gamma > 0$) and bad ($\gamma < 0$) ones based on its well-posedness [14]. By adding nonlinear term of the form $\lambda(u^2u_x)_x$ to BE, their new modified form is obtained as follows

$$u_{tt} - \alpha u_{xx} - \beta(u^2)_{xx} + \gamma u_{xxxx} + \lambda(u^2u_x)_x = 0.$$

BE has been exactly and numerically solved using different approaches as well as the modified decomposition method is applied to develop soliton solutions and periodic solutions [20, 33], where as a

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simplified version of the Hirota technique [17] is used to extract several exact solutions for the BE [32], the solitary wave ansatze method is employed to get solitary wave solution of the BE with power law nonlinearity [11], the extended homoclinic test method is applied to obtain a homoclinic breather-wave solution with convective effect [12], the Hirota direct bilinear method and the variational iteration techniques have employed to find the soliton solution of the good BE [30, 35], the $\exp(-\varphi(\xi))$-expansion method is applied to find exact traveling wave solutions of the (1+1)-dimensional and (2+1)-dimensional BE [3, 18], the soliton solutions of the (2+1)-dimensional BE are obtained by using (G/G')-expansion method [10], exact traveling wave solutions of the (2+1)-dimensional BE are derived by homogeneous balance method [1], exact periodic solitary wave solutions for the (2+1)-dimensional BE are obtained by using the extended ansatz function method [24]. In [19, 29], the BE is solved numerically and discovered that the used technique is unconditionally stable. These methods are also applicable to obtain soliton solutions of many non-linear evolutionary equations as well as the non-linear Schrödinger equation [36], the fuzzy Burgers equation [16], KdV and Fornberg-Witham equations [2], modified KdV equation [15] and so on.

In this article, we consider the following the loaded BE and the loaded modified BE:

$$u_{tt} - \alpha u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \varphi(t)u(0,t)u_{xx} = 0, \tag{1.1}$$

$$u_{tt} - \alpha u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \lambda(u^2 u_x)_x + \varphi(t)u(0,t)u_{xx} = 0, \tag{1.2}$$

where $u(x,t)$ is an unknown function, $x \in \mathbb{R}$, $t \geqslant 0$, α, β, γ and λ are any constants, $\varphi(t)$ is the given real continuous function.

We construct exact traveling wave solutions of the loaded BE and the loaded modified BE by the functional variable method. These solutions will be extremely useful in carrying out further analysis in the context of shallow water waves that arises in the context of oceanography. The performance of this method is reliable and effective and gives the exact solitary wave solutions and periodic wave solutions of the loaded BE and its modifications. The traveling wave solutions obtained via this method are expressed by hyperbolic functions and the trigonometric functions. The graphical representations of some obtained solutions are demonstrated to better understand their physical features, including bell-shaped solitary wave solutions, singular soliton solutions and solitary wave solutions of kink type. This method presents a wider applicability for handling nonlinear wave equations.

In recent years, in connection with intensive research of problems optimal management of the agroecosystem, for example, the problem of long-term forecasting and regulation of the level of groundwater and soil moisture, there has been a significant increase in interest in loaded equations. Among the works devoted to loaded equations, one should especially note the works of Kneser [22], Lichtenstein [25], Nakhushev [28], and others. It is known that the loaded differential equations contain some of the traces of an unknown function. In [9, 27], the term of "loaded equation" was used for the first time, the most general definitions of the loaded differential equation were given and also a detailed classifications of the differential loaded equations as well as their numerous applications were presented. A complete description of solutions of the nonlinear loaded equations and their applications can be found in papers [4–8, 21, 23, 31, 34].

The article is organized as follows. In Section 2, we present some basic information about the description of the functional variable method. Section 3 is devoted to solutions of the loaded Boussinesq equation. In Section 4, we present the graphical representation of the loaded Boussinesq equation. Solutions of the loaded modified Boussinesq equation are given in Section 5. In Section 6, we give the physical interpretations of the loaded modified Boussinesq equation. Finally, conclusions are presented in Section 7.

2. Description of the functional variable method

Consider nonlinear evolution equations with independent variables x, y and t is of the form

$$F(u, u_x, u_y, u_t, u_{xx}, u_{tt}, u_{yy}, u_{xy}, u_{xt}, u_{yt}, \ldots) = 0, \tag{2.1}$$

where F is a polynomial in $u = u(x, y, t)$ and its partial derivatives. In [37, 38], Zerarka and others have summarized the functional variable method in the following.

Step 1. We use the wave transformation

$$\xi = px + qy - kt,$$

where p and q are constants, k is the speed of the traveling wave.

Next, we can introduce the following transformation for a traveling wave solution of Eq. (2.1):

$$u(x, y, t) = u(\xi), \tag{2.2}$$

and the chain

$$\frac{\partial u}{\partial x} = p\frac{du}{d\xi}, \frac{\partial u}{\partial y} = q\frac{du}{d\xi}, \frac{\partial u}{\partial t} = -k\frac{du}{d\xi}, \ldots . \tag{2.3}$$

Using (2.2) and (2.3), the nonlinear partial differential (2.1) can be transformed into an ordinary differential equation of the form

$$P(u, u', u'', u''', \ldots) = 0, \tag{2.4}$$

where P is a polynomial in $u(\xi)$ and its total derivatives, while $u' = \frac{du}{d\xi}$.

Step 2. Then we make a transformation in which the unknown function u is considered as a functional variable in the form

$$u' = F(u), \tag{2.5}$$

then, the solution can be found by the relation

$$\int \frac{du}{F(u)} = \xi + \xi_0, \tag{2.6}$$

here ξ_0 is a constant of integration which is set equal to zero for convenience. Some successive differentiations of u in terms of F are given as

$$\begin{aligned}
u'' &= \frac{1}{2}\frac{d\left(F^2(u)\right)}{du},\\
u''' &= \frac{1}{2}\frac{d^2\left(F^2(u)\right)}{du^2}\sqrt{F^2(u)},\\
u'''' &= \frac{1}{2}\left[\frac{d^3\left(F^2(u)\right)}{du^3}F^2(u) + \frac{d^2\left(F^2(u)\right)}{du^2}\frac{d\left(F^2(u)\right)}{du}\right],\\
&\vdots
\end{aligned} \tag{2.7}$$

Step 3. The ordinary differential Eq. (2.4) can be reduced in terms of u, F and its derivatives upon using the expressions of (2.6) and (2.7) into (2.1) gives

$$H\left(u, \frac{dF(u)}{du}, \frac{d^2F(u)}{du^2}, \frac{d^3F(u)}{du^3}, \ldots\right) = 0. \tag{2.8}$$

The key idea of this particular form Eq. (2.8) is of special interest because it admits analytical solutions for a large class of nonlinear wave type equations. After integration, Eq. (2.8) provides the expression of F and this, together with Eq. (2.5), give appropriate solutions to the original problem.

3. Solutions of the loaded Boussinesq equation

We will show how to find the exact solution of the loaded BE by the functional variable method. Using the wave variable

$$u(x, t) = u(\xi), \ \xi = px - kt,$$

that will convert Eq. (1.1) to an ordinary differential equation

$$k^2u'' - \alpha p^2u'' - \beta p^2(u^2)'' - \gamma p^4u^{(IV)} + p^2\varphi(t)u(0,t)u'' = 0, \tag{3.1}$$

integrating twice Eq. (3.1) with respect to ξ, and putting the constant of integration zero, we have

$$u'' = \frac{1}{\gamma p^2}\left(\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)u - \beta u^2\right). \tag{3.2}$$

Following Eq. (2.7), it is easy to deduce from Eq. (3.2) an expression for the function $F(u)$

$$\frac{1}{2}\frac{d\left(F^2(u)\right)}{du} = \frac{1}{\gamma p^2}\left(\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)u - \beta u^2\right). \tag{3.3}$$

Integrating Eq. (3.3) and setting the constant of integration to zero yields

$$F(u) = \frac{u}{p}\sqrt{\frac{2\beta}{3\gamma}}\sqrt{\mu(t)-u}, \tag{3.4}$$

where $\mu(t) = \frac{3}{2\beta}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)$. From (2.5) and (3.4) we deduce that

$$\frac{du}{u\sqrt{\mu(t)-u}} = \sqrt{\frac{2\beta}{3\gamma}}\frac{d\xi}{p}. \tag{3.5}$$

After integrating Eq. (3.5), with zero constant of integration, we have following exact solution

$$u(x,t) = -\frac{6}{\beta}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)\frac{e^{\sqrt{\frac{1}{\gamma}\left(\frac{k^2-\alpha p^2}{p^2}+\varphi(t)u(0,t)\right)}\frac{px-kt}{p}}}{\left(1-e^{\sqrt{\frac{1}{\gamma}\left(\frac{k^2-\alpha p^2}{p^2}+\varphi(t)u(0,t)\right)}\frac{px-kt}{p}}\right)^2}. \tag{3.6}$$

It is obvious that the function $u(0,t)$ can be easily found based on expression (3.6).

We have several types of traveling wave solutions of the loaded BE (1.1) as follows.

1) When $\frac{1}{\gamma}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right) > 0$, we have the solitary wave solution

$$u(x,t) = -\frac{3}{2\beta}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)\left(\mathrm{cth}^2\left(\sqrt{\frac{1}{\gamma}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)}\frac{px-kt}{2p}\right) - 1\right). \tag{3.7}$$

2) When $\frac{1}{\gamma}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right) < 0$, we have the periodic wave solution

$$u(x,t) = \frac{3}{2\beta}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)\left(\mathrm{ctg}^2\left(\sqrt{\frac{1}{\gamma}\left(\frac{k^2-\alpha p^2}{p^2} + \varphi(t)u(0,t)\right)}\frac{px-kt}{2p}\right) + 1\right). \tag{3.8}$$

If $k=-1$, $\alpha=1$, $\beta=-6$, $\gamma=1$, $p=1$ and $\varphi(t)=t$, we obtain from (3.7) the following solitary wave solution

$$u(x,t) = \frac{tu(0,t)}{4}\left(\mathrm{cth}^2\left(\frac{\sqrt{tu(0,t)}(x+t)}{2}\right) - 1\right),$$

where $u(0,t) = \frac{1}{t^3}\ln^2\left(\frac{t+2\pm\sqrt{t^2+4t}}{2}\right)$.

If $k=-1$, $\alpha=1$, $\beta=\frac{3}{2}$, $\gamma=1$, $p=1$ and $\varphi(t)=-t^2$, we obtain from (3.8) the following the periodic wave solution

$$u(x,t) = t^2u(0,t)\left(\mathrm{ctg}^2\left(t\sqrt{u(0,t)}\frac{x+t}{2}\right) + 1\right),$$

where $u(0,t) = \frac{6\left(12-t^2\pm\sqrt{144-24t^2-3t^4}\right)}{t^6}$.

4. Graphical representation of the loaded Boussinesq equation

We present some graphs of solitary and periodic waves constructed by taking suitable values of the involved unknown parameters to visualize the underlying mechanism to the original physical phenomena. Using mathematical software Matlab, three-dimensional plots of the obtained solutions have been shown in Figures 1 and 2. A soliton or solitary wave in the concept of mathematical physics defined as a self-reinforcing wave package that retains its shape. It propagates at a constant amplitude and velocity. Solitons are solutions of a common class of nonlinearly partially differential equations with weak linearity describing physical systems. The existence of periodic traveling waves usually depends on the parameter values in a mathematical equation. If there is a periodic traveling wave solution, then there is typically a family of such solutions, with different wave speeds.

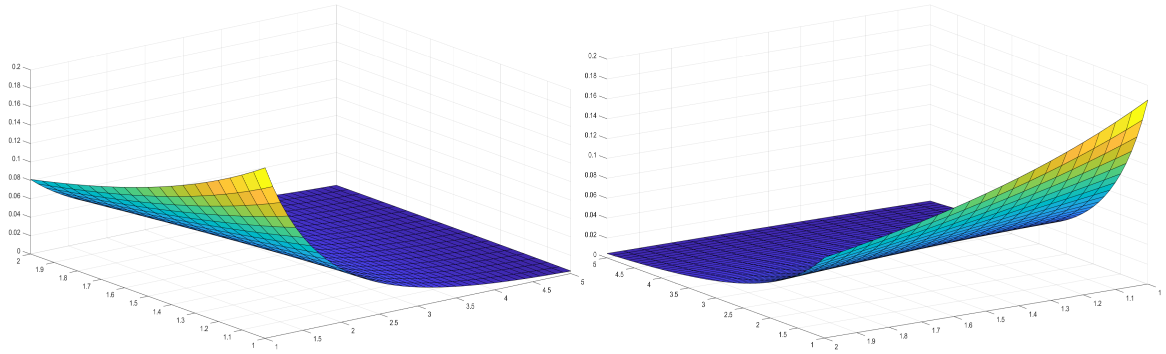

Figure 1: Solitary wave solution of the loaded BE for $k = -1$, $\alpha = 1$, $\beta = -6$, $\gamma = 1$, $p = 1$, and $\varphi(t) = t$.

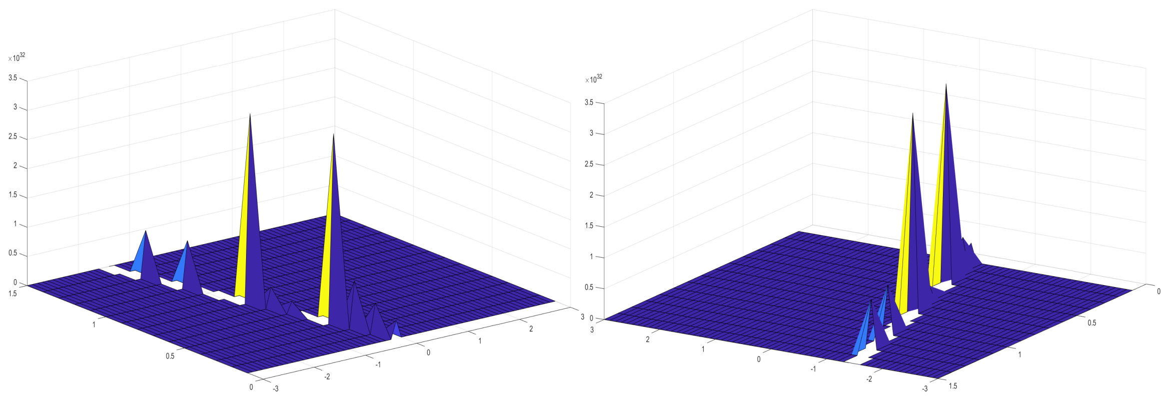

Figure 2: Periodic wave solution of the loaded BE for $k = -1$, $\alpha = 1$, $\beta = \frac{3}{2}$, $\gamma = 1$, $p = 1$, and $\varphi(t) = -t^2$.

5. Solutions of the loaded modified Boussinesq equation

Assume that Eq. (1.2) has an exact solution in the form of a traveling wave

$$u(x,t) = u(\xi), \ \ \xi = px - kt,$$

the Eq. (1.2) can be converted to an ordinary differential equation

$$k^2u'' - \alpha p^2 u'' - \beta p^2 (u^2)'' - \gamma p^4 u^{(IV)} + \lambda p^2 (u^2 u')' + p^2 \varphi(t) u(0,t) u'' = 0. \tag{5.1}$$

Twice integrating (5.1), setting the constant of integrating to zero, we obtain

$$u'' = \frac{1}{\gamma p^4}\left(k^2 u - \alpha p^2 u - \beta p^2 u^2 + \frac{\lambda p^2}{3} u^3 + p^2 \varphi(t) u(0,t) u\right). \tag{5.2}$$

Following Eq. (2.7), it is easy to deduce from Eq. (5.2) an expression for the function $F(u)$

$$\frac{1}{2}\frac{d\left(F^2(u)\right)}{du} = \frac{1}{\gamma p^4}\left(k^2u - \alpha p^2 u - \beta p^2 u^2 + \frac{\lambda p^2}{3}u^3 + p^2\varphi(t)u(0,t)u\right). \tag{5.3}$$

Integrating Eq. (5.3) with respect to u and after the mathematical manipulations, we have

$$F(u) = \frac{u}{p^2}\sqrt{Au^2 + Bu + C}, \tag{5.4}$$

where $A = \frac{\lambda p^2}{6\gamma}$, $B = -\frac{2\beta p^2}{3\gamma}$, $C = \frac{k^2 - \alpha p^2 + p^2\varphi(t)u(0,t)}{\gamma}$.

From (2.5) and (5.4), we deduce that

$$\frac{du}{u\sqrt{Au^2 + Bu + C}} = \frac{d\xi}{p^2}. \tag{5.5}$$

After integrating Eq. (5.5), with zero constant of integration, we have following exact solution

$$u(x,t) = \frac{\frac{2\left(k^2 - \alpha p^2 + p^2\varphi(t)u(0,t)\right)}{\gamma} e^{-\frac{px-kt}{p^2}\sqrt{\frac{k^2-\alpha p^2+p^2\varphi(t)u(0,t)}{\gamma}}}}{\left(e^{-\frac{px-kt}{p^2}\sqrt{\frac{k^2-\alpha p^2+p^2\varphi(t)u(0,t)}{\gamma}}} + \frac{\beta p^2}{3\gamma}\right)^2 - \frac{\lambda p^2(k^2 - \alpha p^2 + p^2\varphi(t)u(0,t))}{6\gamma^2}}. \tag{5.6}$$

It is obvious that the function $u(0,t)$ can be easily found based on expression (5.6).

If $k = 1$, $\alpha = 1$, $\beta = 3$, $\gamma = 1$, $p = -1$, $\lambda = 24$, and $\varphi(t) = 1$, we obtain from (5.6) the following solitary wave solution

$$u(x,t) = 2u(0,t)\frac{e^{(x+t)\sqrt{u(0,t)}}}{\left(e^{(x+t)\sqrt{u(0,t)}} + 1\right)^2 - 4u(0,t)},$$

where $u(0,t) = \left(\frac{t \pm \sqrt{8-3t^2}}{2(2-t^2)}\right)^2$.

If $k = 1$, $\alpha = 1$, $\beta = -3$, $\gamma = 1$, $p = -1$, $\lambda = -6$, and $\varphi(t) = -1$, we obtain from (5.6) the following periodic wave solution

$$u(x,t) = \frac{-2u(0,t)e^{(x+t)i\sqrt{u(0,t)}}}{\left(e^{(x+t)i\sqrt{u(0,t)}} - 1\right)^2 + u(0,t)},$$

where $u(0,t) = \frac{2}{2t^2-1}$.

6. Physical interpretations of the loaded modified Boussinesq equation

We show how to find the solutions of the loaded modified BE in 3D plot formats to make it easier to imagine. Graphical representation is an effective tool for communication and it exemplifies evidently the solutions of the problems. The graphical illustrations of the solutions are depicted in the Figures 3 and 4. Solitary and periodic wave solutions represent an important type of solutions for nonlinear partial differential equations as many nonlinear partial differential equations have been found to have a variety of solitary wave solutions. The solitary wave solutions obtained in this article are encouraging, applicable, and could be helpful in analyzing long wave propagation on the surface of a fluid layer under the action of gravity, iron sound waves in plasma, and vibrations in a nonlinear string.

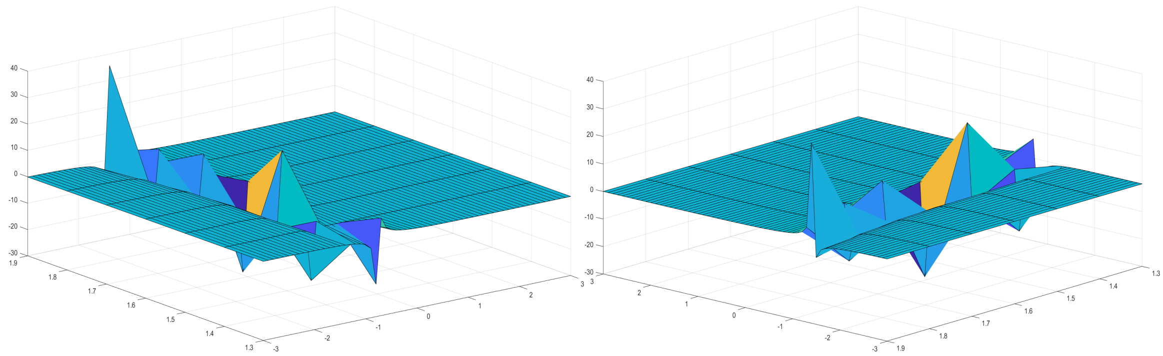

Figure 3: Solitary wave solution of the loaded modified BE for $k = 1$, $\alpha = 1$, $\beta = 3$, $\gamma = 1$, $p = -1$, $\lambda = 24$, and $\varphi(t) = 1$.

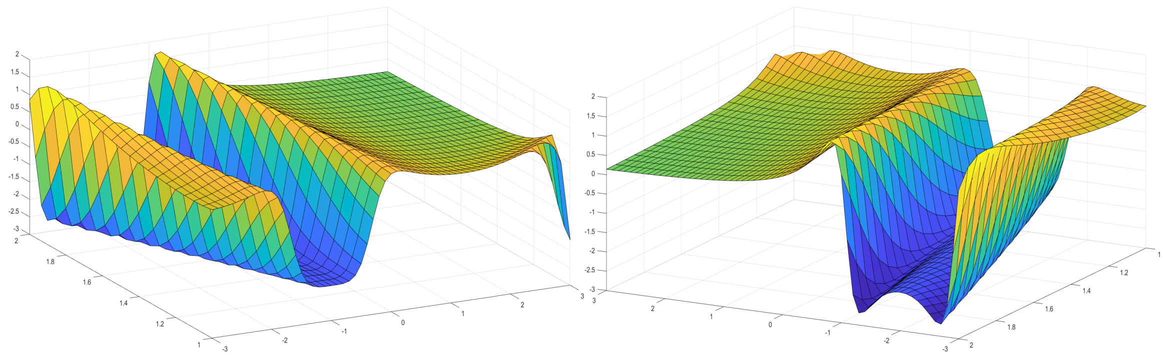

Figure 4: Real part of periodic wave solution of the loaded BE for $k = 1$, $\alpha = 1$, $\beta = -3$, $\gamma = 1$, $p = -1$, $\lambda = -6$, and $\varphi(t) = -1$.

7. Conclusions

The functional variable method has been successfully used to obtain several traveling wave solutions of the loaded BE and loaded modified BE. In this article, we have shown that, this method can provide a useful way to efficiently find the exact structures of solutions to a variety of nonlinear wave equations. The method yields a general solution with free parameters which can be identified by the above conditions. The solution procedure can be easily implemented in Matlab program. We conclude that the functional variable method is significant and important for finding the exact traveling wave solutions of nonlinear evolution equations which can be converted to a second-order ODE through the traveling wave transformation. The proposed method can be applied to many other nonlinear evolution equations in mathematical physics.

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