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Properties of $(3,2)$-fuzzy subalgebras/ideals of subtraction algebras

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Abstract

In this paper, we introduce and study the notions of a $(3,2)$-fuzzy subalgebra and a $(3,2)$-fuzzy ideal of a subtraction algebra in fuzzy setting.

Keywords: $(3,2)$-fuzzy set, $(3,2)$-fuzzy subalgebra, $(3,2)$-fuzzy ideal.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [13]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 3, 4]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision making [5–7]. Yager [12] offered a new fuzzy set called a Pythgorean fuzzy set, which is the generalization of intuitionistic fuzzy sets. Fermatean fuzzy sets were introduced by Senapati and Yager [11], and they also defined basic operations over the Fermatean fuzzy sets. The concept of $(3,2)$-fuzzy sets are introduced and studied in [8]. In this paper, we introduce and study the notions of a $(3,2)$-fuzzy subalgebra and a $(3,2)$-fuzzy ideal of a subtraction algebra in fuzzy setting.

2. Preliminaries

Definition 2.1. An algebra $(X,-)$ with a single binary operation $-$ is called a subtraction algebra if for all $x,y,z \in X$ the following conditions hold:

1. $x-(y-x)=x$,
2. $x-(x-y)=y-(y-x)$,

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3. $(x-y)-z=(x-z)-y$.

The subtraction determines an order relation on $X: a \leqslant b \Rightarrow a-b=0$, where $0=a-a$ is an element that does not depend on the choice of $a \in X$. In a subtraction algebra, the following are true [9, 10]:

1. $(x-y)-y=x-y$,
2. $x-0=x$ and $0-x=0$,
3. $(x-y)-x=0$,
4. $(x-y)-(y-x)=x-y$,
5. $x \leqslant y \Rightarrow x-z \leqslant y-z$ and $z-y \leqslant z-x$ for all $z \in X$.

Definition 2.2 ([8])**.** Let X be a nonempty set. The $(3,2)$-*fuzzy set* on X is defined to be a structure

$$\mathcal{C}_X := \{\langle x, f(x), g(x)\rangle \mid x \in X\}, \tag{2.1}$$

where $f: X \to [0,1]$ is the degree of membership of x to $\mathcal{C}$ and $g: X \to [0,1]$ is the degree of non-membership of x to $\mathcal{C}$ such that $0 \leqslant (f(x))^3 + (g(x))^2 \leqslant 1$.

Definition 2.3 ([9, 10])**.** A non-empty subset A of a subtraction algebra X is called a subalgebra of X if $x-y \in A$ for any $x, y \in A$.

Definition 2.4 ([9, 10])**.** A non-empty subset I of a subtraction algebra X is called an ideal of X if

1. $0 \in I$,
2. $(\forall x, y \in X)(x-y \in I, y \in I \to x \in I)$.

3. On $(3,2)$-fuzzy ideals of subtraction algebras

Definition 3.1. A $(3,2)$-fuzzy set A in X is called a $(3,2)$-fuzzy subalgebra of X if it satisfies:

$$(\forall\, x, y \in X)\left(\begin{array}{l} \mu_A^3(x-y) \geqslant \min\{\mu_A(x), \mu_A(y)\} \\ \nu_A^2(x-y) \leqslant \max\{\nu_A^2(x), \nu_A^2(y)\} \end{array}\right). \tag{3.1}$$

Proposition 3.2. *Every* $(3,2)$-*fuzzy subalgebra of* X *satisfies*

$$(\forall\, x, \in X)\left(\begin{array}{l} \mu_A^3(0) \geqslant \mu_A^3(x) \\ \nu_A^2(0) \leqslant \nu_A^2(x) \end{array}\right). \tag{3.2}$$

Proof. Straightforward. □

Example 3.3. Let $X=\{0,1,2,3\}$ be a subtraction algebra with the table 1:

Table 1

-	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	1
3	3	2	1	0

Define a $(3,2)$-fuzzy set A in X as follows:

$$\mu_A : X \to [0,1], \quad x \to \begin{cases} 0.54 & \text{if } x \in \{0,3\} \\ 0.13 & \text{if } x \in \{1,2\}, \end{cases}$$

and

$$\nu_A : X \to [0,1], \quad x \to \begin{cases} 0.11 & \text{if } x \in \{0,3\} \\ 0.53 & \text{if } x \in \{1,2\}. \end{cases}$$

It is easy to check that A is a $(3,2)$-fuzzy subalgebra of X.

Theorem 3.4. *Let A be a $(3,2)$-fuzzy set in X and let $\alpha,\beta \in [0,1]$ with $0 \leqslant \alpha+\beta \leqslant 2$. Then A is a $(3,2)$-fuzzy subalgebra of X if and only if all of (α,β)-level set $A^{(\alpha,\beta)} = \{x \in X : \mu^3(x) \geqslant \alpha, \nu^2(x) \leqslant \beta\}$ are subalgebras of X when $A^{(\alpha,\beta)} \neq \emptyset$.*

Proof. Assume that A is a $(3,2)$-fuzzy subalgebra of X. Let $\alpha,\beta \in [0,1]$ be such that $0 \leqslant \alpha+\beta \leqslant 2$ and $A^{(\alpha,\beta)} \neq \emptyset$. Let $x,y \in A^{(\alpha,\beta)}$. Then $\mu_A^3(x) \geqslant \alpha, \mu_A^3(y) \geqslant \alpha$ and $\nu_A^2(x) \leqslant \beta$, $\nu_A^2(y) \leqslant \beta$. Using (3.1), we have $\mu_A^3(x-y) \geqslant \min\{\mu_A^3(x),\mu_A^3(y)\} \geqslant \alpha$ and $\nu_A^2(x-y) \leqslant \max\{\nu_A^2(x),\nu_A^2(y)\} \leqslant \beta$. Hence $x-y \in A^{(\alpha,\beta)}$. Therefore $A^{(\alpha,\beta)}$ is a subalgebra of X. Conversely, all of (α,β)-level set $A^{(\alpha,\beta)}$ are subalgebras of X when $A^{(\alpha,\beta)} \neq \emptyset$. Assume that there exist $a_m, b_m \in X$ and $a_n, b_n \in X$ such that $\mu_A^3(a_m - b_m) < \min\{\mu_A^3(a_m),\mu_A^3(b_m)\}$ and $\nu_A^2(a_n-b_n) > \max\{\nu_A^2(a_n),\nu_A^2(b_n)\}$. Then $\mu_A^3(a_m-b_m) < \alpha_1 \leqslant \min\{\mu_A^3(a_m),\mu_A^3(b_m)\}$ and $\nu_A^2(a_n-b_n) > \beta_1 \geqslant \max\{\nu_A^2(a_n),\nu_A^2(b_n)\}$ for some $\alpha_1 \in (0,1]$ and $\beta_1 \in [0,1)$. Hence $a_m, b_m \in A^{(\alpha_1,\beta_1)}$, and $a_n, b_n \in A^{(\alpha_1,\beta_1)}$. But $a_m - b_m \notin A^{(\alpha_1,\beta_1)}$ and $a_n - b_n \notin A^{(\alpha_1,\beta_1)}$, which is a contradiction. Hence $\mu_A^3(x-y) \geqslant \min\{\mu_A^3(x),\mu_A^3(y)\}$, and $\nu_A^2(x-y) \leqslant \max\{\nu_A^2(x),\nu_A^2(y)\}$ for any $x,y \in X$. Therefore A is a $(3,2)$-fuzzy subalgebra of X. □

Theorem 3.5. *If $\{A_i : i \in \mathbb{N}\}$ is a family of $(3,2)$-fuzzy subalgebras of X, then $(\{A_i : i \in \mathbb{N}\}, \subset)$ forms a complete distributive lattice.*

Proof. The proof follows from the fact that $[0,1]$ is a completely distributive lattice with respect to the usual ordering. □

Theorem 3.6. *Let A be a $(3,2)$-fuzzy subalgebra of X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} \mu_A^3(a_n) = 1$ and $\lim_{n\to\infty} \nu_A^2(a_n) = 0$, then $\mu_A^3(0) = 1$ and $\nu_A^2(0) = 0$.*

Proof. By Proposition 3.2, we have $\mu_A^3(0) \geqslant \mu_A^3(x)$ and $\nu_A^2(0) \leqslant \nu_A^2(x)$ for all $x \in X$. Hence we have $\mu_A^3(0) \geqslant \mu_A^3(a_n)$ and $\nu_A^2(0) \leqslant \nu_A^2(a_n)$ for every positive integer n. Therefore $1 = \lim_{n\to\infty} \mu_A^3(a_n) \leqslant \mu_A^3(0) \leqslant 1$ and $0 \leqslant \nu_A^2(0) \leqslant \lim_{n\to\infty} \nu_A^2(a_n) = 0$. Thus we have $\mu_A^3(0) = 1$ and $\nu_A^2(0) = 0$. □

Definition 3.7. A $(3,2)$-fuzzy set A in X is called a $(3,2)$-fuzzy ideal of X if it satisfies (3.2) and

$$(\forall\, x,y \in X) \begin{pmatrix} \mu_A^3(x) \geqslant \min\{\mu_A^3(x-y),\mu_A^3(y)\} \\ \nu_A^2(x) \leqslant \max\{\nu_A^2(x-y),\nu_A^2(y)\} \end{pmatrix}. \tag{3.3}$$

Proposition 3.8. *Every $(3,2)$-fuzzy ideal of X is a $(3,2)$-fuzzy subalgebra of X.*

Proof. Let A be a $(3,2)$-fuzzy ideal of X. Put $x = x-y$ and $y = x$ in (3.3). Then we have $\mu_A^3(x-y) \geqslant \min\{\mu_A^3((x-y)-x),\mu_A^3(x)\}$ and $\nu_A^2(x-y) \leqslant \max\{\nu_A^2((x-y)-x),\nu_A^2(x)\}$. It follows from (3.2) that

$$\begin{aligned} \mu_A^3(x-y) &\geqslant \min\{\mu_A^3((x-x)-y),\mu_A^3(x)\} \\ &= \min\{\mu_A^3(0),\mu_A^3(x)\} \\ &\geqslant \min\{\mu_A^3(x),\mu_A^3(y)\}, \end{aligned}$$

and

$$\begin{aligned}\nu_A^2(x-y) &\leqslant \max\{\nu_A^2((x-y)-x), \nu_A^2(x)\}\\ &= \max\{\nu_A^2(0), \nu_A^2(x)\}\\ &\leqslant \max\{\nu_A^2(x), \nu_A^2(y)\},\end{aligned}$$

for any $x, y \in X$. Thus A is a $(3, 2)$-fuzzy subalgebra of X. □

The following Example shows that the converse of Proposition 3.8 is not true in general.

Example 3.9.
(a). Let $X = \{0, 1, 2, 3\}$ be a subtraction algebra with the table 2:

Table 2

-	0	1	2	3
0	0	0	0	0
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Define a $(3, 2)$-fuzzy set A in X as follows:

$$\mu_A : X \to [0,1],\quad x \to \begin{cases} 0.72 & \text{if } x \in \{0,1\}\\ 0.11 & \text{if } x \in \{2,3\},\end{cases}$$

and

$$\nu_A : X \to [0,1],\quad x \to \begin{cases} 0.13 & \text{if } x \in \{0,1\}\\ 0.71 & \text{if } x \in \{2,3\}.\end{cases}$$

It is easy to check that A is a $(3, 2)$-fuzzy ideal of X.

(b). Let $X = \{0, 1, 2, 3\}$ be a subtraction algebra as in Example 3.3. Define a $(3, 2)$-fuzzy set B in X as follows:

$$\mu_A : X \to [0,1],\quad x \to \begin{cases} 0.53 & \text{if } x = 0\\ 0.22 & \text{if } x \in \{1,2\},\\ 0.13 & \text{if } x = 3,\end{cases}$$

and

$$\nu_A : X \to [0,1],\quad x \to \begin{cases} 0.11 & \text{if } x = 0\\ 0.25 & \text{if } x \in \{1,2\},\\ 0.46 & \text{if } x = 3.\end{cases}$$

It is easy to check that B is a $(3, 2)$-fuzzy subalgebra of X. But it is not a $(3, 2)$-fuzzy ideal of X, since $\mu_B^3(3) = 0.13^3 \ngeq \min\{\mu_B^3(3-1), \mu_B^3(1) = \max\{\mu_B^3(2), \mu_B^3(1)\} = 0.22^3$.

Proposition 3.10. *Every $(3, 2)$-fuzzy ideal A of X satisfies the following:*

$$(\forall\, x, y \in X)\left(\ x \leqslant y \ \Rightarrow \begin{cases} \mu_A^3(x) \geqslant \mu_A^3(y)\\ \nu_A^2(x) \leqslant \nu_A^2(y)\end{cases}\right). \tag{3.4}$$

$$(\forall\, x, y, z \in X)\left(\ x - y \leqslant z \ \Rightarrow \begin{cases} \mu_A^3(x) \geqslant \min\{\mu_A^3(y), \mu_A^3(z)\}\\ \nu_A^2(x) \leqslant \max\{\nu_A^2(y), \nu_A^2(z)\}\end{cases}\right). \tag{3.5}$$

Proof.

(1). Let $x, y \in X$ be such that $x \leqslant y$. Then $x - y = 0$. Using (3.2) and (3.3), we have

$$\mu_A^3(x) \geqslant \min\{\mu_A^3(x-y), \mu_A^3(y)\} = \min\{\mu_A^3(0), \mu_A^3(y)\} = \mu_A^3(y)$$

and

$$\nu_A^2(x) \leqslant \max\{\nu_A^2(x-y), \nu_A^2(y)\} = \max\{\nu_A^2(0), \nu_A^2(y)\} = \nu_A^2(y).$$

(2). Let $x, y, z \in X$ be such that $x - y \leqslant z$. By (3.3) and (3.2), we get

$$\begin{aligned} \mu_A^3(x-y) &\geqslant \min\{\mu_A^3((x-y)-z), \mu_A^3(z)\} \\ &= \min\{\mu_A^3(0), \mu_A^3(z)\} \\ &= \mu_A^3(z), \end{aligned}$$

$$\begin{aligned} \nu_A^2(x-y) &\leqslant \max\{\nu_A^2((x-y)-z), \nu_A^2(z)\} \\ &= \max\{\nu_A^2(0), \nu_A^2(z)\} \\ &= \nu_A^2(z). \end{aligned}$$

Hence

$$\mu_A^3(x) \geqslant \min\{\mu_A^3(x-y), \mu_A^3(y)\} \geqslant \min\{\mu_A^3(y), \mu_A^3(z)\},$$

and

$$\nu_A^2(x) \leqslant \max\{\nu_A^2(x-y), \nu_A^2(y)\} \leqslant \max\{\nu_A^2(y), \nu_A^2(z)\}$$

for any $x, y, z \in X$. □

Definition 3.11. Let A and B be $(3, 2)$-fuzzy sets of a set X. The union of A and B is defined to be a $(3, 2)$-fuzzy set $A \cup B = \{<x, \mu_{A\cup B}(x), \nu_{A\cup B}(x)> : x \in X\}$, where $\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \nu_{A\cup B}(x) = \min\{\nu_A(x), \nu_B(x)\}$ for all $x \in X$. The intersection of A and B is defined to be a $(3, 2)$-fuzzy set $A \cap B = \{<x, \mu_{A\cap B}(x), \nu_{A\cap B}(x)> : x \in X\}$, where $\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \nu_{A\cap B}(x) = \max\{\nu_A(x), \nu_B(x)\}$ for all $x \in X$.

Theorem 3.12. *The intersection of two $(3, 2)$-fuzzy ideals of X is also a $(3, 2)$-fuzzy ideal of X.*

Proof. Let A and B be $(3, 2)$-fuzzy ideals of X. For any $x \in X$, we have $\mu_{A\cap B}^3(0) = \min\{\mu_A^3(0), \mu_B^3(0)\} \geqslant \min\{\mu_A^3(x), \mu_B^3(x)\} = \mu_{A\cap B}^3(x)$ and $\nu_{A\cap B}^2(0) = \max\{\nu_A^2(0), \nu_B^2(0)\} \leqslant \max\{\nu_A^2(x), \nu_B^2(x)\} = \nu_{A\cap B}^2(x)$. Let $x, y \in X$. Then we have

$$\begin{aligned} \mu_{A\cap B}^3(x) &= \min\{\mu_A^3(x), \mu_B^3(x)\} \\ &\geqslant \min\{\min\{\mu_A^3(x-y), \mu_A^3(y)\}, \min\{\mu^3(x-y), \mu_B^3(y)\}\} \\ &= \min\{\min\{\mu_A^3(x-y), \mu^3(x-y)\}, \min\{\mu_A^3(y), \mu^3 y)\}\} \\ &= \min\{\mu_{A\cap B}^3(x-y), \mu_{A\cap B}^3(y)\} \end{aligned}$$

and

$$\begin{aligned} \nu_{A\cap B}^2(x) &= \max\{\nu_A^2(x), \nu_B^2(x)\} \\ &\leqslant \max\{\max\{\nu_A^2(x-y), \nu_A^2(y)\}, \max\{\nu_B^2(x-y), \nu_B^2(y)\}\} \\ &= \max\{\max\{\nu_A^2(x-y), \nu_B^2(x-y)\}, \max\{\nu_A^2(y), \nu_B^2(y)\}\} \\ &= \max\{\nu_{A\cap B}^2(x-y), \nu_{A\cap B}^2(y)\}. \end{aligned}$$

Hence $A \cap B$ is a $(3, 2)$-fuzzy ideal of X. □

Proposition 3.13. *Let* A *be a* $(3,2)$*-fuzzy ideal of* X*. Then* $X_\mu = \{x \in X : \mu_A^3(x) = \mu_A^3(0)\}$ *and* $X_\nu = \{x \in X : \nu_A^2(x) = \nu_A^2(0)\}$ *are ideals of* X*.*

Proof. Clearly, $0 \in X_\mu$. Let $x - y, y \in X_\mu$. Then $\mu_A^3(x-y) = \mu_A^3(0)$ and $\mu_A^3(y) = \mu_A^3(0)$. It follows from (3.3) that $\mu_A^3(x) \geqslant \min\{\mu_A^3(x-y), \mu_A^3(y)\} = \mu_A^3(0)$. By (3.2), we get $\mu_A^3(x) = \mu_A^3(0)$. Hence $x \in X_\mu$. Therefore X_μ is an ideal of X. By a similar way, X_ν is an ideal of X. □

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