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On a minimal set of generators for the algebra $H^*(BE_d; F_2)$ and its applications

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Abstract

We investigate the Peterson hit problem for the polynomial algebra $\mathcal{P}_d$, viewed as a graded left module over the mod-2 Steenrod algebra, $\mathcal{A}$. For $d > 4$, this problem is still unsolved, even in the case of $d = 5$ with the help of computers. In this article, we study the hit problem for the case $d = 6$ in the generic degree $6(2^r - 1) + 6.2^r$, with r an arbitrary non-negative integer. Furthermore, the behavior of the sixth Singer algebraic transfer in degree $6(2^r - 1) + 6.2^r$ is also discussed at the end of this paper.

Keywords: Polynomial algebra, Steenrod algebra, graded rings.

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1. Introduction

Let X be a topological space. Cohomology operations are generated by the natural transformations of degree i which are so-called Steenrod squares

$$Sq^i : H^*(X, F_2) \longrightarrow H^{*+i}(X, F_2),$$

where $H^*(X, F_2)$ is the singular cohomology of X with coefficients in the two-element field F_2, and i is arbitrary non-negative integers. In 1952, Serre [13] proved that the Steenrod squares generate all stable cohomology operations with the usual addition and the composition of maps. The algebra of stable cohomology operations with coefficients in F_2 is known as the modulo 2 Steenrod algebra, $\mathcal{A}$. Then, for each topological space X, $H^*(X, F_2)$ is an $\mathcal{A}$-module.

Hence, the Steenrod algebra is able to be defined algebraically as a quotient algebra of F_2-free graded associative algebra generated by the symbols Sq^i of degree i where i is a non-negative integer, by the two-sided ideal generated by the relation $Sq^0 = 1$ and the Adem's relations

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j, \; 0 < a < 2b.$$

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Let E_d be an elementary abelian 2-group of rank d. Let us denote by BE_d the classifying space of E_d. It may be thought of as the product of d copies of real project space RP^∞. Then, using the Künneth formula for cohomology, one has an isomorphism of F_2-algebras

$$\mathcal{P}_d := H^*(BE_d; F_2) \cong F_2[x_1] \otimes_{F_2} \ldots \otimes_{F_2} F_2[x_d] \cong F_2[x_1, x_2, \ldots, x_d],$$

where $x_i \in H^1(BE_d; F_2)$ for every i.

As is well-known, $\mathcal{P}_d$ is a module over the mod-2 Steenrod algebra $\mathcal{A}$. The action of $\mathcal{A}$ on $\mathcal{P}_d$ is determined by the formula

$$Sq^k(x_j) = \begin{cases} x_j, & k = 0, \\ x_j^2, & k = 1, \\ 0, & k > 1, \end{cases}$$

and the Cartan formula $Sq^k(uv) = \sum_{i=0}^k Sq^i(u) Sq^{k-i}(v)$, where $u, v \in \mathcal{P}_d$ (see Steenrod and Epstein [16]).

The *Peterson hit problem* is to find a minimal generating set for $\mathcal{P}_d$ regarded as a module over the mod-2 Steenrod algebra. If we treat F_2 as a trivial $\mathcal{A}$-module, the hit problem is analogous to the problem of finding a basis for the F_2-graded vector space $F_2 \otimes_{\mathcal{A}} \mathcal{P}_d$.

This issue has first been studied by Peterson [7], Singer [14], Wood [29], Priddy [11], who show its relationship to several classical problems in cobordism theory, modular respresentation theory, Adams spectral sequence for the stable homotopy of spheres, stable homotopy type of the classifying space of finite groups.

Let $\alpha(n)$ be the number of digits 1 in the binary expansion of a natural n. The function $\mu : N \longrightarrow N$ is defined as follows:

$$\mu(0) = 0, \text{ and } \mu(n) = \min\{m \in N \ : \ n = \sum_{i=1}^{m} (2^{n_i} - 1), n_i > 0\} = \min\{m \in N \ : \ \alpha(n+m) \leqslant m\}.$$

Peterson [7] hypothesized that as a module over the Steenrod algebra $\mathcal{A}$, $\mathcal{P}_d$ is generated by monomials of degree m obeying the inequality $\alpha(m+d) \leqslant d$, and proved it for $d \leqslant 2$. After then, Wood [29] proved this in general. This is a fantastic tool for figuring out $\mathcal{A}$-generators for $\mathcal{P}_d$.

The squaring operation of Kameko is one of the most essential tools in the study of the hit problem

$$\widetilde{Sq}^0_* := \widetilde{S^d_{d+2m}} : (F_2 \otimes_{\mathcal{A}} \mathcal{P}_d)_{2m+d} \to (F_2 \otimes_{\mathcal{A}} \mathcal{P}_d)_m,$$

which is induced by an F_2-linear map $S_d : \mathcal{P}_d \to \mathcal{P}_d$, given by

$$S_d(x) = \begin{cases} y, & \text{if } x = x_1 x_2 \cdots x_k y^2, \\ 0, & \text{otherwise}, \end{cases}$$

for any monomial $x \in \mathcal{P}_d$. Clearly, $\widetilde{S^d_{d+2m}}$ is an F_2-epimorphism.

From the results of Wood [29], Kameko [4], and Sum [18], the hit problem is reduced to the case of degree n of the form $n = r(2^t - 1) + 2^t m$, where r, m, t are non-negative integers such that $0 \leqslant \mu(m) < r \leqslant d$.

Recently, the hit problem and its applications have been interested and studied by many authors (see Silverman [15], Repka-Selick [12], Janfada-Wood [2, 3], Nam [6], Sum [17, 18], Mothebe-Kaelo-Ramatebele [5], Phuc-Sum [8], Sum-Tin [20], Tin-Sum [22], Tin [23–26] and others).

The F_2-vector space $F_2 \otimes_{\mathcal{A}} (\mathcal{P}_d$ was entirely calculated for $d \leqslant 4$ (see Peterson [7] for $d = 1, 2$, Kameko [4] for $d = 3$, Sum [18] for $d = 4$), but it remains unresolved for $d \geqslant 5$, even with the aid of computers in the case of $d = 5$.

In this paper, we study the hit problem for the case $d = 6$ in the generic degree $6(2^r - 1) + 6.2^r$, with r an arbitrary non-negative integer. The main goal of the current paper is to explicitly determine an admissible monomial basis of the F_2-vector space $F_2 \otimes_{\mathcal{A}} \mathcal{P}_6$ in these degrees.

One of the primary applications of the hit problem is in surveying a homomorphism proposed by Singer [14], which is a homomorphism from the homology of the Steenrod algebra to the subspace of $F_2 \otimes_{\mathcal{A}} \mathcal{P}_d$ consisting of all the $GL(d; F_2)$-invariant classes.

Noting that the general linear group $GL(d; F_2)$ acts naturally on $\mathcal{P}_d$ by matrix substitution. Due to the fact that the two actions of $\mathcal{A}$ and $GL(d; F_2)$ upon $\mathcal{P}_d$ commute with each other, there is an inherited action of $GL(d; F_2)$ on $F_2 \otimes_{\mathcal{A}} \mathcal{P}_d$.

Recall that $\widetilde{\mathcal{P}_1}$ is the submodule of $F_2[x_1, x_1^{-1}]$ spanned by all powers x_1^i with $i \geqslant -1$. The usual $\mathcal{A}$-action on $\mathcal{P}_1 = F_2[x_1]$ is cannonically extended to an $\mathcal{A}$-action on $F_2[x_1, x_1^{-1}]$. Hence, $\widetilde{\mathcal{P}_1}$ is an $\mathcal{A}$-submodule of $F_2[x_1, x_1^{-1}]$. The inclusion $\mathcal{P}_1 \subset \widetilde{\mathcal{P}_1}$ gives rise to a short exact sequence of $\mathcal{A}$-modules:

$$0 \longrightarrow \mathcal{P}_1 \longrightarrow \widetilde{\mathcal{P}_1} \longrightarrow \sum{}^{-1} F_2 \longrightarrow 0.$$

Let e_1 be the corresponding element in $\mathrm{Ext}^1_{\mathcal{A}}(\sum^{-1} F_2, \mathcal{P}_1)$. Using the cross and Yoneda products, Singer set

$$e_d := (e_1 \times \mathcal{P}_{d-1}) \circ (e_1 \times \mathcal{P}_{d-2}) \circ \dots (e_1 \times \mathcal{P}_1) \circ e_1 \in \mathrm{Ext}^d_{\mathcal{A}}(\sum{}^{-d} F_2, \mathcal{P}_d).$$

Then, he defined

$$\begin{aligned} \widetilde{\varphi_d} : \mathrm{Tor}^{\mathcal{A}}_d(F_2, \sum{}^{-1} F_2) &\longrightarrow \mathrm{Tor}^{\mathcal{A}}_0(F_2, \mathcal{P}_d) = F_2 \otimes_{\mathcal{A}} \mathcal{P}_d, \\ z &\longmapsto e_d \cap z. \end{aligned}$$

Remarkably, $\mathrm{Im}\widetilde{\varphi_d}$ is a submodule of $(F_2 \otimes_{\mathcal{A}} \mathcal{P}_d)^{GL(d;F_2)}$. So, $\widetilde{\varphi_d}$ induces the homomorphism

$$\varphi_d : \mathrm{Tor}^{\mathcal{A}}_d(F_2, \sum{}^{-1} F_2) \longrightarrow (F_2 \otimes_{\mathcal{A}} \mathcal{P}_d)^{GL(d;F_2)}.$$

Let $F_2 \otimes_{GL(d;F_2)} PH_m((\mathbb{R}\mathcal{P}^\infty)^d)$ be dual to $(F_2 \otimes_{\mathcal{A}} \mathcal{P}_d)_m^{GL(d;F_2)}$. By passing to the dual, we have an algebraic homomorphism called Singer's algebraic transfer

$$\psi_d : F_2 \otimes_{GL(d;F_2)} PH_*((\mathbb{R}\mathcal{P}^\infty)^d) \longrightarrow \mathrm{Ext}^{d,d+*}_{\mathcal{A}}(F_2, F_2).$$

This is a useful tool in describing the cohomology groups of the Steenrod algebra, $\mathrm{Ext}^{d,d+*}_{\mathcal{A}}(F_2, F_2)$. At the conclusion of this article, the behavior of the sixth Singer algebraic transfer in degree $6(2^r - 1) + 6.2^r$ is also discussed.

Next, in Section 2, we recall some needed information on admissible monomials in $\mathcal{P}_d$. The main results are presented in Section 3.

2. Preliminaries

We will review some key facts from Sum [18], Kameko [4], and Singer [14] in this section, which will be used in the next section. Let us denote by $N_d = \{1, 2, \dots, d\}$ and

$$X_J = X_{\{j_1, j_2, \dots, j_s\}} = \prod_{j \in N_d \setminus J} x_j, \quad J = \{j_1, j_2, \dots, j_s\} \subset N_d.$$

In particular, $X_{N_d} = 1$, $X_\emptyset = x_1 x_2 \dots x_d$, $X_j = x_1 \dots \hat{x}_j \dots x_d$, $1 \leqslant j \leqslant d$, and $X := X_d \in \mathcal{P}_{d-1}$.

Let $\alpha_t(n)$ be the t-th coefficient in dyadic expansion of n. Then, $n = \sum_{t \geqslant 0} \alpha_t(n).2^t$ where $\alpha_t(n) \in \{0, 1\}$. Let $x = x_1^{a_1} x_2^{a_2} \dots x_d^{a_d} \in \mathcal{P}_d$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant d$. Set

$$J_t(x) = \{j \in N_d : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geqslant 0$. Then, we have $x = \prod_{t \geqslant 0} X_{J_t(x)}^{2^t}$.

Definition 2.1. For a monomial x belongs to $\mathcal{P}_d$, define two sequences associated with x by

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_d(x)),$$

where $\omega_i(x) = \sum_{1 \leqslant j \leqslant d} \alpha_{i-1}(\nu_j(x)) = \deg X_{J_{i-1}(x)}$, $i \geqslant 1$. The sequences $\omega(x)$ and $\sigma(x)$ are, respectively called the weight vector and the exponent vector of x.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order. Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ be a sequence of non-negative integers. The sequence ω is called the weight vector if $\omega_i = 0$ for $i \gg 0$. Then, we define $\deg \omega = \sum_{i>0} 2^{i-1}\omega_i$. Denote by $\mathcal{P}_d(\omega)$ the subspace of $\mathcal{P}_d$ spanned by all monomials y such that $\deg y = \deg \omega$, $\omega(y) \leqslant \omega$, and by $\mathcal{P}_d^-(\omega)$ the subspace of $\mathcal{P}_d$ spanned by all monomials $y \in \mathcal{P}_d(\omega)$ such that $\omega(y) < \omega$.

Definition 2.2. Let $\mathcal{A}^+$ be an ideal of $\mathcal{A}$ generated by all Steenrod squares of positive degrees, and u, v two polynomials of the same degree in $\mathcal{P}_d$. We define the equivalence relations " $\equiv$ " and " $\equiv_\omega$ " on $\mathcal{P}_d$ by stating that

(i) $u \equiv v$ if and only if $u - v \in \mathcal{A}^+\mathcal{P}_d$;
(ii) $u \equiv_\omega v$ if and only if $u, v \in \mathcal{P}_d(\omega)$ and $u - v \in \big(\mathcal{A}^+\mathcal{P}_d \cap \mathcal{P}_d(\omega) + \mathcal{P}_d^-(\omega)\big)$.

Then, we have an F_2-qoutient space of $\mathcal{P}_d$ by the equivalence relation " $\equiv_\omega$ " as follows:

$$\mathcal{A}\mathcal{P}_d(\omega) = \mathcal{P}_d(\omega)/((\mathcal{A}^+\mathcal{P}_d \cap \mathcal{P}_d(\omega)) + \mathcal{P}_d^-(\omega)).$$

If a polynomial u in $\mathcal{P}_d$ can be expressed as a finite sum $u = \sum_{i \geqslant 0} Sq^{2^i}(f_i)$ for suitable polynomials $f_i \in \mathcal{P}_d$, it is called a *hit*. That means u belongs to $\mathcal{A}^+\mathcal{P}_d$.

Definition 2.3. Let u, v be monomials of the same degree in $\mathcal{P}_d$. We say that $u < v$ if one of the following holds:

(i) $\omega(u) < \omega(v)$;
(ii) $\omega(u) = \omega(v)$, and and $\sigma(u) < \sigma(v)$.

Definition 2.4. Let u be a monomial in $\mathcal{P}_d$. The monomial u is said to be inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_i < u$ for $i = 1, 2, \ldots, m$ and $u - \sum_{i=1}^m v_i \in \mathcal{A}^+\mathcal{P}_d$. If u is not inadmissible, we say it is admissible.

It is crucial to note that the set of all admissible monomials of degree n in $\mathcal{P}_d$ is a minimal set of $\mathcal{A}$-generators for $\mathcal{P}_d$ in degree n. And therefore, $(F_2 \otimes_{\mathcal{A}} \mathcal{P}_d)_n$ is an F_2-vector space with a basis consisting of all the classes represent by the elements in $(\mathcal{P}_d)_n$.

Definition 2.5. Let u be a monomial in $\mathcal{P}_d$. We say u is strictly inadmissible if there exist monomials $v_1, v_2, \ldots, v_m$ such that $v_j < u$, for $j = 1, 2, \ldots, m$ and $u = \sum_{j=1}^m v_j + \sum_{i=1}^{2^s-1} Sq^i(f_i)$ with $s = \max\{k : \omega_k(u) > 0\}$, $f_i \in \mathcal{P}_d$.

Observe that if u is strictly inadmissible monomial, then it is inadmissible monomial, as defined by the Definitions 2.4 and 2.5. In general, the inverse is not true.

Theorem 2.6 (Kameko [4], Sum [18]). *Let u, v, w be monomials in $\mathcal{P}_d$ such that $\omega_t(u) = 0$ for $t > k > 0$, $\omega_r(w) \neq 0$ and $\omega_t(w) = 0$ for $t > r > 0$. Then,*

(i) uw^{2^k} *is inadmissible if w is inadmissible;*
(ii) wv^{2^r} *is strictly inadmissible if w is strictly inadmissible.*

Definition 2.7. Let $z = x_1^{a_1}x_2^{a_2}\ldots x_d^{a_d}$ in $\mathcal{P}_d$. The monomial z is called a spike if $a_j = 2^{t_j} - 1$ for t_j a non-negative integer and $j = 1, 2, \ldots, d$. Moreover, z is called the minimal spike, if it is a spike such that $t_1 > t_2 > \ldots > t_{r-1} \geqslant t_r > 0$ and $t_j = 0$ for $j > r$.

The following is a Singer's criterion on the hit monomials in $\mathcal{P}_d$.

Theorem 2.8 (Singer [14]). *Assume that z is the minimal spike of degree n in $\mathcal{P}_d$, and $u \in (\mathcal{P}_d)_n$ satisfying the condition $\mu(n) \leqslant d$. If $\omega(u) < \omega(z)$, then u is hit.*

The $\mathcal{A}$-submodules of $\mathcal{P}_d$ that spanned all the monomials $x_1^{s_1}x_2^{s_2}\ldots x_d^{s_d}$ such that $s_1\ldots s_d = 0$, and $s_1\ldots s_d > 0$, respectively, will be denoted by $\mathcal{P}_d^0$ and $\mathcal{P}_d^+$. It is easy to check that $\mathcal{P}_d^0$ and $\mathcal{P}_d^+$ are the $\mathcal{A}$-submodules of $\mathcal{P}_d$. Then, we have a direct summand decomposition of the F_2-vector spaces:

$$F_2 \otimes_{\mathcal{A}} \mathcal{P}_d = (F_2 \otimes_{\mathcal{A}} \mathcal{P}_d^0) \oplus (F_2 \otimes_{\mathcal{A}} \mathcal{P}_d^+).$$

From now on, we set $\mathcal{A}\mathcal{P}_d := F_2 \otimes_{\mathcal{A}} \mathcal{P}_d$, $\mathcal{A}\mathcal{P}_d^0 := F_2 \otimes_{\mathcal{A}} \mathcal{P}_d^0$, $\mathcal{A}\mathcal{P}_d^+ := F_2 \otimes_{\mathcal{A}} \mathcal{P}_d^+$. Let us denote by $\mathcal{D}_d^{\otimes}(m)$ the set of all admissible monomials of degree m in $\mathcal{P}_d$. For f an element of $\mathcal{P}_d$, we denote by $[f]$ the class in $\mathcal{A}\mathcal{P}_d$ represented by f. The cardinality of a set U is denoted by $|U|$.

3. The main results

First, we study the hit problem for the polynomial algebra of six variables in the generic degree $m_r := 6(2^r - 1) + 6.2^r$, with r an arbitrary non-negative integer. For $r = 0$, and $1 \leqslant i, j, k, \ell, t, m, s, r \leqslant 6$, we set

$$\mathcal{F} = \left\{ \prod_{i=1}^{6} x_i;\ x_i^3x_j^3;\ x_ix_jx_kx_\ell^3;\ x_jx_kx_\ell x_tx_m^2;\ x_tx_mx_s^2x_r^2 :\ t < m, s < r \right\}.$$

An easy computation shows that the following proposition, which is an immediate consequence of the result in [18].

Proposition 3.1. *The set $\mathcal{M} = \{[a_i] :\ a_i \in \mathcal{F}, 1 \leqslant i \leqslant 190\}$ is a basis of F_2-vector space $(\mathcal{A}\mathcal{P}_6)_{6(2^0-1)+6.2^0}$. Consequently, $|\mathcal{M}| = 190$.*

For $r = 1$, then $m_1 = 6(2^1 - 1) + 6.2^1$. Consider the homomorphism $\mathcal{T}_j : \mathcal{P}_5 \to \mathcal{P}_6$, for $1 \leqslant j \leqslant 6$ by substituting:

$$\mathcal{T}_j(x_i) = \begin{cases} x_i, & \text{if } 1 \leqslant i \leqslant j-1, \\ x_{i+1}, & \text{if } j \leqslant i < 6. \end{cases}$$

Then, the F_2-vector space $(\mathcal{A}\mathcal{P}_6)_{6(2^1-1)+6.2^1}$ is explicitly determined by the following theorem.

Theorem 3.2. *Let $\widetilde{\omega_1} := (2,2,1,1)$, $\widetilde{\omega_2} := (2,2,3)$, $\widetilde{\omega_3} := (2,4,2)$, $\widetilde{\omega_4} := (4,1,1,1)$, $\widetilde{\omega_5} := (4,1,3)$, $\widetilde{\omega_6} := (4,5,1)$, $\widetilde{\omega_7} := (4,3,2)$. Then :*

(i) $\mathrm{Im}\widetilde{S^6_{m_1}}$ *is isomorphic to a subspace of $(\mathcal{A}\mathcal{P}_6)_{m_1}$ generated by all the classes represented by the admissible monomials of the form $\prod_{i=1}^6 x_ia_j^2$, for all $1 \leqslant j \leqslant 190$. Consequently,* $\dim \mathrm{Im}\widetilde{S^6_{m_1}} = 190$.

(ii) *The set $\left\{[b_i] :\ b_i \in \bigcup_{j=1}^6 \mathcal{T}_j(\mathcal{C}_5^{\otimes}(18)), 1 \leqslant i \leqslant 2865\right\}$ is a basis of the F_2-vector space $(\mathcal{A}\mathcal{P}_6^0)_{m_1}$. This implies that $(\mathcal{A}\mathcal{P}_6^0)_{m_1}$ has dimension* 2865.

(iii) *We have $\left(\mathrm{Ker}\widetilde{S^6_{m_1}} \cap (\mathcal{A}\mathcal{P}_6^+)_{m_1}\right) \cong \bigoplus_{i=1}^7 \mathcal{A}\mathcal{P}_6^+(\widetilde{\omega_i})$. Moreover, the space $\left(\mathrm{Ker}\widetilde{S^6_{m_1}} \cap (\mathcal{A}\mathcal{P}_6^+)_{m_1}\right)$ is* 886-*dimensional.*

Proof. Since $\mathcal{P}_d = \oplus_{m\geqslant 0}(\mathcal{P}_d)_m$ is the graded polynomial algebra, and the homomorphism $\widetilde{S^6_{m_1}}$ is an F_2-epimorphism, it follows that

$$(\mathcal{A}\mathcal{P}_6)_{m_1} \cong (\mathcal{A}\mathcal{P}_6^0)_{m_1} \bigoplus \left(\mathrm{Ker}\widetilde{S^6_{m_1}} \cap (\mathcal{A}\mathcal{P}_6^+)_{m_1}\right) \bigoplus \mathrm{Im}\widetilde{S^6_{m_1}}.$$

The proof of Part (i) of the above theorem is straightforward. It is an immediate consequence of Proposition 3.1.

Recall that, Phuc [9] demonstrated that the space $(\mathcal{A}\mathcal{P}_5)_{6(2^1-1)+6.2^1}$ is an $\mathbb{F}_2$-vector space of dimension 730 with a basis consisting of all the classes represented by the monomials a_k, $1 \leqslant k \leqslant 730$. Consequently, $|\mathcal{D}_5^{\otimes}(6(2^1-1)+6.2^1)| = 730$. An easy computation shows that

$$\left|\bigcup_{j=1}^{6} \mathcal{T}_j(\mathcal{D}_5^{\otimes}(6(2^1-1)+6.2^1))\right| = 2865,$$

and the set

$$\{b_i : \ b_i \in \bigcup_{j=1}^{6} \mathcal{T}_j(a_t), 1 \leqslant t \leqslant 730, 1 \leqslant i \leqslant 2865\}$$

is a minimal set of generators for $\mathcal{A}$-modules $\mathcal{P}_6^0$ in degree $6(2^1-1)+6.2^1$. This implies $(\mathcal{A}\mathcal{P}_6^0)_{6(2^1-1)+6.2^1}$ has dimension 2865. Part (ii) is proved.

Remark 3.3. We set $\mathcal{H}_{(6,t)} = \{I = (i_1, i_2, \ldots, i_t) : 1 \leqslant i_1 < \ldots < i_t \leqslant 6\}$, with $1 \leqslant t < 6$. For $H \in \mathcal{H}_{(6,t)}$, consider the homomorphism $\varphi_H : \mathcal{P}_t \to \mathcal{P}_6$ of algebras by substituting $\varphi_H(x_\ell) = x_{i_\ell}$ with $1 \leqslant \ell \leqslant t$. Then, φ_H is an $\mathcal{A}$-modules monomorphism. From the result in [5], one has

$$\mathcal{A}\mathcal{P}_6^0 = \bigoplus_{1\leqslant t\leqslant 5} \bigoplus_{H\in\mathcal{H}_{(6,t)}} (Q\varphi_H(\mathcal{P}_t^+)),$$

where $Q\varphi_H(\mathcal{P}_t^+) = \mathbb{F}_2 \otimes_{\mathcal{A}} \varphi_H(\mathcal{P}_t^+)$. Then, $\dim(Q\varphi_H(\mathcal{P}_t^+))_n = \dim(\mathcal{A}\mathcal{P}_t^+)_n$, and $|\mathcal{H}_{(6,t)}| = \binom{6}{t}$. Combining with the results in Wood [29], one gets

$$\dim(\mathcal{A}\mathcal{P}_6^0)_n = \sum_{\mu(n)\leqslant t\leqslant 6} \binom{6}{t} \dim(\mathcal{A}\mathcal{P}_t^+)_n.$$

Since $\mu(6(2^1-1)+6.2^1) = 2$, the vector space $(\mathcal{A}\mathcal{P}_1)_{6(2^1-1)+6.2^1}$ is trivial. Using the results in Peterson [7], Kameko [4], Sum [18], and Phuc [9], we have

$$\dim(\mathcal{A}\mathcal{P}_t^+)_{6(2^1-1)+6.2^1} = \begin{cases} 3, & \text{if } \ t = 2,\\ 12, & \text{if } \ t = 3,\\ 60, & \text{if } \ t = 4,\\ 280, & \text{if } \ t = 5.\end{cases}$$

From the above results, we get

$$\dim(\mathcal{A}\mathcal{P}_6^0)_{6(2^1-1)+6.2^1} = \binom{6}{2}.3 + \binom{6}{3}.12 + \binom{6}{4}.60 + \binom{6}{5}.280 = 2865.$$

Next, we prove Part (iii) of the theorem by explicitly determining the admissible monomial basis of the $\mathbb{F}_2$-vector space $\big(\mathrm{Ker}\widetilde{S^6_{m_1}} \cap (\mathcal{A}\mathcal{P}_6^+)_{m_1}\big)$.

Denote $\mathcal{D}_6^{\otimes}(\omega) := \mathcal{D}_6^{\otimes}(m) \cap \mathcal{P}_6(\omega)$. It is easy to see that $\mathcal{D}_6^{\otimes}(m) = \bigcup\limits_{\deg \omega = m} \mathcal{D}_6^{\otimes}(\omega)$. Put

$$Q\mathcal{P}_6^{\omega} := \langle\{[x] \in \mathcal{A}\mathcal{P}_6 : \omega(x) = \omega, \text{ and } x \in \mathcal{D}_6^{\otimes}(\omega)\}\rangle.$$

It is simple to verify that the map $\mathcal{A}\mathcal{P}_6(\omega) \longrightarrow Q\mathcal{P}_6^{\omega}$, $[x]_\omega \longrightarrow [x]$ is an isomorphism of $\mathbb{F}_2$-vector spaces. Hence, $Q\mathcal{P}_6^{\omega} \subset \mathcal{A}\mathcal{P}_6$ can be used to identify the vector space $\mathcal{A}\mathcal{P}_6(\omega)$. As a result of this, one obtains

$$(\mathcal{AP}_6)_m = \bigoplus_{\deg\omega=m} Q\mathcal{P}_6^{\omega} \cong \bigoplus_{\deg\omega=m} \mathcal{AP}_6(\omega).$$

From this, it follows that $(\mathcal{AP}_6^+)_{m_1} = \bigoplus_{\deg\omega=m_1} \mathcal{AP}_6^+(\omega)$.

Assume that x belongs to $(\mathcal{D}_6^{\otimes}(6(2^1-1)+6.2^1)\cap\mathcal{P}_6^+)$ such that $[x]$ does not an element of $\mathrm{Im}\widetilde{S^6_{18}}$. It is easy to check that $y = x_1^{15}x_2^3$ is the minimal spike of degree eighteen in $\mathcal{P}_6$ and $\omega(y) = (2,2,1,1)$. Since x is an admissible monomial, by Theorem 2.8 it shows that $\omega_1(x) \geqslant \omega_1(y)$. Moreover, $\deg(x)$ is an even number, it implies $\omega_1(x) = 2$, or $\omega_1(x) = 4$, or $\omega_1(x) = 6$.

If $\omega_1(x) = 2$, then $x = x_ix_ju^2$ with u a monomial of degree eight in $\mathcal{P}_6$ and $1 \leqslant i < j \leqslant 6$. By Theorem 2.6, u is an admissible monomial. Moreover, using Theorem 2.8, we also have $\omega_1(u) \geqslant \omega_2(y) = 2$. Hence, $\omega_1(u) = 6$, or $\omega_1(u) = 4$, or $\omega_1(u) = 2$.

If $\omega_1(u) = 6$, then $\omega_2(x) = 6$. Using the results in Sum [17], we see that x is strictly inadmissible. And therefore, x is inadmissible. This contradicts the fact that x belongs to $\mathcal{D}_6^{\otimes}(18)$. In case of $\omega_1(u) = 4$, then $u = x_mx_rx_sx_tv^2$ with $1 \leqslant m < r < s < t \leqslant 6$, where $v \in \mathcal{D}_6^{\otimes}(2)$, and $\omega(v) = (2,0)$. From this, we obtain $\omega(x) = \widetilde{\omega_3}$.

If $\omega_1(u) = 2$, then $u = x_nx_mw^2$ with $1 \leqslant n < m \leqslant 6$, where $w \in \mathcal{D}_6^{\otimes}(3)$. Since $w \in \mathcal{D}_6^{\otimes}(3)$, yields $\omega(w) = (3,0)$ or $\omega(w) = (1,1)$. So, either $\omega(x) = \widetilde{\omega_1}$, or $\omega(x) = \widetilde{\omega_2}$.

If $\omega_1(x) = 4$, then $u = x_ix_jx_kx_\ell f^2$, where f is an admissible monomial of degree seven in $\mathcal{P}_6$ and $1 \leqslant i < j < k < \ell \leqslant 6$. An easy computation, using the result in [21], we obtain the set

$$\left\{[x_ix_jx_kx_\ell x_tx_m^2] : 1 \leqslant i,j,k,\ell,t,m \leqslant 6,\ t<m\right\} \cup \left\{[q] : q \in \bigcup_{m=1}^{6} \mathcal{T}_m(\mathcal{D}_5^{\otimes}(7))\right\}$$

is a basis of F_2-vector space $(\mathcal{AP}_6)_7$. Since $f \in \mathcal{D}_6^{\otimes}(7)$, it yields that $\omega(f) = (5,1)$ or $\omega(f) = (3,2)$, or $\omega(f) = (1,3)$, or $\omega(f) = (1,1,1)$. So, $\omega(x) = \widetilde{\omega_i}$ for $4 \leqslant i \leqslant 7$.

If $\omega_1(x) = 6$, then $x = \prod_{i=1}^{6} x_i g^2$ with g an admissible monomial of degree six in $\mathcal{P}_6$. By Theorem 2.6, g is an admissible monomial, and therefore $[g] \neq 0$. Thus, we have $[g] = \widetilde{S^6_{18}}\big([x]\big) \neq 0$. This contradicts the fact that $[x]$ belongs to $\mathrm{Ker}\widetilde{S^6_{18}}$.

From the above results, we get $\omega(x) = \widetilde{\omega_i}$, for all $1 \leqslant i \leqslant 7$. Furthermore, one gets

$$\mathrm{Ker}\widetilde{S^6_{18}} \cap (\mathcal{AP}_6^+)_{18} \cong \bigoplus_{i=1}^{7} \mathcal{AP}_6^+(\widetilde{\omega_i}).$$

We will denote by $\mathcal{D}_d^+(\omega)$ the set of all admissible monomials in $\mathcal{P}_d^+(\omega)$. In order to explicitly determine the space $\mathrm{Ker}\widetilde{S^6_{18}} \cap (\mathcal{AP}_6^+)_{18}$, we show all admissible monomials in $\mathcal{P}_6^+(\widetilde{\omega_{(i)}})$, for all $1 \leqslant i \leqslant 7$. The proof is divided into the following steps.

Step 1. Consider the weight vector $\omega = \widetilde{\omega_1}$. Assume that x is an admissible monomial in $\mathcal{P}_6$ such that $\omega(x) = \widetilde{\omega_1}$, then $x = x_ix_jy^2$, where $y \in \mathcal{D}_6^{\otimes}(1,1,1)$, and $1 \leqslant i < j \leqslant 6$. We set

$$\mathcal{M}_6^1 := \{x_ix_jy^2 : \omega(y) = (1,1,1),\ 1 \leqslant i < j \leqslant 6\}\ \cap\ \mathcal{P}_6^+.$$

It is easy to see that $\mathrm{Span}\{\mathcal{M}_6^1\} = \mathcal{P}_6^+(\widetilde{\omega_1})$, and if u is an element in $\mathcal{M}_6^1$, then u has the form $x_ix_jx_k^2x_\ell^4x_m^8x_n^2$, with $k < \ell < m$, where (i,j,k,ℓ,m,n) is an permutation of $(1,2,3,4,5,6)$.

Clearly, the monomials $x_1^2x_ix_jx_\ell^2x_k^4x_m^8$ are inadmissible (more precisely by Sq^1), where (i,j,k,ℓ,m) is an arbitrary permutation of $(2,3,4,5,6)$. Furthermore, for $1 < i;j < \ell$, one has

$$x_1x_i^2x_j^2x_\ell x_k^4x_m^8 = Sq^8(x_1x_ix_j^2x_\ell x_k^2x_m^4) + Sq^1(x_1^2x_ix_jx_\ell x_k^4x_m^8) + \text{ smaller than.}$$

From this, the monomials $x_1x_i^2x_j^2x_\ell x_k^4x_m^8$ are inadmissible.

As may be seen from the preceding findings, $\mathcal{P}_6^+(\widetilde{\omega_1})$ is generated by 9 elements $c_{i;1}$, with $1 \leqslant i \leqslant 9$, as follows:

1. $x_1^1x_2^1x_3^2x_4^2x_5^4x_6^8$, 2. $x_1^1x_2^1x_3^2x_4^4x_5^2x_6^8$, 3. $x_1^1x_2^1x_3^2x_4^4x_5^8x_6^2$, 4. $x_1^1x_2^2x_3^1x_4^2x_5^4x_6^8$
5. $x_1^1x_2^2x_3^1x_4^4x_5^2x_6^8$, 6. $x_1^1x_2^2x_3^1x_4^4x_5^8x_6^2$, 7. $x_1^1x_2^2x_3^4x_4^1x_5^2x_6^8$, 8. $x_1^1x_2^2x_3^4x_4^1x_5^8x_6^2$, r9. $x_1^1x_2^2x_3^4x_4^8x_5^1x_6^2$.

We then prove the set $\{[c_{i;1}] : 1 \leqslant i \leqslant 9\}$ is linearly independent in $\mathcal{AP}_6(\widetilde{\omega_1})$. Denote $\mathcal{N}_d = \{(i;I) : I = (i_1, i_2, \ldots, i_t), 1 \leqslant i < i_1 < \ldots < i_t \leqslant d, 0 \leqslant t < d\}$, where by convention $I = \emptyset$ if $t = 0$. Write $t = \ell(I)$ for the length of I.

For each $(i;I) \in \mathcal{N}_6$, consider the homomorphism $\Omega_{(i;I)} : \mathcal{P}_6 \to \mathcal{P}_5$ which is defined as:

$$\Omega_{(i;I)}(x_k) = \begin{cases} x_i, & \text{if } 1 \leqslant k \leqslant i-1, \\ \sum_{s\in I} x_{s-1}, & \text{if } k = i, \\ x_{k-1}, & \text{if } i < k \leqslant 6. \end{cases}$$

We use them to prove that a given set of monomials is the set of admissible monomials in $\mathcal{P}_6$ by showing that they are linearly independent in $\mathcal{AP}_6$.

Assume that there is a linear relation

$$\mathcal{S}_1 = \sum_{1\leqslant i\leqslant 9} \gamma_i c_{i;1} \equiv 0, \quad \text{where } \gamma_i \in F_2. \tag{3.1}$$

From a result in [9], one has $\dim \mathcal{AP}_5^+(\widetilde{\omega_1}) = 25$, with a basis consisting of all the classes represented by the monomials $a_k, 1 \leqslant k \leqslant 25$, which are determined as follows:

1. $x_1^1x_2^1x_3^2x_4^2x_5^{12}$, 2. $x_1^1x_2^1x_3^2x_4^{12}x_5^2$, 3. $x_1^1x_2^2x_3^1x_4^2x_5^{12}$, 4. $x_1^1x_2^2x_3^1x_4^{12}x_5^2$,
5. $x_1^1x_2^2x_3^{12}x_4^1x_5^2$, 6. $x_1^1x_2^1x_3^2x_4^4x_5^{10}$, 7. $x_1^1x_2^2x_3^1x_4^4x_5^{10}$, 8. $x_1^1x_2^2x_3^4x_4^1x_5^{10}$,
9. $x_1^1x_2^1x_3^2x_4^6x_5^8$, 10. $x_1^1x_2^1x_3^6x_4^2x_5^8$, 11. $x_1^1x_2^2x_3^1x_4^6x_5^8$, 12. $x_1^1x_2^6x_3^1x_4^2x_5^8$,
13. $x_1^1x_2^2x_3^5x_4^2x_5^8$, 14. $x_1^1x_2^2x_3^5x_4^8x_5^2$, 15. $x_1^1x_2^2x_3^3x_4^4x_5^8$, 16. $x_1^1x_2^3x_3^2x_4^4x_5^8$,
17. $x_1^1x_2^3x_3^4x_4^2x_5^8$, 18. $x_1^1x_2^3x_3^4x_4^8x_5^2$, 19. $x_1^3x_2^1x_3^2x_4^4x_5^8$, 20. $x_1^3x_2^1x_3^4x_4^2x_5^8$,
21. $x_1^3x_2^1x_3^4x_4^8x_5^2$, 22. $x_1^1x_2^2x_3^4x_4^3x_5^8$, 23. $x_1^1x_2^2x_3^4x_4^8x_5^3$, 24. $x_1^3x_2^4x_3^1x_4^2x_5^8$, 25. $x_1^1x_2^2x_3^4x_4^9x_5^2$.

Acting the homomorphism $\Omega_{(5;6)}$ on both sides of (3.1), and explicitly computing $\Omega_{(5;6)}(\mathcal{S}_1)$ in terms of $a_k, 1 \leqslant k \leqslant 25$ in $\mathcal{P}_5(\text{mod}(\mathcal{A}^+\mathcal{P}_5))$, we obtain

$$\Omega_{(5;6)}(\mathcal{S}_1) \equiv_{\widetilde{\omega_1}} \gamma_1 a_1 + (\gamma_2+\gamma_3)a_6 + \gamma_4 a_3 + (\gamma_5+\gamma_6)a_7 + (\gamma_7+\gamma_8)a_8 + \gamma_9 a_{23} \equiv_{\widetilde{\omega_1}} 0.$$

From the above equation, we can derive that $\gamma_1 = \gamma_4 = \gamma_9 = 0$.

Similarly, the homomorphism $\Omega_{(4;5)}$ sends the relation (3.1) to the following relation in $\mathcal{P}_5(\text{mod } (\mathcal{A}^+\mathcal{P}_5))$

$$\Omega_{(4;5)}(\mathcal{S}_1) \equiv_{\widetilde{\omega_1}} \gamma_2 a_9 + \gamma_3 a_2 + \gamma_5 a_{11} + \gamma_6 a_4 + \gamma_7 a_{22} + \gamma_8 a_{25} \equiv_{\widetilde{\omega_1}} 0.$$

From the above results, one gets $\gamma_i = 0$, for all $1 \leqslant i \leqslant 9$.

In summary, the set $\{[c_{i;1}] : 1 \leqslant i \leqslant 9\}$ is a basis of the F_2-vector space $\mathcal{AP}_6^+(\widetilde{\omega_1})$. Consequently, $|\mathcal{D}_6^+(\widetilde{\omega_1})| = 9$.

Step 2. Consider the weight vector $\omega = (4,1,1,1)$. Let us denote by

$$\mathcal{M}_6^2 := \{x_ix_jx_kx_\ell z^2 : \omega(z) = (1,1,1),\ 1 \leqslant i < j < k < \ell \leqslant 6\} \cap \mathcal{P}_6^+.$$

It is easy to see that $\mathcal{P}_6^+(\widetilde{\omega_4}) = \text{Span}\{\mathcal{M}_6^2\}$, and if ν is an element in $\mathcal{M}_6^2$, then ν has the form: $x_ix_jx_k^2x_\ell^4x_m^4x_n^6$, $x_ix_j^3x_k^2x_\ell^4x_m^4x_n^4$, $x_ix_j^2x_k^5x_\ell^2x_m^4x_n^4$, where (i,j,k,ℓ,m,n) is an permutation of $(1,2,3,4,5,6)$.

By direct calculations, using Theorem 2.6, we remove the inadmissible monomials in $\mathcal{M}_6^2$, and we see that $\mathcal{P}_6^+(\widetilde{\omega_4})$ is generated by 50 elements $c_{i;4}$, $1 \leqslant i \leqslant 50$, as follows:

1. $x_1^1x_2^1x_3^1x_4^1x_5^2x_6^{12}$, 2. $x_1^3x_2^1x_3^1x_4^1x_5^4x_6^8$, 3. $x_1^1x_2^3x_3^1x_4^1x_5^4x_6^8$, 4. $x_1^1x_2^1x_3^3x_4^1x_5^4x_6^8$, 5. $x_1^1x_2^1x_3^1x_4^3x_5^4x_6^8$,
6. $x_1^1x_2^1x_3^1x_4^2x_5^1x_6^{12}$, 7. $x_1^3x_2^1x_3^1x_4^4x_5^1x_6^8$, 8. $x_1^1x_2^3x_3^1x_4^4x_5^1x_6^8$, 9. $x_1^1x_2^1x_3^3x_4^4x_5^1x_6^8$, 10. $x_1^1x_2^1x_3^1x_4^2x_5^5x_6^8$,
11. $x_1^1x_2^1x_3^1x_4^2x_5^{12}x_6^1$, 12. $x_1^3x_2^1x_3^1x_4^4x_5^8x_6^1$, 13. $x_1^1x_2^3x_3^1x_4^4x_5^8x_6^1$, 14. $x_1^1x_2^1x_3^3x_4^4x_5^8x_6^1$, 15. $x_1^1x_2^1x_3^1x_4^2x_5^4x_6^9$,
16. $x_1^1x_2^1x_3^2x_4^1x_5^1x_6^{12}$, 17. $x_1^3x_2^1x_3^4x_4^1x_5^1x_6^8$, 18. $x_1^1x_2^3x_3^4x_4^1x_5^1x_6^8$, 19. $x_1^1x_2^1x_3^2x_4^5x_5^1x_6^8$, 20. $x_1^1x_2^1x_3^2x_4^1x_5^5x_6^8$,
21. $x_1^1x_2^1x_3^2x_4^1x_5^{12}x_6^1$, 22. $x_1^3x_2^1x_3^4x_4^1x_5^8x_6^1$, 23. $x_1^1x_2^3x_3^4x_4^1x_5^8x_6^1$, 24. $x_1^1x_2^1x_3^2x_4^5x_5^8x_6^1$, 25. $x_1^1x_2^1x_3^2x_4^1x_5^4x_6^9$,
26. $x_1^1x_2^1x_3^2x_4^{12}x_5^1x_6^1$, 27. $x_1^3x_2^1x_3^4x_4^8x_5^1x_6^1$, 28. $x_1^1x_2^3x_3^4x_4^8x_5^1x_6^1$, 29. $x_1^1x_2^1x_3^2x_4^4x_5^9x_6^1$, 30. $x_1^1x_2^1x_3^2x_4^4x_5^1x_6^9$,
31. $x_1^1x_2^2x_3^1x_4^1x_5^1x_6^{12}$, 32. $x_1^3x_2^4x_3^1x_4^1x_5^1x_6^8$, 33. $x_1^1x_2^2x_3^5x_4^1x_5^1x_6^8$, 34. $x_1^1x_2^2x_3^1x_4^5x_5^1x_6^8$, 35. $x_1^1x_2^2x_3^1x_4^1x_5^5x_6^8$,
36. $x_1^1x_2^2x_3^1x_4^1x_5^{12}x_6^1$, 37. $x_1^3x_2^4x_3^1x_4^1x_5^8x_6^1$, 38. $x_1^1x_2^2x_3^5x_4^1x_5^8x_6^1$, 39. $x_1^1x_2^2x_3^1x_4^5x_5^8x_6^1$, 40. $x_1^1x_2^2x_3^1x_4^1x_5^4x_6^9$,
41. $x_1^1x_2^2x_3^1x_4^{12}x_5^1x_6^1$, 42. $x_1^3x_2^4x_3^1x_4^8x_5^1x_6^1$, 43. $x_1^1x_2^2x_3^5x_4^8x_5^1x_6^1$, 44. $x_1^1x_2^2x_3^1x_4^4x_5^9x_6^1$, 45. $x_1^1x_2^2x_3^1x_4^4x_5^1x_6^9$,
46. $x_1^1x_2^2x_3^{12}x_4^1x_5^1x_6^1$, 47. $x_1^3x_2^4x_3^8x_4^1x_5^1x_6^1$, 48. $x_1^1x_2^2x_3^4x_4^9x_5^1x_6^1$, 49. $x_1^1x_2^2x_3^4x_4^1x_5^9x_6^1$, 50. $x_1^1x_2^2x_3^4x_4^1x_5^1x_6^9$.

We now show that the set $\{[c_{i;4}] : 1 \leqslant i \leqslant 50\}$ is linearly independent in $\mathcal{AP}_6(\widetilde{\omega_4})$. Assume that there is a linear relation

$$\mathcal{S}_2 = \sum_{1 \leqslant i \leqslant 50} \gamma_i c_{i;4} \equiv 0, \quad \text{where } \gamma_i \in \mathbb{F}_2, i \in \mathbb{N}_{50}. \tag{3.2}$$

Recall that $\dim \mathcal{AP}_5^+(\widetilde{\omega_4}) = 40$, with a basis consisting of all the classes represented by the monomials $a_k, 26 \leqslant k \leqslant 65$, which are determined as follows:

26. $x_1^1x_2^1x_3^1x_4^1x_5^{14}$, 27. $x_1^1x_2^1x_3^1x_4^{14}x_5^1$, 28. $x_1^1x_2^1x_3^{14}x_4^1x_5^1$, 29. $x_1^1x_2^{14}x_3^1x_4^1x_5^1$, 30. $x_1^1x_2^1x_3^1x_4^2x_5^{13}$,
31. $x_1^1x_2^1x_3^2x_4^1x_5^{13}$, 32. $x_1^1x_2^1x_3^2x_4^{13}x_5^1$, 33. $x_1^1x_2^2x_3^1x_4^1x_5^{13}$, 34. $x_1^1x_2^2x_3^1x_4^{13}x_5^1$, 35. $x_1^1x_2^2x_3^{13}x_4^1x_5^1$,
36. $x_1^1x_2^1x_3^1x_4^3x_5^{12}$, 37. $x_1^1x_2^1x_3^3x_4^1x_5^{12}$, 38. $x_1^1x_2^1x_3^3x_4^{12}x_5^1$, 39. $x_1^1x_2^3x_3^1x_4^1x_5^{12}$, 40. $x_1^1x_2^3x_3^1x_4^{12}x_5^1$,
41. $x_1^1x_2^3x_3^{12}x_4^1x_5^1$, 42. $x_1^3x_2^1x_3^1x_4^1x_5^{12}$, 43. $x_1^3x_2^1x_3^1x_4^{12}x_5^1$, 44. $x_1^3x_2^1x_3^{12}x_4^1x_5^1$, 45. $x_1^1x_2^1x_3^2x_4^5x_5^9$,
46. $x_1^1x_2^2x_3^1x_4^5x_5^9$, 47. $x_1^1x_2^2x_3^5x_4^1x_5^9$, 48. $x_1^1x_2^2x_3^5x_4^9x_5^1$, 49. $x_1^1x_2^1x_3^3x_4^4x_5^9$, 50. $x_1^1x_2^3x_3^1x_4^4x_5^9$,
51. $x_1^1x_2^3x_3^4x_4^1x_5^9$, 52. $x_1^1x_2^3x_3^4x_4^9x_5^1$, 53. $x_1^3x_2^1x_3^1x_4^4x_5^9$, 54. $x_1^2x_2^1x_3^4x_4^1x_5^9$, 55. $x_1^3x_2^1x_3^4x_4^9x_5^1$,
56. $x_1^1x_2^1x_3^3x_4^5x_5^8$, 57. $x_1^1x_2^3x_3^1x_4^5x_5^8$, 58. $x_1^1x_2^3x_3^5x_4^1x_5^8$, 59. $x_1^1x_2^3x_3^5x_4^8x_5^1$, 60. $x_1^3x_2^1x_3^1x_4^5x_5^8$,
61. $x_1^3x_2^1x_3^5x_4^1x_5^8$, 62. $x_1^3x_2^1x_3^5x_4^8x_5^1$, 63. $x_1^3x_2^5x_3^1x_4^1x_5^8$, 64. $x_1^3x_2^5x_3^1x_4^8x_5^1$, 65. $x_1^3x_2^5x_3^8x_4^1x_5^1$.

Acting the homomorphism $\Omega_{(4;5)}$ on both sides of (3.2), and explicitly computing $\Omega_{(4;5)}(\mathcal{S}_2)$ in terms of $a_k, 26 \leqslant k \leqslant 65$ in $\mathcal{P}_5(\mathrm{mod}(\mathcal{A}^+\mathcal{P}_5))$, we obtain $\gamma_i = 0$, for all $i \in L = \{11, 12, 13, 14\}$. Therefore, the relation (3.2) becomes

$$\mathcal{U} = \sum_{i \in \mathbb{N}_{50} \setminus L} \gamma_i c_{i;4} \equiv 0, \tag{3.3}$$

By the same calculation as above, we explicitly compute $\Omega_{(j;I)}(\mathcal{S})$, $(j; I) \in \mathcal{N}_6$, in terms of $a_k, 26 \leqslant k \leqslant 65$ in $\mathcal{P}_5(\mathrm{mod}(\mathcal{A}^+\mathcal{P}_5))$, and from the relation $\varphi_{(j;I)}(\mathcal{U}) \equiv 0$, with $\ell(I) = 1$, we get $\gamma_i = 0$ for all $i \in \mathbb{N}_{50} \setminus L$. That means, the set $\{[c_{i;4}] : 1 \leqslant i \leqslant 50\}$ is a basis of the $\mathbb{F}_2$-vector space $\mathcal{AP}_6^+(\widetilde{\omega_4})$. Consequently, $|\mathcal{D}_6^+(\widetilde{\omega_4})| = 50$.

Step 3. Consider the weight vector $\omega = \widetilde{\omega_i}$, with $i \in J = \{2, 3, 5, 6, 7\}$. Let us denote by $\mathcal{D}_6^+(\omega_J) := \cup_{i \in J} \mathcal{D}_6^+(\widetilde{\omega_i})$. For each $i \in J$, by the same method as in the previous section, we explicitly determine the $\mathbb{F}_2$-vector spaces $\mathcal{AP}_6^+(\widetilde{\omega_i})$. By direct calculations, using Theorem 2.6, one gets $|\mathcal{D}_6^+(\omega_J)| = \sum_{i \in J} \dim \mathcal{AP}_6^+(\widetilde{\omega_i}) = 827$.

Hence, one gets $\dim \left(\mathrm{Ker}\widetilde{S^6_{m_1}} \cap (\mathcal{AP}_6^+)_{m_1}\right) = 886$. Part (iii) has been established. So, the theorem is proved. $\square$

From the above results, we obtain the following corollary.

Corollary 3.4. *There exist exactly* 3941 *admissible monomials in* $\mathcal{P}_6$ *of degree* $6(2^1-1)+6.2^1$. *Consequently,* $|\mathcal{D}_6^{\otimes}(6(2^1-1)+6.2^1)| = 3941$.

Next, we consider the degree $m_r := 6(2^r-1)+6.2^r$, for any $r \geqslant 2$. Since the homomorphism $\widetilde{S^6_{m_2}} : (\mathcal{AP}_6)_{m_2} \longrightarrow (\mathcal{AP}_6)_{m_1}$ is an F_2-epimorphism, it shows that

$$(\mathcal{AP}_6)_{42} \cong (\mathcal{AP}_6^0)_{42} \bigoplus \left(\mathrm{Ker}\widetilde{S^6_{42}} \cap (\mathcal{AP}_6^+)_{42}\right) \bigoplus \mathrm{Im}\widetilde{S^6_{42}}.$$

Consider the homomorphism $\Gamma : \mathcal{P}_6 \to \mathcal{P}_6$ is an F_2-homomorphism determined by $\Gamma(x) = \prod_{i=1}^6 x_i x^2$, for $x \in \mathcal{P}_6$. Thus, we have the following theorem.

Theorem 3.5. *The following statements are true.*

(i) $\mathrm{Im}\widetilde{S^6_{m_2}}$ *is isomorphic to a subspace of* $(\mathcal{AP}_6)_{m_2}$ *generated by all the classes represented by the admissible monomials of the form* $\Gamma(u)$ *for every* u *belongs to* $\mathcal{D}_6^{\otimes}(18)$. *Consequently,* $\dim \mathrm{Im}\widetilde{S^6_{42}} = 3941$.

(ii) *The set* $\left\{[d_i] : d_i \in \bigcup_{j=1}^6 \mathcal{T}_j(\mathcal{D}_5^{\otimes}(42)), 1 \leqslant i \leqslant 13020\right\}$ *is a basis of the* F_2*-vector space* $(\mathcal{AP}_6^0)_{m_2}$. *This implies that* $(\mathcal{AP}_6^0)_{42}$ *has dimension* 13020.

Proof. The proof of Part (i) of the above theorem is straightforward. It occurs as a direct result of Corollary 3.4. Observe, from the result in Corollary 3.4, it shows that

$$\dim \mathrm{Im}\widetilde{S^6_{42}} = \left|\left\{\prod_{i=1}^{6} x_i x^2 : x \in \mathcal{D}_6^{\otimes}(18)\right\}\right| = 3941.$$

Consider the degree $m_r := 6(2^r-1)+6.2^r$, for $r = 2$. By using the MAGMA computer algebra system, Phuc showed in [10] that the F_2-vector space $(\mathcal{AP}_5)_{42}$ has 2520-dimensional (see [10], pp.4), where $\dim(\mathcal{AP}_5^0)_{42} = 700$, and $\dim(\mathcal{AP}_5^+)_{42} = 1820$. Assume that the set $\{e_i \in (\mathcal{P}_5)_{42} : 1 \leqslant i \leqslant 2520\}$ is a minimal set of generators for $\mathcal{A}$-modules $\mathcal{P}_5$ in degree forty-two.

That means, $\mathcal{D}_5^{\otimes}(42) = \{e_i \in (\mathcal{P}_5)_{42} : 1 \leqslant i \leqslant 2520\}$. It is easy to check that $\left|\bigcup_{j=1}^6 \mathcal{T}_j(\mathcal{C}_5^{\otimes}(42))\right| = 13020$, and the set

$$\left\{u_i : u_i \in \bigcup_{j=1}^{6} \mathcal{T}_j(e_k), 1 \leqslant k \leqslant 2520, 1 \leqslant i \leqslant 13020\right\}$$

is a minimal set of generators for $\mathcal{A}$-module $\mathcal{P}_6^0$ in degree forty-two. This implies that $(\mathcal{AP}_6^0)_{42}$ has dimension 13020. The second part has been established.

Remark 3.6. By the same argument as the previous part, we set

$$\mathcal{H}_{(d,t)} = \left\{I = (i_1, i_2, \ldots, i_t) : 1 \leqslant i_1 < \ldots < i_t \leqslant d\right\}, \ \text{ with } 1 \leqslant t < d.$$

For each $H \in \mathcal{H}_{(d,t)}$, consider the homomorphism $F_H : \mathcal{P}_t \to \mathcal{P}_d$ of algebras by substituting $F_H(x_\ell) = x_{i_\ell}$ with $1 \leqslant \ell \leqslant t$. Then, F_H is an $\mathcal{A}$-modules monomorphism. From the result in [5], we have a direct summand decomposition of the F_2-vector subspaces:

$$\mathcal{AP}_d^0 = \bigoplus_{1\leqslant t\leqslant d-1} \bigoplus_{H\in\mathcal{H}_{(d,t)}} (QF_H(\mathcal{P}_t^+)),$$

where $QF_H(\mathcal{P}_t^+) = F_2 \otimes_{\mathcal{A}} F_H(\mathcal{P}_t^+)$. Hence, $\dim(QF_H(\mathcal{P}_t^+))_m = \dim(\mathcal{AP}_t^+)_m$ and $|\mathcal{H}_{(d,t)}| = \binom{d}{t}$. Combining with the results in Wood [29], one gets

$$\dim(\mathcal{A}\mathcal{P}^0_d)_m = \sum_{\mu(m)\leqslant t\leqslant d} \binom{d}{t} \dim(\mathcal{A}\mathcal{P}^+_t)_m.$$

Since $\mu(6(2^2-1)+6.2^2) = 4$, then for $t < 4$ the vector space $(\mathcal{A}\mathcal{P}_t)_{6(2^2-1)+6.2^2}$ is trivial. On the other hand, using the result in Sum [18] we have $\dim(\mathcal{A}\mathcal{P}^+_4)_{42} = 140$.

From the above results, one obtains

$$\dim(\mathcal{A}\mathcal{P}^0_6)_{42} = \binom{6}{4}.\dim(\mathcal{A}\mathcal{P}^+_4)_{42} + \binom{6}{5}.\dim(\mathcal{A}\mathcal{P}^+_5)_{42} = 13020.$$

The theorem has been established. □

Putting $\widetilde{\omega_{[1]}} := (4,3,2,1,1)$, $\widetilde{\omega_{[2]}} := (4,3,2,3)$, $\widetilde{\omega_{[3]}} := (4,3,4,2)$, $\widetilde{\omega_{[4]}} := (4,5,5,1)$, $\widetilde{\omega_{[5]}} := (4,5,3,2)$, $\widetilde{\omega_{[6]}} := (4,5,1,3)$, $\widetilde{\omega_{[7]}} := (4,5,1,1,1)$, then we have the following theorem.

Theorem 3.7. *Suppose that* u *belongs to* $(\mathcal{D}^{\otimes}_6(42) \cap \mathcal{P}^+_6)$ *such that* $\widetilde{S^6_{42}}([u])$ *is not an element of* $\mathrm{Im}\widetilde{S^6_{42}}$. *Then* $\omega(u) = \widetilde{\omega_{[i]}}$ *for* $1 \leqslant i \leqslant 7$. *Furthermore, we have an isomorphism of the* $\mathbb{F}_2$*-vector spaces:*

$$\big(\mathrm{Ker}\widetilde{S^6_{42}} \cap (\mathcal{A}\mathcal{P}^+_6)_{42}\big) \cong \bigoplus_{i=1}^{7} \mathcal{A}\mathcal{P}^+_6\big(\widetilde{\omega_{[i]}}\big).$$

Proof. Let ω be a weight vector of degree m, we set

$$Q\mathcal{P}^{\omega}_d := \mathrm{Span}\big\{[u] \in \mathcal{A}\mathcal{P}_d : \omega(u) = \omega, \text{ and } u \in \mathcal{D}^{\otimes}_6(\omega)\big\}.$$

By the same arguments as in the proof of the previous theorem, it is easy to check that the map $\mathcal{A}\mathcal{P}_d(\omega) \longrightarrow Q\mathcal{P}^{\omega}_d$, $[u]_\omega \longrightarrow [u]$ is an isomorphism of $\mathbb{F}_2$-vector spaces. Thus, $Q\mathcal{P}^{\omega}_d \subset \mathcal{A}\mathcal{P}_d$ can be used to identify the vector space $\mathcal{A}\mathcal{P}_d(\omega)$. As a result of this, one gets

$$(\mathcal{A}\mathcal{P}_d)_m = \bigoplus_{\deg \omega = m} Q\mathcal{P}^{\omega}_d \cong \bigoplus_{\deg \omega = m} \mathcal{A}\mathcal{P}_d(\omega).$$

Hence, it follows that $(\mathcal{A}\mathcal{P}^+_6)_{42} = \bigoplus_{\deg \omega = 42} \mathcal{A}\mathcal{P}^+_6(\omega)$.

Assume that u is an admissible monomial of degree forty-two in $\mathcal{P}_6$ such that $[u]$ belongs to $\mathrm{Ker}\widetilde{S^6_{42}}$. Observe that $v = x_1^{31}x_2^7x_3^3x_4$ is the minimal spike of degree forty-two in $\mathcal{P}_6$, and $\omega(v) = \widetilde{\omega_{[1]}}$. Using Theorem 2.8, one obtains $\omega_1(u) \geqslant \omega_1(v) = 4$. Since the degree of u is even, one gets either $\omega_1(u) = 4$, or $\omega_1(u) = 6$.

If $\omega_1(u) = 4$ then $u = X_{\{i,j\}}w^2$, with w a monomial of degree nineteen in $\mathcal{P}_6$, and $1 \leqslant i < j \leqslant 6$. By Theorem 2.6, it shows that w is admissible. Clearly, $y = x_1^{15}x_2^3x_3$ is the minimal spike of degree nineteen in $\mathcal{P}_6$, and $\omega(y) = (3,2,1,1)$. Using Theorem 2.8, we have $\omega_1(w) \geqslant 3$. Since the degree of w is even, one gets either $\omega_1(w) = 3$, or $\omega_1(w) = 5$.

Case 1. If $\omega_1(w) = 3$ then $w = x_kx_\ell x_t f^2$, where f is an admissible monomial of degree eight in $\mathcal{P}_6$, and $1 \leqslant k < \ell < t \leqslant 6$. Since $f \in \mathcal{D}^{\otimes}_6(8)$, and using the result in [5], one has $\omega(f)$ belongs to $\{(2,1,1),(2,3),(4,2),(6,1)\}$.

Remarkably, if w is a monomial in $\mathcal{P}_6$ such that $\omega(w) = (3,6,1)$ then w is strictly inadmissible (see Sum [17], Prop. 4.3). Hence, w is inadmissible. Thus, $\omega(w)$ belongs to $\{(3,2,1,1),(3,2,3),(3,4,2)\}$. So $\omega(u) = \widetilde{\omega_{[i]}}$ for $i = 1,2,3$.

Case 2. If $\omega_1(w) = 5$ then $w = X_{\{k\}}g^2$, with g a monomial of degree seven in $\mathcal{P}_6$, $1 \leqslant k \leqslant 6$. Using the results in the previous section, we see that if g belongs to $\mathcal{D}_6^{\otimes}(7)$, one has $\omega(g)$ belongs to $\{(5,1),(3,2),(1,3),(1,1,1)\}$. Then, $\omega(w)$ belongs to $\{(5,5,1),(5,3,2),(5,1,3),(5,1,1,1)\}$. So $\omega(u) = \widetilde{\omega_{[i]}}$ for $i = 4,5,6,7$.

If $\omega_1(u) = 6$ then $x = X_{\emptyset}h^2$, with h a monomial of degree eighteen in $\mathcal{P}_6$. Since u is admissible, by Theorem 2.6, it shows that h is also admissible, and $[h] \neq 0$. From this, it implies $[h] = \mathrm{Ker}\widetilde{S^6_{42}}([u]) \neq 0$. This contradicts the fact that $[x] \in \mathrm{Ker}\widetilde{S^6_{42}}$.

In summary, $\omega(u) = \widetilde{\omega_{[i]}}$ for all $1 \leqslant i \leqslant 7$. From the above results, one obtains

$$\mathrm{Ker}\widetilde{S^6_{42}} \cap (\mathcal{AP}_6^+)_{42} \cong \bigoplus_{i=1}^{7} \mathcal{AP}_6^+(\widetilde{\omega_{[i]}}).$$

The theorem has been established. □

For each integer $r > 2$, we consider the degree $m_r = 6(2^r - 1) + 6.2^r$. Let m be an arbitrary non-negative integer, and let $\xi(m)$ be the greatest integer ν such that m is divisible by 2^ν. That means $m = 2^\nu k$, with k an odd integer. Put

$$\lambda(d, m) = \max\{0, d - \alpha(d+m) - \xi(d+m)\}.$$

Then, the map

$$(\widetilde{Sq^0_*})^{s-t} : (\mathcal{AP}_d)_{d(2^s-1)+2^s m} \longrightarrow (\mathcal{AP}_d)_{d(2^t-1)+2^t m}$$

is an isomorphism of $GL(d;F_2)$-modules for every $s \geqslant t$ if and only if $t \geqslant \lambda(d, m)$ (see Tin-Sum [22]).

For $d = m = 6$, $m_r = 6(2^r - 1) + 6.2^r$, then $\alpha(d+m) = \alpha(12) = 2$, and $\xi(d+m) = \xi(2^2.3) = 2$. And therefore $\lambda(n, d) = 2$. Using the above result, we have an isomorphism of F_2-vector space

$$(\mathcal{AP}_6)_{6(2^r-1)+2^r 6} \cong (\mathcal{AP}_6)_{m_2}$$

for all $r \geqslant 2$. Hence, the set $\{[x] : x \in \Gamma^{r-2}(\mathcal{D}_6^{\otimes}(m_2))\}$ is a basis of the F_2-vector space $\mathcal{AP}_6$ in degree $6(2^r - 1) + 6.2^r$ for any interger $r \geqslant 2$. So, we obtain the following theorem.

Theorem 3.8. *The set* $\{[x] : x \in \Gamma^{r-2}(\mathcal{D}_6^{\otimes}(m_2))\}$ *is a basis of the* F_2*-vector space* $\mathcal{AP}_6$ *in degree* $6(2^r - 1) + 6.2^r$, *for any* $r \geqslant 2$.

Remark. It could be seen from the work of Singer the meaning and necessity of the hit problem. In [14], Singer defined the algebraic transfer, which is a homomorphism

$$\psi_d : F_2 \otimes_{GL(d;F_2)} PH_*((\mathbb{R}\mathcal{P}^\infty)^d) \longrightarrow \mathrm{Ext}_{\mathcal{A}}^{d,d+*}(F_2, F_2),$$

where $F_2 \otimes_{GL(d;F_2)} PH_m((\mathbb{R}\mathcal{P}^\infty)^d)$ is dual to $(\mathcal{AP}_d)_m^{GL(d;F_2)}$, and $\mathrm{Ext}_{\mathcal{A}}^{d,d+*}(F_2, F_2)$ is the cohomology groups of the Steenrod algebra.

Singer has indicated the importance of the algebraic transfer by showing that ψ_d is an isomorphism with $d = 1, 2$ and at some other degrees with $d = 3, 4$, but he also disproved this for ψ_5 at degree 9, and then gave the following conjecture.

Conjecture 3.9. *The algebraic transfer* ψ_d *is a monomorphism for any* $d \geqslant 0$.

Boardman [1] then corroborated this by demonstrating that ψ_3 is likewise an isomorphism using the modular representation theory of linear groups. Singer's conjecture, however, remains open for $d \geqslant 4$.

In [19] and [24], we based on the results for the hit problem to verify Singer's conjecture is true for $n = 5$ and the generic degrees $d_s = 5(2^s - 1) + 2^s m$, where $m \in \{1, 2, 3\}$. Continuing this work, using the results of the hit problem, we will investigate and validate Singer's conjecture for the sixth algebraic transfer in the aforementioned degrees by combining the computations of the cohomology groups of the Steenrod algebra $\mathrm{Ext}_{\mathcal{A}}^{6,6(2^s-1)+6.2^s+6}(\mathbb{F}_2, \mathbb{F}_2)$.

Remarkably, by using the result in Tin-Sum [22] (see Theorem 3, pp. 2), we also obtain an isomorphism of $GL(6;\mathbb{F}_2)$-modules

$$(\mathcal{A}\mathcal{P}_6)^{GL(6;\mathbb{F}_2)}_{6(2^r-1)+6.2^r} \cong (\mathcal{A}\mathcal{P}_6)^{GL(6;\mathbb{F}_2)}_{6(2^2-1)+2.2^2}, \quad \text{for all } r \geqslant 2.$$

Hence, one obtains

$$\mathbb{F}_2 \otimes_{GL(6;\mathbb{F}_2)} PH_{6(2^r-1)+6.2^r}((\mathbb{R}\mathcal{P}^\infty)^6) \cong (\mathbb{F}_2 \otimes_{GL(6;\mathbb{F}_2)} PH_{6(2^2-1)+6.2^2}((\mathbb{R}\mathcal{P}^\infty)^6)),$$

for all $r \geqslant 2$.

And therefore, we need only to compute the dimension of spaces $\mathbb{F}_2 \otimes_{GL(6;\mathbb{F}_2)} PH_{6(2^r-1)+6.2^r}((\mathbb{R}\mathcal{P}^\infty)^6)$ for $r \leqslant 2$. This is an open problem.

Furthermore, Walker and Wood have recently published volumes on the hit problem and its applications to representations of general linear groups in the books [27] and [28]. This is yet another application of the hit problem that has to be investigated further in the future.

4. Conclusion

In the article, we study the hit problem for the polynomial algebra of six variables, viewed as a module over the Steenrod algebra in the generic degree $6(2^r - 1) + 6.2^r$ with r an arbitrary positive integer, and its application to the sixth algebraic transfer of Singer. In the future, we will verify the Singer conjecture for the sixth algebraic transfer in degree $6(2^r - 1) + 6.2^r$, with r an arbitrary positive integer, by combining the computations of the cohomology groups of the Steenrod algebra in these cases.

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