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On the oscillation and non-oscillation of solutions of forced second order differential equations

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Abstract

The oscillatory behavior of solutions of a class of second order forced non-linear differential equations is discussed. Several oscillation and non-oscillation criteria are established using Riccati transformations technique. Four examples are given to illustrate our results.

Keywords: Differential equations, interval oscillation, forcing terms, damped equations.

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1. Introduction

Consider the damped second order differential equation

$$[a(t)\Omega(y(t))(y'(t))^{\delta}]' + P(t)(y'(t))^{\delta} + q(t)\rho(y(t)) = 0, \tag{1.1}$$

and the forced differential equation

$$[a(t)\Omega(y(t))(y'(t))^{\delta}]' + P(t)(y'(t))^{\delta} + q(t)\rho(y(t)) = e(t) \text{ for } t \geqslant t_0 > 0, \tag{1.2}$$

where $\delta \geqslant 1$ is a ratio of odd positive integers, and $P(t), \Omega(y(t))$, and $q(t) \in C\ ([t_0, \infty); R)$. The functions $\rho(t)$ and $e(t) \in C\ (R; R)$ with $y\rho(y) > 0$ and $a(t) \in C^1([t_0, \infty); (0, \infty))$. As usual, we restrict our attention to those solutions $y(t)$ of the differential equations which exist on $[t_0, \infty)$. Each equation is called oscillatory if all its solutions are oscillatory. A non-trivial solution of the differential equation is called oscillatory if it has an infinite number of zeros; otherwise, it is said to be non-oscillatory. During the last few decades, there has been considerable interest in studying the oscillatory behavior of solutions of different classes of second order differential equations with and without damping or forcing term. In 1993, El Sheikh [3] studied the oscillatory behavior of solutions of the undamped second order differential equation

$$[a(t)\Omega(y(t))y'(t)]' + q(t)\rho(y(t)) = 0, \tag{1.3}$$

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with $\Omega(y(t)) \leqslant \eta(t)$, where $\eta(t)$ is a positive function. More recently Cakmak [2] and Rogovchenko [15] discussed the oscillation of the damped equation

$$[a(t)y'(t)]' + P(t)y'(t) + q(t)\rho(y(t)) = 0.$$

Rogovchenko et al. [18], Mustafa et al. [12], and Zhang et al. [28] used Riccati transformation and the integral average technique to derive sufficient conditions for the oscillation of solutions of the differential equation

$$[a(t)\Omega(y(t))y'(t)]' + P(t)y'(t) + q(t)\rho(y(t)) = 0. \tag{1.4}$$

In [18, 28] the authors discussed the oscillation of solutions of Eq. (1.4) with the assumption that $m_1 \leqslant \Omega(y(t)) \leqslant m_2$, where m_1 and m_2 are constants. In 2007, Jiang et al. [7] discussed the oscillation of the forced second order differential equation

$$[a(t)\Omega(y(t))\Phi(y')]' + q(t)\rho(y(t)) = e\ (t)\ .$$

In 2013, Tunç et al. [23] discussed the interval oscillation criteria for the unforced second order nonlinear differential equations of the form

$$[a(t)K_1(y,y')]' + P(t)K_2(y,y')y'(t) + q(t)\rho(y(t)) = 0.$$

In [24] Tunç et al. studied the oscillatory behavior of the forced second order differential equations with mixed nonlinearities of the form

$$[a(t)\left|y'(t)\right|^{\delta-1} y'(t)]' + P(t)\left|y(t)\right|^{\delta-1} y(t) + \sum_{j=1}^{m} q_j \left|y(t)\right|^{\beta_j-1} y(t) = e(t),$$

where $0 < \delta < \beta_1 < \beta_2 < \cdots < \beta_m$ are real numbers. In [6, 13, 22] the authors studied the more general forced differential equation

$$[a(t)K_1(y,y')]' + P(t)K_2(y,y')y'(t) + q(t)\rho(y(t)) = e(t).$$

In 2017, Ogrekci et al. [13] introduced a new functional $A_{s_i}^{t_i}(h,t)$ as $A_{s_i}^{t_i}(h,n)$ to overcome problems of singularity and in-applicability of the functional $A_{s_i}^{t_i}(h,t)$ at the points s_i , t_i mentioned in [25].

The aim of this paper is to extend and complement some known oscillation criteria published in the literature. In Section 2, we study the oscillatory behavior of Eq. (1.1) in the case $(\delta = 1)$ which is equivalent to Eq. (1.4) that was discussed by Cakmak [2]. We extend the range of the function $\Omega(y(t))$ to be more general than those considered by the authors in [18, 28]. Moreover, we relax the restriction of [1] on the damping term $P(t)$. Then we establish some oscillation criteria for (1.3), (1.4), and (1.1), which partially generalize some of those given by [3, 8, 18]. In Section 3, we discuss the interval oscillation type [10] for the more general forced Eq. (1.2) using Ogrekci's technique [13]. Further, we establish sufficient condition for the non-oscillation of an undamped forced differential equation. In the last section, we give some illustrative examples.

Throughout the paper we assume that

(C_1) $\frac{\rho(y(t))}{y^{\delta}} \geqslant \mu$, $\mu > 0$ for $y(t) \neq 0$;
(C_2) $0 < \xi(t) \leqslant \Omega(y(t)) \leqslant \eta(t)$.

Denote $D = \{(t,s) : t_0 \leqslant s \leqslant t \leqslant \infty\}$ and $D_0 = \{(t,s) : t_0 \leqslant s < t \leqslant \infty\}$.

Following [11], we say that $H \in C\ (D,\ [0,\infty))$ belongs to the class ω_δ if it satisfies the conditions:

1. $H(t,t) = 0$ for $t \geqslant t_0$, $H(t,s) > 0$ for all $(t,s) \in D_0$;
2. $\partial H/\partial t = h_1(t,s)\ (H(t,s))^{\frac{\delta}{\delta+1}}$ and $\partial H/\partial s = -h_2(t,s)\ (H(t,s))^{\frac{\delta}{\delta+1}}$.

The following lemmas will be needed.

Lemma 1.1 ([11]). *Let $\delta \geqslant 1$ be a ratio of two odd numbers. Then*

$$A^{1+\frac{1}{\delta}} - (A-B)^{1+\frac{1}{\delta}} \leqslant \frac{B^{\frac{1}{\delta}}}{\delta}[(\delta+1)A - B], \tag{1.5}$$

$$C^{\frac{\delta+1}{\delta}} - \frac{\delta+1}{\delta} C D^{\frac{1}{\delta}} \geqslant -\frac{1}{\delta} D^{\frac{\delta+1}{\delta}}. \tag{1.6}$$

Lemma 1.2 ([6]). *If A and B are non-negative, then*

$$\frac{1}{p}A^p + \frac{1}{q}B^q \geqslant AB, \qquad \textit{for} \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{1.7}$$

2. Non-linear second-order differential equation with damping term

In this section, we first discuss the oscillatory behavior of solutions of Eq. (1.1) in the particular case $(\delta = 1)$ and establish new criteria which generalize those of [2], and relax the restriction of [1] about the sign of the damping term. Then, we study the oscillatory behavior of the more general second-order differential equation (1.1).

Theorem 2.1. *Suppose that the conditions* (C_1) *and* (C_2) *hold. Assume that* $q(t) \geqslant 0$*, and there exists a differentiable function* $g(t) \in C^1([t_0, \infty); R_+)$ *such that*

$$\int_{t_0}^{\infty} \frac{ds}{a(s)g(s)\eta(s)} = \infty, \tag{2.1}$$

and

$$\limsup_{t\to\infty} \{\frac{a(t)g'(t)\xi(t) - P(t)g(t)}{2} + \int_{t_0}^{t} \mu g(s)q(s) - \frac{[a(s)g'(s)\eta(s) - P(s)g(s)]^2}{4\, a(s)g(s)\eta(s)} ds\} = \infty. \tag{2.2}$$

Then Eq. (1.4) *is oscillatory.*

Proof. Suppose the contrary that there exists a non-oscillatory solution $y(t)$ of Eq. (1.4). Without loss of generality, we may assume that $y(t) \neq 0$ for all $t \geqslant t_0$. Define,

$$\varpi(t) = -g(t)\frac{a(t)\Omega(y(t))y'(t)}{y(t)}. \tag{2.3}$$

In view of (1.4), we have

$$\varpi'(t) = \frac{g'(t)\varpi(t)}{g(t)} - P(t)\frac{\varpi(t)}{a(t)\Omega(y(t))} + g(t)q(t)\frac{\rho(y(t))}{y(t)} + \frac{\varpi^2(t)}{a(t)g(t)\Omega(y(t))},$$

for all $t \geqslant t_0$. Thus by (C_1), we get

$$\begin{aligned}\varpi'(t) \geqslant{}& \frac{1}{a(t)g(t)\Omega(y(t))}[\varpi(t) + \frac{a(t)g'(t)\Omega(y(t)) - P(t)g(t)}{2}]^2 - \frac{[a(t)g'(t)\Omega(y(t)) - P(t)g(t)]^2}{4\, a(t)g(t)\Omega(y(t))} \\ &+ \mu g(t)q(t).\end{aligned} \tag{2.4}$$

Therefore by (C_2), we have

$$\varpi'(t) \geqslant \mu g(t)q\,(t) + \frac{1}{a(t)g(t)\eta(t)}[\varpi(t) + \frac{a(t)g'(t)\xi(t) - P(t)g(t)}{2}]^2 - \frac{[a(t)g'(t)\eta(t) - P(t)g(t)]^2}{4\, a(t)g(t)\eta(t)}.$$

Putting

$$G(t) = \varpi(t) + \frac{a(t)g'(t)\xi(t) - P(t)g(t)}{2}, \tag{2.5}$$

then

$$\varpi'(t) \geqslant \mu g(t)q(t) + \frac{G^2(t)}{a(t)g(t)\eta(t)} - \frac{[a(t)g'(t)\eta(t) - P(t)g(t)]^2}{4\, a(t)g(t)\eta(t)}.$$

By integrating from t_0 to t, we obtain

$$G(t) \geqslant \varpi(t_0) + \int_{t_0}^{t} \frac{G^2(s)}{a(s)g(s)\eta(s)} ds + \{\frac{a(t)g'(t)\xi(t) - P(t)g(t)}{2} + \int_{t_0}^{t} \mu g(s)q(s) - \frac{[a(s)g'(s)\eta(s) - P(s)g(s)]^2}{4\, a(s)g(s)\eta(s)} ds\}.$$

This with (2.2) leads to

$$G(t) \geqslant \int_{t_0}^{t} \frac{G^2(s)}{a(s)g(s)\eta(s)} ds,$$

for sufficiently large $t \geqslant t_1$. Putting

$$M(t) = \int_{t_0}^{t} \frac{G^2(s)}{a(s)g(s)\eta(s)} ds,$$

then it follows that

$$G(t) > M(t) > 0,$$

and

$$M'(t) = \frac{G^2(t)}{a(t)g(t)\eta(t)} > \frac{M^2(t)}{a(t)g(t)\eta(t)}.$$

By dividing by $M^2(t)$ and integrating from t_1 to t, we get

$$\int_{t_1}^{t} \frac{1}{a(s)g(s)\eta(s)} ds < \frac{1}{M(t_1)} - \frac{1}{M(t)}.$$

But since $M(t) > 0$, then

$$\int_{t_1}^{t} \frac{1}{a(s)g(s)\eta(s)} ds < \frac{1}{M(t_1)},$$

which is a contradiction with (2.1) as $t \to \infty$. Then Eq. (1.4) is oscillatory. □

Remark 2.2. In the special case $\Omega(y(t)) = 1$, $\mu = 1$, Theorem 2.1 includes the criteria (2.1) and (2.2) of [2], while if $\Omega(y(t)) = a(t) = g(t) = 1$ and $\rho(y) = y$, then the Theorem includes those of [20]. Moreover if $\Omega(y(t)) = a(t) = 1$, and $\rho(y) = y$, Theorem 2.1 includes those of [1].

Theorem 2.3. *If*

$$\int_{t_0}^{\infty} \{\int_{t_0}^{s} a(\tau)g(\tau)\eta(\tau) d\tau\}^{-1} ds = \infty \tag{2.6}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} \{\frac{a(s)g'(s)\xi(s) - P(s)g(s)}{2} + \int_{t_0}^{s} \mu g(\tau)q(\tau) - \frac{[a(\tau)g'(\tau)\eta(\tau) - P(\tau)g(\tau)]^2}{4\, a(\tau)g(\tau)\eta(\tau)} d\tau\} ds = \infty, \tag{2.7}$$

then Eq. (1.4) *is oscillatory.*

Proof. Going through as in Theorem 2.1, we get

$$G(t) \geqslant \varpi(t_0) + \int_{t_0}^{t} \frac{G^{\,2}(s)}{a(s)g(s)\eta(s)} ds + \{\frac{a(s)g'(s)\xi(s) - P(s)g(s)}{2} + \int_{t_0}^{t} \mu g(\tau)q(\tau) - \frac{[a(\tau)g'(\tau)\eta(\tau) - P(\tau)g(\tau)]^2}{4\, a(\tau)g(\tau)\eta(\tau)} d\tau\}.$$

By integrating from t_0 to t and dividing by t,

$$\frac{1}{t}\int_{t_0}^{t} G(s)ds \geqslant \frac{1}{t}\int_{t_0}^{t} \varpi(t_0) + \frac{1}{t}\int_{t_0}^{t} \frac{G^{\,2}(s)}{a(s)g(s)\eta(s)} ds + \frac{1}{t}\int_{t_0}^{t} \{\frac{a(s)g'(s)\xi(s) - P(s)g(s)}{2} + \int_{t_0}^{s} \mu g(\tau)q(\tau) - \frac{[a(\tau)g'(\tau)\eta(\tau) - P(\tau)g(\tau)]^2}{4\, a(\tau)g(\tau)\eta(\tau)} d\tau\}ds.$$

Thus, we can choose t_1 sufficiently large such that for $t \geqslant t_1$, we have

$$\int_{t_0}^{t} G(s)ds - \int_{t_0}^{t} M(s)ds \geqslant 0.$$

Putting

$$A(t) = \int_{t_0}^{t} M(s)ds,$$

and using (1.7), we get

$$\begin{aligned} A^2(t) \leqslant \{\int_{t_0}^{t} G(s)ds\}^2 &= \{\int_{t_0}^{t} \sqrt{a(s)g(s)\eta(s)} \cdot \frac{G(s)}{\sqrt{a(s)g(s)\eta(s)}}\}^2 ds \\ &\leqslant \{\int_{t_0}^{t} a(s)g(s)\eta(s)ds\}\{\int_{t_0}^{t} \frac{G^{\,2}(s)}{a(s)g(s)\eta(s)} ds\} \\ &\leqslant M(t)\int_{t_0}^{t} a(s)g(s)\eta(s)ds \leqslant A'(t)\int_{t_0}^{t} a(s)g(s)\eta(s)ds. \end{aligned}$$

Dividing by $A^2(t)\int_{t_0}^{t} a(s)g(s)\eta(s)ds$ and integrating from t_1 to t, we obtain

$$\int_{t_1}^{t} \{\int_{t_0}^{s} a(\tau)g(\tau)\eta(\tau)d\tau\}^{-1} ds \leqslant \frac{1}{A(t_1)} - \frac{1}{A(t)} \leqslant \frac{1}{A(t_1)}.$$

Then as $t \to \infty$, we find that $\int_{t_1}^{t}(\int_{t_0}^{s} a(\tau)g(\tau)\eta(\tau)d\tau)^{-1}ds \neq \infty$. This contradicts (2.6), and so Eq. (1.4) is oscillatory. □

Remark 2.4. In the special case $\Omega(y(t)) = 1$, the criteria (2.6) and (2.7) of Theorem 2.3 include the criteria (2.15) and (2.16) of the paper [2].

The following result, improves the result obtained by Li et al. [11] for the unforced Eq. (1.1).

Theorem 2.5. *Suppose that* (C_1) *and* (C_2) *hold. Suppose further that* $Q(t)$ *is non-negative and does not vanish eventually. If for some* $\beta \geqslant 1$*, a positive function* $\Upsilon(t) \in C^1([t_0,\infty), R)$*, and some* $H \in \omega_\delta$*, we have*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\Theta_1(s) - \frac{\beta^{\delta}}{(\delta+1)^{\delta+1}} a(s)\eta(s)\nu_1(s)h_2^{\delta+1}(t,s)ds = \infty, \tag{2.8}$$

where

$$\nu_1 = \exp\{-(\delta+1)\int^{t} \frac{\Upsilon^{\frac{1}{\delta}}(s)}{\xi^{\frac{1}{\delta}}(s)} - \frac{P(s)}{(\delta+1)a(s)\eta(s)} ds\},$$

and

$$\Theta_1(t)=\nu_1(t)\{-P(t)\frac{\Upsilon(t)}{\xi(t)}+\mu q(t)+\frac{a(t)\Upsilon^{1+\frac{1}{\delta}}(t)}{\eta(t)^{\frac{1}{\delta}}}-[a(t)\Upsilon(t)]'\},$$

then Eq. (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a non-oscillatory solution of Eq. (1.1). Assume that $y(t)\neq 0$ for all $t\geqslant t_0$ and define a generalized Riccati transformation of the form

$$u_1(t)=\nu_1(t)a(t)[\Omega(y(t))(\frac{y'(t)}{y(t)})^{\delta}+\Upsilon(t)].$$

Then

$$u_1'(t)=\frac{\nu_1'(t)u(t)}{\nu_1(t)}+\nu_1(t)[-P(t)(\frac{y'(t)}{y(t)})^{\delta}-q(t)\frac{\rho(y(t))}{y(t)^{\delta}}]$$
$$-\delta\nu_1(t)a(t)\Omega(y(t))\{\frac{1}{\Omega(y(t))}(\frac{u_1(t)}{\nu_1(t)a(t)}-\Upsilon(t))\}^{\frac{\delta+1}{\delta}}+\nu_1(t)[a(t)\Upsilon(t)]',$$

i.e.,

$$u_1'(t)=\frac{\nu_1'(t)u(t)}{\nu_1(t)}+\nu_1(t)[-P(t)(\frac{y'(t)}{y(t)})^{\delta}-q(t)\frac{\rho(y(t))}{y(t)^{\delta}}]$$
$$+\nu_1(t)[a(t)\Upsilon(t)]'-\delta\frac{\nu_1(t)a(t)}{[\Omega(y(t))]^{\frac{1}{\delta}}}\{\frac{u_1(t)}{\nu_1(t)a(t)}-\Upsilon(t)\}^{\frac{\delta+1}{\delta}}.$$

This with the inequality (1.5) leads to

$$u_1'(t)=\frac{\nu_1'(t)u(t)}{\nu_1(t)}+\nu_1(t)[-P(t)(\frac{y'(t)}{y(t)})^{\delta}-q(t)\frac{\rho(y(t))}{y(t)^{\delta}}]+\nu_1(t)[a(t)\Upsilon(t)]'$$
$$+\frac{\delta\nu_1(t)a(t)}{[\Omega(y(t))]^{\frac{1}{\delta}}}\{\frac{\Upsilon^{\frac{1}{\delta}}(t)}{\delta}[(\delta+1)(\frac{u_1(t)}{\nu_1(t)a(t)}-\Upsilon(t))]\}-[\frac{u_1(t)}{\nu_1(t)a(t)}]^{1+\frac{1}{\delta}}.$$

Using the condition (C_2), we have

$$u_1'(t)\leqslant-\delta[\frac{u_1^{1+\delta}(s)}{\eta(s)\nu_1(s)a(s)}]^{\frac{1}{\delta}}-\Theta_1(t).$$

Multiplying by $H(t,s)$ and integrating from T_1 to t, in view of the properties of $H(t,s)$, we get

$$\int_{T_1}^{t}H(t,s)\Theta_1(s)ds\leqslant H(t,T_1)u_1(T_1)-\delta\int_{T_1}^{t_i}H(t,s)[\frac{u_1^{1+\delta}(s)}{\eta(s)\nu_1(s)a(s)}]^{\frac{1}{\delta}}ds-\int_{T_1}^{t_i}u_1(s)h_2(t,s)[H(t,s)]^{\frac{\delta}{\delta+1}}ds.$$

Thus for some $\beta\geqslant 1$, we have

$$\int_{T_1}^{t}H(t,s)\Theta_1(s)ds+\int_{T_1}^{t_i}u_1(s)h_2(t,s)[H(t,s)]^{\frac{\delta}{1+\delta}}ds+\frac{\delta}{\beta}\int_{T_1}^{t_i}H(t,s)[\frac{u_1^{1+\delta}(s)}{\eta(s)\nu_1(s)a(s)}]^{\frac{1}{\delta}}ds$$
$$\leqslant H(t,T_1)u_1(T_1)-\frac{\delta}{\beta}(\beta-1)\int_{T_1}^{t_i}H(t,s)[\frac{u_1^{1+\delta}(s)}{\eta(s)\nu_1(s)a(s)}]^{\frac{1}{\delta}}ds. \tag{2.9}$$

Applying the inequality (1.6) with

$$C=[\frac{\delta H u_1^{\frac{1+\delta}{\delta}}}{\beta\,\nu_1^{\frac{1}{\delta}}a^{\frac{1}{\delta}}\eta^{\frac{1}{\delta}}}]^{\frac{\delta}{1+\delta}},$$

and
$$D=-[\frac{\delta\beta^{\delta}\eta(s)\nu_1(s)a(s)h_2^{1+\delta}(t,s)}{(1+\delta)^{1+\delta}}]^{\frac{\delta}{1+\delta}},$$
we have
$$u_1(s)h_2(t,s)[H(t,s)]^{\frac{1+\delta}{\delta}}+\frac{\delta}{\beta}H(t,s)[\frac{u_1^{1+\delta}(s)}{\eta(s)\nu_1(s)a(s)}]^{\frac{1}{\delta}}\leqslant-\frac{\beta^{\delta}\eta(s)\nu_1(s)a(s)h_1^{1+\delta}(t,s)}{(1+\delta)^{1+\delta}}.$$
Substituting into (2.9), we have
$$\int_{T_1}^{t}H(t,s)\Theta_1(s)-\frac{\beta^{\delta}\eta(s)\nu_1(s)a(s)h_2^{1+\delta}(t,s)}{(1+\delta)^{1+\delta}}ds\leqslant H(t,T_1)u_1(T_1)-\frac{\delta}{\beta}(\beta-1)\int_{T_1}^{t_i}H(t,s)[\frac{u_1^{1+\delta}(s)}{\eta(s)\nu_1(s)a(s)}]^{\frac{1}{\delta}}ds.$$
By the monotonicity of $H(t,s)$ for all $t\geqslant T_1$, we have
$$\begin{aligned}\int_{T_1}^{t}H(t,s)\Theta_1(s)-\frac{\beta^{\delta}h_2^{1+\delta}\eta(s)\nu_1(s)a(s)}{(1+\delta)^{1+\delta}}ds&\leqslant H(t,T_1)\left|u_1(T_1)\right|\\&\leqslant H(t,t_0)\left|u_1(T_1)\right|\leqslant H(t,t_0)\{\left|u_1(T_1)\right|+\int_{t_0}^{T}\left|\Theta_1(s)\right|ds\}.\end{aligned}$$
Thus
$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\Theta_1(s)-\frac{\beta^{\delta}}{(\delta+1)^{\delta+1}}a(s)\eta(s)\nu_1(s)h_2^{1+\delta}(t,s)ds\leqslant\left|u_1(T_1)\right|+\int_{t_0}^{T}\Theta_1(s)ds<\infty,$$
which contradicts (2.8). Then Eq. (1.1) is oscillatory. □

Remark 2.6.

(1) Theorem 2.4 includes Theorem 2 of [11] in the special case $\Omega(y(t))=1$.

(2) If $\delta=1$, then the criterion (2.8) of Theorem 2.4 partially improves that given by [18].

Corollary 2.7. *If*
$$\limsup_{t\to\infty}\frac{1}{R^{n-1}(t)}\int_{t_0}^{t}\mu q(s)[R(t)-R(s)]^{n-1}ds=\infty,\quad \textit{for } n\geqslant 3, t\geqslant t_0, \tag{2.10}$$
where
$$R(t)=\int_{t_0}^{t}\frac{ds}{\eta(s)a(s)}, \tag{2.11}$$
then Eq. (1.3) is oscillatory.

Proof. Let $H(t,s)=[R(t)-R(s)]^{n-1}$ for $t>s\geqslant t_0$, then
$$h_2(t,s)=\frac{n-1}{a(s)\eta(s)}[R(t)-R(s)]^{\frac{n-3}{2}}$$
and
$$\int_{t_0}^{t}a(s)\eta(s)h_2^2(t,s)=\int_{t_0}^{t}\frac{(n-1)^2}{a(s)\eta(s)}[R(t)-R(s)]^{n-3}ds=\frac{(n-1)^2}{n-2}[R(t)]^{n-2}.$$
By (2.11) we directly obtain (2.10). Then by Theorem 2.4, Eq. (1.3) oscillates. □

Remark 2.8. In the case $\mu=1$, Corollary 2.5 includes Theorem 3.3 of [3], and Theorem 2 of [26]. Moreover in the case $\Omega(y(t))=1$ and $H(t,s)=[R(t)-R(s)]^{\lambda}ds$ for $\lambda>1$, the criterion (2.10) includes the Criterion (2.2) of [8], while for $\mu=1$ and $H(t,s)=(t-s)^{\lambda}$, Theorem 2.4 improves corollary (2.4) of [8].

3. Forced second-order differential equations

Now, we are going to discuss the oscillation of solutions of the more general forced equation (1.2). We are concerned with the interval oscillation type [10]. Throughout the section, we assume that

(C_3) for any $T \geqslant t_0$, or there exist $T \leqslant \nu_1 < b_1 \leqslant \nu_2 < b_2$, such that: $e(t) \leqslant 0$ for $t \in [\nu_1, b_1]$ and $e(t) \geqslant 0$ for $t \in [\nu_2, b_2]$;

(C_4) $P(t) > 0$ and $q(t) \geqslant 0$ on $t \in [\nu_1, b_1] \cup [\nu_2, b_2]$.

Theorem 3.1. *Suppose that* (C_1)-(C_4) *hold. If there exist some* $c_i \in (\nu_i, b_i), i = 1, 2$ *such that*

$$\begin{aligned}&\frac{1}{H(c_i,\nu_i)}\int_{\nu_i}^{c_i}\{H(s,\nu_i)\Theta_2(s)-\delta_1 H_1(s,\nu_i)a(s)\nu_2(s)\eta(s)\}ds\\&\quad+\frac{1}{H(b_i,c_i)}\int_{c_i}^{b_i}\{H(b_i,s)\Theta_2(s)-\delta_1 H_2(b_i,s)a(s)\nu_2(s)\eta(s)\}ds>0,\end{aligned} \tag{3.1}$$

for $i = 1, 2$, *then Eq.* (1.2) *is oscillatory, where*

$$H_1(t,s)=\left|(\delta+1)H^{\frac{1}{1+\delta}}(t,s)\frac{\Upsilon^{\frac{1}{\delta}}(t)}{\xi^{\frac{1}{\delta}}(t)}+h_1(t,s)\right|,$$

$$H_2(t,s)=\left|(\delta+1)H^{\frac{1}{1+\delta}}(t,s)\frac{\Upsilon^{\frac{1}{\delta}}(t)}{\xi^{\frac{1}{\delta}}(t)}-h_2(t,s)\right|,$$

$$\nu_2=\exp\{\int^{t}\frac{P(s)}{a(s)\eta(s)}ds\},\quad \delta_1=\frac{1}{(\delta+1)^{\delta}},$$

and

$$Theta_2(t)=\nu_2(t)\{-P(t)\frac{\Upsilon(t)}{\xi(t)}+\mu q(t)+\frac{(\delta+1)a(t)\Upsilon(t)^{1+\frac{1}{\delta}}}{\eta(t)^{\frac{1}{\delta}}}-[a(t)\Upsilon(t)]'\}.$$

Proof. Let $y(t)$ be a non-oscillatory solution of Eq. (1.2). We may assume that $y(t) \neq 0$ for all $t \geqslant t_0$ and define the Riccati transformation

$$u_2(t)=\nu_2(t)a(t)\{\Omega(y(t))(\frac{y'(t)}{y(t)})^{\delta}+\Upsilon(t)\}. \tag{3.2}$$

By differentiating (3.2) in view of (1.2), we get

$$\begin{aligned}u_2'(t)=&\frac{\nu_2'(t)u_2(t)}{\nu_2(t)}+\nu_2(t)\{\frac{e(t)}{y^{\delta}}-P(t)(\frac{y'(t)}{y(t)})^{\delta}-q(t)\frac{\rho(y(t))}{y(t)^{\delta}}\}\\&-\delta\nu_2(t)a(t)\Omega(y(t))\{\frac{1}{\Omega(y(t))}[\frac{u(t)}{\nu_2(t)a(t)}-\Upsilon(t)]\}^{\frac{\delta+1}{\delta}}+\nu_2(t)(a(t)\Upsilon(t))'.\end{aligned}$$

Thus by (1.5), we have

$$-\{\frac{u_2(t)}{\nu_2(t)a(t)}-\Upsilon(t)\}^{\frac{\delta+1}{\delta}}\leqslant\frac{\Upsilon(t)^{\frac{1}{\delta}}}{\delta}[(\delta+1)(\frac{u_2(t)}{\nu_2(t)a(t)}-\Upsilon(t)]-[\frac{u_2(t)}{\nu_2(t)a(t)}]^{1+\frac{1}{\delta}},$$

i.e.,

$$\begin{aligned}u_2'(t)=&\frac{\nu_2'(t)u_2(t)}{\nu_2(t)}+\nu_2(t)\{\frac{e(t)}{y^{\delta}}-P(t)(\frac{y'(t)}{y(t)})^{\delta}-q(t)\frac{\rho(y(t))}{y(t)^{\delta}}\}+\nu_2(t)[a(t)\Upsilon(t)]'\\&+\frac{\delta\nu_2(t)a(t)}{\Omega(y(t))^{\frac{1}{\delta}}}\{\frac{\Upsilon(t)^{\frac{1}{\delta}}}{\delta}[(\delta+1)(\frac{u_2(t)}{\nu_2(t)a(t)}-\Upsilon(t))]-[\frac{u_2(t)}{\nu_2(t)a(t)}]^{1+\frac{1}{\delta}}.\end{aligned} \tag{3.3}$$

Therefore by (C_2), we have

$$u_2'(t) \leqslant -\delta[\frac{u_2^{1+\delta}(t)}{\eta(t)\nu_2(t)a(t)}]^{\frac{1}{\delta}} + \nu_2(t)[\frac{e(t)}{y^\delta}] - \Theta_2(t) + \frac{(\delta+1)u_2(t)\Upsilon^{\frac{1}{\delta}}(t)}{\xi(t)^{\frac{1}{\delta}}}.$$

Now, consider the case $y(t) > 0$ for all $t \geqslant T_0$. Since $e(t) \leqslant 0$ on the interval $[\nu_1, b_1]$, we get

$$u_2'(t) \leqslant -\delta[\frac{u_2^{1+\delta}(t)}{\eta(t)\nu_2(t)a(t)}]^{\frac{1}{\delta}} - \Theta_2(t) + \frac{(\delta+1)u_2(t)\Upsilon^{\frac{1}{\delta}}(t)}{\xi^{\frac{1}{\delta}}(t)}.$$

Multiplying by $H(t,s)$ and integrating (with t replaced by s) over $[c_i, t)$ for $t \in [c_i, b_i)$, $i = 1,2$ using the properties of $H(t,s)$, we have

$$\begin{aligned}\int_{c_i}^{t} H(t,s)\Theta_2(s)ds &\leqslant H(t,c_i)u_2(c_i) - \delta\int_{c_i}^{t_i} H(t,s)[\frac{u_2^{1+\delta}(s)}{\eta(s)\nu_2(s)a(s)}]^{\frac{1}{\delta}}ds \\ &\quad + \int_{c_i}^{t_i} u_2(s)[(\delta+1)H(t,s)(\frac{\Upsilon(t)}{\xi(t)})^{\frac{1}{\delta}} - h_2(t,s)H^{\frac{\delta}{\delta+1}}(t,s)]ds \\ &\leqslant H(t,c_i)u_2(c_i) + \int_{c_i}^{t_i} u_2(s)H^{\frac{\delta}{\delta+1}}(t,s)H_2(t,s) - \delta H(t,s)[\frac{u_2^{1+\delta}(s)}{\eta(s)\nu_2(s)a(s)}]^{\frac{1}{\delta}}ds.\end{aligned} \tag{3.4}$$

For a given t and s, let

$$F(u_2) = u_2(s)H^{\frac{\delta}{\delta+1}}(t,s)H_2(t,s) - \delta H(t,s)[\frac{u_2^{1+\delta}(s)}{\eta(s)\nu_2(s)a(s)}]^{\frac{1}{\delta}}.$$

Then

$$F'(u_2) = H^{\frac{\delta}{\delta+1}}(t,s)H_2(t,s) - (\delta+1)H(t,s)[\frac{u_2(s)}{\eta(s)\nu_2(s)a(s)}]^{\frac{1}{\delta}}.$$

So the maximum of $F(u_2)$ is obtained at

$$u_2(t) = [\frac{H_2(t,s)}{(\delta+1)[H(t,s)]^{\frac{1}{\delta+1}}}]^{\delta}\eta(s)\nu_2(s)a(s),$$

i.e.,

$$F(u_2) \leqslant \delta_1[H_2(t,s)]^{\delta+1}\eta(s)\nu_2(s)a(s),$$

$$\text{Thus } \int_{c_i}^{t} H(t,s)\Theta_2(s)ds \leqslant H(t,c_i)u_2(c_i) + \int_{c_i}^{t} \delta_1 H_2^{\delta+1}(t,s)\eta(s)\nu_2(s)a(s)ds.$$

Letting $t \to b_i^-$ in (3.4), we obtain

$$\int_{c_i}^{b_i} H(b_i,s)\Theta_2(s)ds \leqslant H(b_i,c_i)u_2(c_i) + \delta_1\int_{c_i}^{b_i} H_2^{\delta+1}(b_i,s)\eta(s)\nu_2(s)a(s)ds. \tag{3.5}$$

On the other hand, if we multiply (3.4) by $H(t,s)$ and integrating (with t replaced by s) over $(t, c_i]$ for $t \in (\nu_i, c_i]$, $s \in (t, c_i], i = 1,2$, and using the properties of $H(t,s)$, we get

$$\begin{aligned}\int_{t}^{c_i} H(s,t)\Theta_2(s)ds &\leqslant -H(c_i,t)u_2(c_i) - \int_{t}^{c_i} (\delta+1)H(s,t)[\frac{u_2^{1+\delta}(s)}{\eta(s)\nu_2(s)a(s)}]^{\frac{1}{\delta}}ds \\ &\quad + \int_{t}^{c_i} u_2(s)\{(\delta+1)H(s,t)[\frac{\Upsilon(s)}{\xi(s)}]^{\frac{1}{\delta}} + h_1(t,s)H^{\frac{\delta}{\delta+1}}(s,t)\}ds \\ &\leqslant -H(c_i,t)u_2(c_i) + \delta_1\int_{t}^{c_i} H_1^{\delta+1}(s,t)\eta(s)\nu_2(s)a(s)ds.\end{aligned} \tag{3.6}$$

Letting $t \to \nu_i^+$ in (3.6), we have

$$\int_{\nu_i}^{c_i} H(s,\nu_i)\Theta_2(s)ds \leqslant -H(c_i,\nu_i)u_2(c_i) + \delta_1 \int_{\nu_i}^{c_i} H_1^{\delta+1}(s,t)\eta(s)\nu_2(s)a(s)ds. \tag{3.7}$$

Finally, dividing (3.5) and (3.7) by $H(b_i,c_i)$ and $H(c_i,\nu_i)$, respectively, and then adding them, we get the following inequality

$$\begin{aligned}&\frac{1}{H(c_i,\nu_i)}\int_{\nu_i}^{c_i} H(s,\nu_i)\Theta_2(s)ds + \frac{1}{H(b_i,c_i)}\int_{c_i}^{b_i} H(b_i,s)\Theta_2(s)ds\\ &\leqslant \frac{1}{H(c_i,\nu_i)}\int_{\nu_i}^{c_i} \delta_1 H_1^{\delta+1}(s,\nu_i)a(s)\nu_2(s)\eta(s)]ds + \frac{1}{H\ (b_i,c_i)}\int_{c_i}^{b_i} \delta_1 H_2^{\delta+1}(b_i,s)a(s)\nu_2(s)\eta(s)ds,\end{aligned}$$

which contradicts (3.1). Then Eq. (1.2) is oscillatory. □

Now, following [13], we shall use the functional

$$A_{s_i}^{t_i}(h,n) = \int_{s_i}^{t_i} |H(t)|^n\, h(t)dt, s_i \leqslant t \leqslant t_i\ , i = 1,2 \text{ and } n \geqslant 0,$$

where $D_2(s_i,t_i) = \{u \in C^1[s_i,t_i] : u(t) \neq 0 \text{ for } t \in (s_i,t_i), u(s_i) = u(t_i\ ) = 0\}$ for $i = 1,2$, and $h \in C([t_0,\infty),[0,\infty))$, $H \in D_2(s_i,t_i)$, where, the linear functional $A_{s_i}^{t_i}(h,n)$ satisfies the conditions

- (C_5) $A_{s_i}^{t_i}(h,n) = A_{s_i}^{t_i}(|H(t)|^k\, h; n-k)$, for $i = 1,2$ and $k \in R$;
- (C_6) $A_{s_i}^{t_i}(h\,',n) \geqslant -A_{s_i}^{t_i}(n\ |H\,'(t)h|; n-1)$, for $i = 1,2$.

Theorem 3.2. *Suppose that* (C_1)-(C_4) *hold. If there exists a function* $H \in D_2(s_i,t_i)$ *and non-negative constants* n *and* α *such that*

$$A_{s_i}^{t_i}(\mu q(t), n+\alpha+1) > A_{s_i}^{t_i}(\delta_2 a(t)\eta(t)G\,_1^{\delta+1}H^{\,\alpha-\delta}, n), \text{ for } i = 1,\ 2, \tag{3.8}$$

then Eq. (1.2) *is oscillatory, where the linear functional* $G_1(t) = \left|(n+\alpha+1)H'(t) - \frac{P(t)}{a(t)\eta(t)}H(t)\right|$ *and* $\delta_2 = \frac{1}{(\delta+1)^\delta}$.

Proof. Let $y(t)$ be a non-oscillatory solution of Eq. (1.2). Assume that $y(t) \neq 0$ for all $t \geqslant t_0$. Consider the Riccati transformation

$$u_3(t) = a(t)\Omega(y(t))[\frac{y'(t)}{y(t)}]^\delta.$$

Then in view of (1.2), we get

$$\begin{aligned} u_3'(t) &= \{\frac{e(t)}{y^\delta} - P(t)[\frac{y'(t)}{y(t)}]^\delta - q(t)\frac{\rho(t)}{y^\delta(t)}\} - \delta a(t)\Omega(y(t))[\frac{y'(t))}{y(t)}]^{\delta+1}\\ &\leqslant \frac{e(t)}{y^\delta} - P(t)\frac{u_3(t)}{a(t)\Omega(y(t))} - \mu q(t) - \delta\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\Omega(y(t))]^{\frac{1}{\delta}}}.\end{aligned}$$

On the intervals $[s_1,t_1]$ if $(y(t) > 0)$ or $[s_2,t_2]$ (if $y(t) < 0), u_3(t)$ satisfies

$$u_3'(t) \leqslant -\mu q(t) - P(t)\frac{u_3(t)}{a(t)\Omega(y(t))} - \delta\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\Omega(y(t))]^{\frac{1}{\delta}}}.$$

Then by (C_2), we have

$$u_3'(t) \leqslant -\mu q(t) - P(t)\frac{u_3(t)}{a(t)\eta(t)} - \delta\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}.$$

Multiplying by $|H(t)|^{n+\alpha+1}$ and integrating from s_i to t_i for $i=1$, we get

$$\begin{aligned}
&\int_{s_i}^{t_i} \mu\, |H(t)|^{n+\alpha+1}\, q(t)dt\\
&\quad\leqslant \int_{s_1}^{t_1} -u_3'(t)\, |H(t)|^{n+\alpha+1}\, dt - \int_{s_1}^{t_1} |H(t)|^{n+\alpha+1} \{P(t)\frac{u_3(t)}{a(t)\eta(t)} + \delta\frac{(u_3)^{1+\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}\}dt\\
&\quad\leqslant (n+\alpha+1)\int_{s_1}^{t_1} u_3(t)\, |H(t)|^{n+\alpha}\, H'(t)dt + \int_{s_1}^{t_1} |H(t)|^{n+\alpha+1}\{-P(t)\frac{u_3(t)}{a(t)\eta(t)} - \delta\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}\}dt\\
&\quad\leqslant \int_{s_1}^{t_1} u_3(t)\{(n+\alpha+1)\, |H(t)|^{n+\alpha}\, H'(t) - \frac{P(t)}{a(t)\eta(t)}\, |H(t)|^{n+\alpha+1}\}dt - \delta\int_{s_1}^{t_1}\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}\, |H(t)|^{n+\alpha+1}\, dt.
\end{aligned}$$

Thus

$$\begin{aligned}
&A_{s_1}^{t_1}(\mu q, n+\alpha+1)\\
&\quad\leqslant A_{s_1}^{t_1}((n+\alpha+1)u_3(t)H'(t) - \frac{P(t)}{a(t)\eta(t)}H, n+\alpha) - A_{s_1}^{t_1}(\delta\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}\, |H(t)|^{\alpha+1}, n+\alpha+1)\\
&\quad\leqslant A_{s_1}^{t_1}(H^{\alpha}u_3(t)G_1 - \delta\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}\, |H(t)|^{\alpha+1}, n).
\end{aligned}$$

Now setting

$$F(u_3) = H^{\alpha}(t)u_3(t)G_1(t) - \delta\frac{u_3^{1+\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}\, |H(t)|^{\alpha+1},\ u_3>0,$$

then

$$F'(u_3) = H^{\alpha}(t)G_1(t) - (\delta+1)\frac{u_3^{\frac{1}{\delta}}(t)}{[a(t)\eta(t)]^{\frac{1}{\delta}}}\, |H(t)|^{\alpha+1}.$$

Let the maximum of $F\ (u_3)$ occurs at u_3^*, then

$$u_3^*(t) = \frac{a(t)\eta(t)}{(\delta+1)^{\delta}}[\frac{G_1(t)}{H(t)}]^{\delta}.$$

Thus

$$F(u_3) \leqslant F_{max} = \frac{G_1^{\delta+1}(t)a(t)\eta(t)}{(\delta+1)^{\delta}}.$$

So,

$$A_{s_i}^{t_i}(\mu q(t), n+\alpha+1) \leqslant\ A_{s_i}^{t_i}(\delta_2 a(t)\eta(t)G\ _1^{\delta+1}(t)H\ ^{\alpha-\delta}(t), n),$$

which contradicts (3.8) for $i=1$. Similarly, if $y(t)<0$ on $[T_0,\infty)$ for some large $T_0 \geqslant t_0$ and on the interval $[s_2, t_2]$. Then we get a contradiction with (3.8). This completes the proof. □

Remark 3.3. In Theorems 3.1 and 3.2, we have to use a restriction on the sign of the damping term $P(t)$. For those who did not impose any restriction on the sign of the damping term, see [9, 17, 19, 21].

Now, we discuss the non-oscillation property of the following forced equation

$$[a(t)\Omega(y(t)(y'(t))^{\delta}]' + q(t)\rho(t)) = e(t),\ \text{for } t\geqslant t_0>0. \tag{3.9}$$

Assume that

(B_1) $a(t),\ q(t), e(t) \in C\ ([t_0,\infty),\ R),\ a(t) > 0, \Omega(y(t)) > 0,\ \rho(t)), \Omega(y(t)) \in (R, R)$;
(B_2) $y\rho(y(t)) > 0$, for $x \neq 0$, and there exists $k > 0$ such that for any $|y_2| \geqslant |y_1| > 0, |\rho_1(t))| \leqslant k|\rho(y_2(t))|$.

Theorem 3.4. *Suppose that the assumptions* (B_1) *and* (B_2) *hold. Then the conditions*

$$\int_t^\infty |e(s)|\, ds = \infty \ \text{and} \ \int_t^\infty \rho(u(s))q(s)ds < \infty, \tag{3.10}$$

are sufficient for any bounded solution $y(t)$ *of Eq.* (3.9) *to be non-oscillatory.*

Proof. Suppose the contrary that $y(t)$ is oscillatory. Then there exists a sufficiently large $t_1 > t_0$ such that $y'(t_1) > 0$ and a sequence $\{t_n\}, t_n \to \infty, t_n > t_1$ with $y'(t_n) = 0$. By integrating Eq. (3.9) from $t_1 \to t_n$, it follows that

$$\int_{t_1}^{t_n} [a(s)\Omega(y(s)(y'(s))^\delta]' ds \ = \int_{t_1}^{t_n} e(s)ds \ - \int_{t_1}^{t_n} q(s)\rho(s)ds,$$

i.e.,

$$-a(t_1)\Omega(y(t_1)(y'(t_1))^\delta = \int_{t_1}^{t_n} e(s)ds \ - \int_{t_1}^{t_n} q(s)\rho(s)ds.$$

This means that

$$\int_{t_1}^{t_n} e(s)ds < \int_{t_1}^{t_n} q(s)\rho(s)ds < \infty,$$

which is a contradiction with the condition (3.10), then any bounded solution of Eq. (3.9) is non-oscillatory. □

Remark 3.5. Theorem 3.3 includes Theorem 5.2 of [3] in the special case $\delta = 1$.

4. Examples

In the first example, we show that the restriction $P(t) < 0$ of [1] is not necessary.

Example 4.1. Consider the differential equation

$$[t\Omega(y(t))y'(t)]' + \frac{1}{t}y'(t) + \frac{1}{t^{\,2}}\rho(t) = 0. \tag{4.1}$$

Taking the function $\Omega(y(t))$ such that $\frac{1}{t^3} \leqslant \Omega(y(t)) \leqslant \frac{1}{t}$, choose

$$g(t) = t.$$

Now, applying the conditions of Theorem 2.1, we get

$$\int_{t_0}^t \frac{ds}{\eta(s)a(s)g(s)} = \int_{t_0}^t \frac{ds}{s} = \infty$$

and

$$\limsup_{t\to\infty}\{\frac{a(t)g'(t)\xi(t) - P(t)g(t)}{2} + \int_{t_0}^t \mu g(s)q(s) - \frac{[a(s)g'(s)\eta(s) - P(s)g(s)]^2}{4\ a(s)g(s)\eta(s)}ds\}$$
$$= \limsup_{t\to\infty}\{\frac{1}{2t^{\,2}} - \frac{1}{2} + \int_{t_0}^t \frac{\mu ds}{s}\} = \infty.$$

Then the conditions of Theorem 2.1 hold and so Eq. (4.1) is oscillatory.

Example 4.2. Consider the differential equation

$$(5e^{t}\Omega(y(t))y')' - 10e^{3t}y'(t) + e^{2t}\rho(t) = 0, t \geqslant 0, \tag{4.2}$$

for

$$e^{2t} \leqslant \Omega(y(t)) \leqslant 2e^{2t}.$$

Taking

$$g(t) = \frac{e^{-t}}{5},$$

it is easy to see that the condition (2.1),

$$\int_{t_0}^{\infty} \frac{ds}{\eta(s)a(s)g(s)} = \int_{0}^{\infty} \frac{ds}{2e^{2s}} = \frac{1}{4} \neq \infty,$$

so we cannot apply Theorem 2.1. Now, applying Theorem 2.3, we obtain

$$\int_{0}^{\infty} [\int_{t_0}^{s} g(\tau)a(\tau)\eta(\tau)d\tau]^{-1}ds = \int_{0}^{\infty} \frac{ds}{-1+e^{2s}} = \infty,$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} \{\frac{a(s)g'(s)\xi(s) - P(s)g(s)}{2} + \int_{t_0}^{t} \mu g(s)q(s) - \frac{[a(s)g'(s)\eta(s) - P(s)g(s)]^2}{4\ a(s)g(s)\eta(s)}ds\}ds$$
$$= \limsup_{t\to\infty} \frac{1}{t}\int_{0}^{t} \frac{(\mu+1)e^{2s} - \mu}{2} = \infty.$$

So Eq. (4.2) is oscillatory.

Example 4.3. Consider the differential equation

$$[\frac{1}{t}\Omega(y(t))(y')^{\delta}(t)]' + \frac{1}{t^{\ 2}}(y'(t))^{\delta} + Q(t)y^{\delta} = 0,\ \delta > 1 \text{ and } t \geqslant 1, \tag{4.3}$$

where

$$Q(t) = \frac{1}{t} + [\frac{(\delta+1)^{-\delta}}{t^{\ \delta+1}}] + \frac{\delta}{t^{\ 2}(\delta+1)^{\delta}},$$

for any function $\Omega(y(t))$ that satisfies $\frac{y(t)}{t^{\delta}} \leqslant \Omega(y(t)) \leqslant \frac{y(t)}{t}$ and $H(t,s) = (t-s)^2$.

Choosing $\rho(t) = (\frac{\frac{1}{t}}{\delta+1})^{\delta}$, then $h_2 = 2(t-s)^{\frac{1-\delta}{1+\delta}}, \nu_1 = 1$ and $\Theta_1(t) = \frac{1}{t}$ for $\beta \geqslant 1$. Now since

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\Theta_1(s) - \frac{\beta^{\delta}}{(\delta+1)^{\delta+1}}a(s)\eta(s)\nu_1(s)h_2^{\delta+1}(t,s)ds$$
$$= \limsup_{t\to\infty} \frac{1}{t^2}\int_{1}^{t} \frac{(t-s)2}{s} - \frac{2^{\delta+1}\beta^{\delta}}{(\delta+1)^{\delta+1}}[\frac{1}{t^{\ 2}}(t-s)^{1-\delta}]ds = \infty,$$

then the conditions of Theorem 2.4 hold and so Eq. (4.3) is oscillatory.

Example 4.4. Consider the forced differential equation

$$[t^{3\lambda+1}(y')^{\delta}]' + t^{3\lambda}(y')^{\delta} + Nt^{3\lambda}y^{\delta} = \sin t,\ \text{for } t \geqslant t_0 > 1, \tag{4.4}$$

where N and $\lambda > 0$. Comparing with Eq. (1.2), it is clear that: $a(t) = t^{3\lambda+1}$, $\Omega(y(t)) = 1$, $P(t) = t^{3\lambda}$, $\rho(y(t)) = t^{3\lambda}$, and $e(t) = \sin t$. Choosing $s_1 = n\pi, t_1 = (n+1)\pi, s_2 = (n+1)\pi$ and $t_2 = (n+2)\pi$, for $n \geqslant 1$, it is easy to verify that

(C_1) $\frac{\rho(y(t))}{y^{\delta}} = \frac{y^{\delta}(t)}{y^{\delta}(t)} \geqslant \mu = 1$, for $y(t) \neq 0$, and all the conditions (C_2)-(C_4) hold. For $\alpha = n = \delta = 1$, then $\delta_2 = \frac{1}{2}$. Suppose that $H(t) = t^{-\lambda}\sin^2(t)$, then

$$A_{s_i}^{t_i}(\mu q(t), n+\alpha+1) = A_{s_i}^{t_i}(q(t),3) = \int_{s_i}^{t_i} |H(t)|^3 q(t)dt = N\int_{\pi}^{2\pi}\sin^6(t)dt = \frac{5\pi}{16}N, \text{ for } i = 1,\ 2,$$

and

$$\begin{aligned} A_{s_1}^{t_1}(\delta_2 a(t)\eta(t)G_1^{\delta+1}(t)H^{\ \alpha-\delta}, n) &= A_{s_2}^{t_2}(\delta_2 a(t)\eta(t)G_1^{\delta+1}(t)H^{\ \alpha-\delta}, n) \\ &= \frac{1}{2}\int_{\pi}^{2\pi}(t^{3\lambda+1}G_1^2(t).[t^{-\lambda}\sin^2(t)])dt = \frac{1}{2}\int_{\pi}^{2\pi} t^{2\lambda+1}\sin^2(t)G_1^2(t)dt. \end{aligned}$$

Since

$$\begin{aligned} G_1(t) = \left|(n+\alpha+1)H'(t) - \frac{P(t)}{a(t)\eta(t)}H(t)\right| &= \left|3[2t^{-\lambda}\sin t\cos t - \lambda t^{-1}\sin^2(t)] - t^{-\lambda-1}\sin^2(t)\right| \\ &= \left|6t^{-\lambda}\sin t\cos t + (-3\lambda-1)t^{-\lambda-1}\sin^2(t)\right|, \end{aligned}$$

then

$$\begin{aligned} &A_{s_1}^{t_1}(\delta_2 a(t)\eta(t)G_1^{\delta+1}(t)H^{\ \alpha-\delta}, n) \\ &\quad = \frac{1}{2}\int_{\pi}^{2\pi} t^{2\lambda+1}\sin^2(t)[36t^{-2\lambda}\sin^2(t)\cos^2(t) + 12(-3\lambda-1)t^{-2\lambda-1}\sin^3(t)\cos(t) \\ &\qquad + (-3\lambda-1)^2t^{-2\lambda-2}\sin^4(t)]dt \\ &\quad \leqslant \frac{1}{2}\int_{\pi}^{2\pi}[36t - 12(3\lambda+1)\sin^5(t)\cos(t) + (3\lambda+1)^2t^{-1}]dt = \frac{27}{2}\Pi^2 + (3\lambda+1)^2\ln 2. \end{aligned}$$

So, the condition (3.8) holds for

$$\frac{5\pi}{16}N > \frac{27}{2}\Pi^2 + (3\lambda+1)^2\ln 2.$$

Thus Eq. (4.4) is oscillatory if $N > [\frac{216}{5}\Pi + \frac{16}{5\pi}(3\lambda+1)^2\ln 2]$ according to Theorem 3.2.

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