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# More properties of δβ-rough continuous functions on topological approximation spaces

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## Abstract

Using the notion of δβ-open set, we intend to do more research on rough continuous functions. The ideas of δβ-totally rough continuous functions and δβ-strongly rough continuous functions are proposed and researched. The notions of δβ-internally and δβ-totally functions are discussed, as well as some of their characterizations. Finally, the composition of δβ-internally and totally functions is discussed.

**Keywords:** Topological approximation spaces, δβ-open sets, totally rough continuous functions, strongly rough continuous functions.

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## 1. Introduction

Several recent papers [5, 12] have discussed near-openness extensions and the opportunity of spending them in a variety of presentations, counting data decreasing and novel judgments and conclusions. Rough set theory [18–20, 35] is a relatively new approach to data reasoning. This idea is based on a specific space that has had a lot of success in real-world applications [1, 5, 23, 24]. Rough sets theory may now be studied by general topologists as a topological road from real-world initiatives to computer science [29, 34].

The basics of rough sets was proposed as an original method to incomplete data dispensation. One of rough set theory's goals is to describe imprecise concepts. Assume we're given a universe $U$, which is a finite non-empty collection of items. A description, such as a list of attributes, is attached to each of $U$'s objects. Pawlak's rough sets defined a universal set's equivalence relation based on their properties. The equality relation on the attributes is specifically used to start this relationship. Both nominal and continuous belongings are present in many real-world applications [4, 38]. The normal roughness model built on the indiscernibility relative was recognized early on to be well suited to the problem of nominal characteristics.

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Other ways were devised to solve the methodology's flaws [28], and several writers obtainable fascinating postponements of the fundamental concept (for instance [2, 3, 15, 17, 22]). It was observed that analyzing a similarity relationship rather than an indiscernibility relationship is crucial. Any relation that produces courses of substances that are undistinguishable or not significantly dissimilar in relationships of the supplied explanation can be used to indicate similarities between things [21, 31–33, 36, 37]. [13, 14, 25, 26, 30] explore additional presentations of rough sets in mainframe (in the ground of material repossessions) that used topological simplifications [6–11, 16, 27]. There are many ideas, applications and topological generalizations that can be used in future work on this paper.

Here, we want to perform additional research on rough continuity and propose new rough function notions. Section 2 goes through the principles of near open sets, specifically $\delta\beta$-open sets. Section 3 introduces and explains the ideas of $\delta\beta$-totally rough continuous functions. Section 4 focuses on $\delta\beta$-exactly rough continuous functions. Section 5's goal is to define and investigate $\delta\beta$-internally functions. Finally, Section 6 provides the paper's findings and future work ideas.

## 2. Some fundamentals of near open sets

This section reviews the meanings of numerous near open sets with a topological structure, which will be important in the following section. Both arbitrary union and finite intersection are closed operations for $\tau\subseteq P(U)$ that contains $\varphi$, $U$. $\tau$ is referred to as a topology on $U$. For every subset $A\subseteq U$, $\overline{A}$, $A^{o}$, and $A^{c}$ indicate "Closure", "Interior", and "Complement" of $A$ in $U$, correspondingly.

**Definition 2.1.** If $A$ is a subset of $(U,\tau)$, then $A$ is referred to as following.

(i) If $A\subseteq\overline{(A^{\circ})}$ (resp., $A\subseteq(\overline{A})^{\circ}$, $A\subseteq\left(\overline{(A^{\circ})}\right)^{\circ}$), then $A$ is a semi-open (resp., pre-open, $\alpha$-open), and its accompaniment is a semi-closed (resp., pre-closed, $\alpha$-closed) if $(\overline{A})^{\circ}\subseteq A$ (resp., $\overline{(A^{\circ})}\subseteq A$, $\overline{\left((\overline{A})^{\circ}\right)}\subseteq A$). A subset that is together semi-open and semi-closed is known to as semi-regular.

(ii) If $A\subseteq\overline{\left((\overline{A})^{\circ}\right)}$, then $A$ is called a semi-pre-open set (or $\beta$-open set), and if $\left(\overline{(A^{\circ})}\right)^{\circ}\subseteq A$, it is named a semi-pre-closed set (or $\beta$-closed set).

(iii) If $A\subseteq(\overline{A})^{\circ}$, it is called a regular-open set, and if $\overline{(A^{\circ})}=A$, it is called a regular-closed set.

(iv) $A$ is called $\delta$-closed set if $A=(\overline{\delta(A)})$ and $(\overline{\delta(A)})=\left\{x\in U:(\overline{G})^{\circ}\cap A\neq\varphi, x\in G, G\in\tau\right\}$.

The connection of all $\alpha$-closed (resp. semi-closed, semi-pre-closed) sets that cover $A$ is named the $\alpha$-closure (resp. semi-closure, semi-pre-closure) and represented by $\alpha(\overline{A})$ (resp. $S(\overline{A})$, $sp(\overline{A})$) of a subset $A$ of $(U,\tau)$. The union of all semi-open subsets of $U$ is the semi-interior of $A$, meant by $s(A^{\circ})$.

**Definition 2.2.** Let $(X,\o)$ be a topological space and $A$ be a subset of it, then $A$ is as follows.

(i) Generalized closed set if $\overline{A}\subseteq G$ whenever $A\subseteq G$ and $G\in\tau$ are both true.

(ii) Semi-generalized closed (sg-closed) set if $s\left(\overline{A}\right)\subseteq G$ whenever $A\subseteq G$ and $G$ is semi-open set. The sg-open set is its complement.

(iii) A semi-closed generalized set if $s(\overline{A})\subseteq G$ whenever $A\subseteq G$ and $G\in\o$.

(iv) $\alpha$-generalized closed set if $s(\overline{A})\subseteq G$ whenever $A\subseteq G$ and $G\in\o$.

(v) If $\alpha(\overline{A})\subseteq G$ whenever $A\subseteq G$ and $G$ is ff-open, we have a generalized $\alpha$-closed set.

(vi) $g\alpha^{**}$-closed if $\overline{A}\subseteq\overline{G^{o}}$ whenever $A\subseteq G$ and $G$ is both $\alpha$-open.

**Definition 2.3** ([28]). The $\delta$-closure of $A$ is defined by $cl_{\delta}(A)=\{x\in X:A\cap(\overline{U})^{\circ}\neq\varphi, U\in\tau \text{ and } x\in U\}$ for any subset $A$ of a topological space $(X,\tau)$. If $A=cl_{\delta}(A)$, a set $A$ is said to be $\delta$-closed. A $\delta$-closed set's accompaniment is a $\delta$-open set.

It's worth noting that $int_{\delta}(A)=[cl_{\delta}(A^{c})]^{c}$.

**Definition 2.4** ([34])**.** If $A\subseteq\overline{(cl_{\delta}\ (A)^{\circ})}$, a subset $A$ of a topological space $(X, \tau)$ is called $\delta\beta$-open.

$\delta\beta O(X)$ stands for the family of all $\delta\beta$-open sets of $X$. The $\delta\beta$-closed set is the complement of the $\delta\beta$-open set. $\delta\beta C(X)$ stands for the family of all $\delta\beta$-closed sets of $X$.

**Definition 2.5** ([34])**.** If we assume $A$ to be a subset of a topological space $(X, \tau)$, we get following.

(i) The $\delta\beta$-interior of $A$ is the combination of all $\delta\beta$-open sets comprised in $A$, and it is represented by $\delta\beta I(A)$.

(ii) The $\delta\beta$-closure of $A$ is the connection of all $\delta\beta$-closed sets covering $A$, and it is represented by $\delta\beta C(A)$.

*Remark* 2.6 ([34]). For any subsection $A$ of a topological space $(X, \tau)$ we have

(i) $\delta\beta I(A) = A\cap\overline{\left((\overline{A})^{\circ}\right)}$;

(ii) $\delta\beta C(A) = A\cup\left(\overline{(A^{\circ})}\ \right)^{\circ}$.

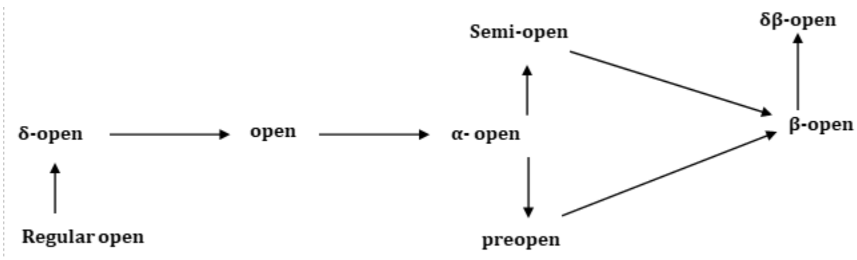


Figure 1: Relationships of some types of near open sets.

*Remark* 2.7. As illustrated in Figure 1, $\delta\beta$-open sets are weaker than any near open sets for example $\delta$-open, regular open, semi-open, $\alpha$-open, pre-open, and $\beta$-open.

**Example 2.8.** Let $(X, \tau)$ be a topological space wherever, $X=\{a, b, c, d, e\}$ and $\tau=\{X, \varphi, \{d\}, \{e\}, \{a, d\}, \{d, e\}, \{a, d, e\}, \{b, c, e\}, \{b, c, d, e\}\}$. We have $\{a, c\}\in\delta\beta O(X)$ but $\{a, c\}\notin\delta O(X)$. Also, $\{b, d, e\}\in\delta\beta O(X)$ but $\{b, d, e\}\notin RO(X)$. $\{a, e\}\in\delta\beta O(X)$ but $\{a, e\}\notin PO(X)$. $\{c\}\in\delta\beta O(X)$ but $\{c\}\notin\beta O(X)$. $\{b\}\in\delta\beta O(X)$ but $\{b\}\notin SO(X)$ and $\{c, d\}\in\delta\beta O\ (X)$ but $\{c, d\}\notin\alpha O(X)$.

*Remark* 2.9 ([2]). The connection of binary $\delta\beta$-open sets may not be a $\delta\beta$-open set, but the arbitrary union of $\delta\beta$-open sets is. As a result, in the space $X$, the domestic of all $\delta\beta$-open sets does not form a topology.

## 3. δβ-totally rough continuous functions

Throughout this section, consider $\mathcal{K}=(X, R_1, \tau_K)$ and $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$ are two approximation spaces that have been topologized. The relation $\underline{R}^{\delta\beta}$ is devoted a relation $R$ w.r.to $\delta\beta$-openness. The goal of this section is to define the term $\delta\beta$-rough continuous and to discuss some of its features.

**Definition 3.1.** The function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is called $\delta\beta$-rough continuous if every internally set in $\mathcal{Q}$ is internally $\delta\beta$-definable set in $\mathcal{K}$, i.e., $f^{-1}(\underline{R_2}N)\subseteq\underline{R_1}^{\delta\beta}f^{-1}(N)$ for every subset $N$ in $\mathcal{Q}$.

**Theorem 3.2.** *For a function* $f:\mathcal{K}\longrightarrow\mathcal{Q}$*, the following are equivalent:*

(i) $f$ *is* $\delta\beta$*-rough continuous;*

(ii) *the opposite copy of each internally-definable set in* $\mathcal{Q}$ *is internally* $\delta\beta$*-definable set in* $\mathcal{K}$*;*

(iii) *the opposite copy of each externally-definable set in* $\mathcal{Q}$ *is externally* $\delta\beta$*-definable set in* $\mathcal{K}$*.*

*Proof.*

(i)$\Longrightarrow$(ii) Let f be $\delta\beta$-rough continuous and let V be an internally $R_2$-definable set in $\mathcal{Q}$. Then $\underline{R_2}V$=V and $f^{-1}(V)$ is a subset of X in $\mathcal{K}$. By (i), we get $f^{-1}(V) = f^{-1}\left(\underline{R_2}V\right) \subseteq \underline{R_1^{\delta\beta}} f^{-1}(V)$. Then $f^{-1}(V) \subseteq \underline{R_1^{\delta\beta}} f^{-1}(V)$. But $\underline{R_1^{\delta\beta}} f^{-1}(V) \subseteq f^{-1}(V)$. Hence $\underline{R_1^{\delta\beta}} f^{-1}(V) = f^{-1}(V)$. Therefore $f^{-1}(V)$ is internally $\delta\beta$-definable set in $\mathcal{K}$.

(ii)$\Longrightarrow$(i) Let V be a subset of Y in $\mathcal{Q}$. Since $\underline{R_2}V \subseteq V$, then $f^{-1}\left(\underline{R_2}V\right) \subseteq f^{-1}(V)$. Since $\underline{R_2}V$ is inside $\delta\beta$-definable set in Q, then by (ii), we get $f^{-1}(\underline{R_2}V)$ is internally $\delta\beta$-definable set in $\mathcal{K}$ limited in $f^{-1}(V)$. Hence $f^{-1}(\underline{R_2}V) \subseteq \underline{R_1^{\delta\beta}} f^{-1}(V)$ since $\underline{R_1^{\delta\beta}} f^{-1}(V)$ is the main internally $R_1$-definable set contained in $f^{-1}(V)$. Thus $f^{-1}(\underline{R_2}V) \subseteq \underline{R_1^{\delta\beta}} f^{-1}(V)$ for every subset V of Y in $\mathcal{Q}$. Consequently f is rough continuous.

(ii)$\Longrightarrow$(iii) Let L be an externally $R_2$-definable set in $\mathcal{Q}$, then we get Y−L is $R_2$-internally definable. Thus by (ii), we have $f^{-1}(Y-L)$ is internally $R_1$-definable set in $\mathcal{K}$. Since $f^{-1}(Y-L)$ then $f^{-1}(Y-L) = X - f^{-1}(L)$ is internally $R_1$-definable set in $\mathcal{K}$. Hence $f^{-1}(L)$ is externally $R_1$-definable set in $\mathcal{K}$.

Likewise we can show (iii)$\Longrightarrow$(ii). □

**Definition 3.3.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is called totally-rough continuous if the inverse image of each internally definable set in $\mathcal{Q}$ is a totally definable set in $\mathcal{K}$, i.e., $f^{-1}\left(\underline{R_2}\, N\right) \subseteq \underline{R_1}^{\delta\beta}(f^{-1}(N))$ for every subset N in $\mathcal{Q}$.

**Definition 3.4.** The function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is named $\delta\beta$-totally rough continuous if the inverse image of each internally $\delta\beta$-definable set in $\mathcal{Q}$ is a totally definable set in $\mathcal{K}$.

**Example 3.5.** Let $\mathcal{K}=(X, R_1, \tau_K)$ and $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$ be two topologized approximation spaces such that $X=\{x_1, x_2, x_3\}$, $Y=\{y_1, y_2, y_3\}$, $R_1=\{(x_1, x_1), (x_2, x_2), (x_2, x_3)\}$ and $R_2=\{(y_1, y_1)\}$. Then, $x_1R_1=\{x_1\}$, $x_2R_1=\{x_2, x_3\}$, $x_3R_1=\varphi$, $S_1=\{\varphi, \{x_1\}, \{x_2, x_3\}\}$, $B_1=\{X, \varphi, \{x_1\}, \{x_2, x_3\}\}$, $\tau_K=\{X, \varphi, \{x_1\}, \{x_2, x_3\}\}=\tau_K^*$, $y_1R_2=\{y_1\}$, $y_2R_2=\varphi$, $y_3R_2=\varphi$, $S_2=\{\varphi, \{y_1\}\}$, $B_2=\{Y, \varphi, \{y_1\}\}$ and $\tau_Q=\{Y, \varphi, \{y_1\}\}$. Hence, $\delta\beta I(Q) = \{Y, \varphi, \{y_1\}, \{y_1, y_2\}, \{y_1, y_3\}\}$. Define a function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ such that $f(x_2) = f(x_3) = y_1$ and $f(x_1) = y_3$. Then, f is $\delta\beta$-totally rough continuous, since the inverse image of all internally $\delta\beta$-definable set in $\mathcal{Q}$ is totally definable set in $\mathcal{K}$.

**Theorem 3.6.** *A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is $\delta\beta$-totally rough continuous if and only if the inverse image of each externally $\delta\beta$-definable set in $\mathcal{Q}$ is a totally-definable set in $\mathcal{K}$.*

*Proof.* Let N be an internally $\delta\beta$-definable set in $\mathcal{Q}$. Formerly, in $\mathcal{Q}$, $N^c$ is externally $\delta\beta$-definable. However, any externally $\delta\beta$-definable set in $\mathcal{Q}$ has an inverse image that is a definable set in $\mathcal{K}$. Then, in $\mathcal{K}$, $f^{-1}(N^c) = \left[f^{-1}(N)\right]^c$ is totally definable. As a result, $f^{-1}(N)$ is totally definable in $\mathcal{K}$. Therefore f is $\delta\beta$-totally rough continuous.

Conversely, let N be any externally $\delta\beta$-definable set in $\mathcal{Q}$, and f be a $\delta\beta$-totally rough continuous. Then, in $\mathcal{Q}$, $N^c$ is internally $\delta\beta$-definable. $f^{-1}(N^c) = \left[f^{-1}(N)\right]^c$ is totally definable in $\mathcal{K}$ because $N^c$ is an internally $\delta\beta$-definable set in $\mathcal{Q}$ and f is a $\delta\beta$-totally rough continuous function. This implies that $f^{-1}(N)$ is totally set in $\mathcal{K}$. □

**Lemma 3.7.** *Every internally definable set in $\mathcal{K}$ is internally $\delta\beta$-definable set in $\mathcal{K}$.*

*Proof.* Suppose A be an internally set in $\mathcal{K}$. Then $\underline{R}A = A^\circ = \cup\{G\in\tau : G\subseteq A\} = \cup\{G\in\delta\beta O(X) : G\subseteq A\}$. Since $\tau\subseteq\delta\beta O(X) = \left(A^{\delta\beta}\right)^\circ = \underline{R}^{\delta\beta}A \subseteq A$, hence, A is internally $\delta\beta$-definable set in $\mathcal{K}$. □

**Theorem 3.8.** *Every $\delta\beta$-totally rough continuous function is totally rough continuous function.*

*Proof.* Let N be any internally definable set in $\mathcal{Q}$, and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a $\delta\beta$-totally rough continuous function. Using Lemma 3.7, we may conclude that N is $\delta\beta$-definable set in $\mathcal{Q}$. However, f is a $\delta\beta$-totally rough continuous function. Then, in $\mathcal{K}$, $f^{-1}(N)$ is totally definable. Every internally definable set in $\mathcal{Q}$ is totally definable set in $\mathcal{K}$, and vice versa. As a result, f is a totally rough continuous function. □

The opposite of Theorem 3.8 does not have to be factual in general.

**Example 3.9** (Continued for Example 3.5)**.** Let the function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be defined by $f(x_1)=y_1$, $f(x_2)=y_2$, and $f(x_3)=y_3$. Then the family of all internally definable sets in $\mathcal{Q}$ is $\tau_{\mathcal{Q}}$ and the family of all internally $\delta\beta$-definable sets in $\mathcal{Q}$ is $\delta\beta I(\mathcal{Q})=\{Y,\varphi,\{y_1\},\{y_1,y_2\},\{y_1,y_3\}\}$. Hence, f is totally rough continuous. But f is not a $\delta\beta$-totally rough continuous function, since $V=\{y_1,y_2\}$ is $\delta\beta$-definable set in $\mathcal{Q}$, and $f^{-1}(V)=\{x_1,x_2\}$ is not totally definable set in $\mathcal{K}$.

**Definition 3.10.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is said to be strongly rough continuous if the inverse image of each subset of Y in $\mathcal{Q}$ is totally definable set in $\mathcal{K}$.

**Theorem 3.11.** *Every strongly rough continuous function is $\delta\beta$-totally rough continuous.*

*Proof.* Let N be any internally $\delta\beta$-definable set in $\mathcal{Q}$ and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a strongly rough continuous function. Because f is a strongly rough continuous function, the set $f^{-1}(N)$ in $\mathcal{K}$ is totally definable. As a result, every internally $\delta\beta$-definable set in $\mathcal{Q}$ has an inverse image in $\mathcal{K}$ that is totally definable. As a result, f is a $\delta\beta$-totally rough continuous function. □

**Example 3.12.** Let $\mathcal{K}=(X,R_1,\tau_{\mathcal{K}})$ and $\mathcal{Q}=(Y,R_2,\tau_{\mathcal{Q}})$ are topologized approximation spaces such that $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$, $R_1=\{(x_1,x_1),(x_2,x_2),(x_3,x_3),(x_3,x_1)\}$, and $R_2=\{(y_1,y_1),(y_2,y_2),(y_2,y_3)\}$. Then $x_1R_1=\{x_1\}$, $x_2R_1=\{x_2\}$, $x_3R_1=\{x_1,x_3\}$, $S_1=\{\{x_1\},\{x_2\},\{x_1,x_3\}\}$, $B_1=\{X,\varphi,\{x_1\},\{x_2\},\{x_1,x_3\}\}$, $\tau_{\mathcal{K}}=\{X,\varphi,\{x_1\},\{x_2\},\{x_1,x_2\},\{x_1,x_3\}\}$, $\tau^*_{\mathcal{K}}=\{X,\varphi,\{x_2\},\{x_3\},\{x_2,x_3\},\{x_1,x_3\}\}$, $y_1R_2=\{y_1\}$, $y_2R_2=\{y_2,y_3\}$, $y_3R_2=\varphi$, $S_2=\{\varphi,\{y_1\},\{y_2,y_3\}\}$, $B_2=\{Y,\varphi,\{y_1\},\{y_2,y_3\}\}$ and $\tau_{\mathcal{Q}}=\{Y,\varphi,\{y_1\},\{y_2,y_3\}\}=\tau^*_{\mathcal{Q}}$. Define a function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be such that $f(x_1)=y_2$, $f(x_2)=y_1$ and $f(x_3)=y_3$. Then f is a $\delta\beta$-totally rough continuous function, $\delta\beta I(\mathcal{Q})=\{Y,\varphi,\{y_1\},\{y_2,y_3\}\}$ and every internally $\delta\beta$-definable set in $\mathcal{Q}$ has an inverse image in $\mathcal{K}$ that is totally definable. However, because $V=\{y_2\}$ is a subset of Y in $\mathcal{Q}$ and $f^{-1}(V)=\{x_1\}$ is not a totally definable set in $\mathcal{K}$, f is not a strongly rough continuous function.

**Definition 3.13.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is said to be totally $\delta\beta$-rough continuous function if the inverse image of each internally definable set in $\mathcal{Q}$ is a totally $\delta\beta$-definable set in $\mathcal{K}$.

**Theorem 3.14.** *Every $\delta\beta$-totally rough continuous function is totally $\delta\beta$-rough continuous.*

*Proof.* Let N be any internally definable set in $\mathcal{Q}$ and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a $\delta\beta$-totally rough continuous function. We have N is internally $\delta\beta$-definable set in $\mathcal{Q}$ by using Lemma 3.7. Because f is a $\delta\beta$-totally rough continuous function, $f^{-1}(N)$ is totally definable in $\mathcal{K}$. $f^{-1}(N)$ is totally $\delta\beta$-definable set in $\mathcal{K}$, according to Theorem 3.11. As a result, each internally definable set in $\mathcal{Q}$ has an inverse image in $\mathcal{K}$ that is totally $\delta\beta$-definable. As a result, f is a totally $\delta\beta$-rough continuous function. □

The opposite of Theorem 3.14 does not have to be overall correct.

**Example 3.15.** Let $K=(X,R_1,\tau_K)$ and $Q=(Y,R_2,\tau_Q)$ be two topologized approximation spaces such that $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$, $R_1=\{(x_1,x_1),(x_2,x_2)\}$ and $R_2=\{(y_1,y_1)\}$. Then $x_1R_1=\{x_1\}$, $x_2R_1=\{x_2\}$, $x_3R_1=\varphi$, $S_1=\{\varphi,\{x_1\},\{x_2\}\}$, $B_1=\{X,\varphi,\{x_1\},\{x_2\}\}$, $\tau_{\mathcal{K}}=\{X,\varphi,\{x_1\},\{x_2\},\{x_1,x_2\}\}$, $\tau^*_{\mathcal{K}}=\{X,\varphi,\{x_3\},\{x_1,x_3\},\{x_2,x_3\}\}$, and $\tau_{\mathcal{Q}}=\{Y,\varphi,\{y_1\}\}$. Hence, $\delta\beta I(\mathcal{K})=\{X,\varphi,\{x_1\},\{x_2\},\{x_1,x_2\},\{x_2,x_3\}\}$ and $\delta\beta I(\mathcal{Q})=\{Y,\varphi,\{y_1\},\{y_1,y_2\},\{y_1,y_3\}\}$. Define a function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be such that $f(x_1)=y_1$, $f(x_2)=f(x_3)=y_2$. Then f is totally $\delta\beta$-rough continuous function, since the inverse image of each internally definable set in $\mathcal{Q}$ is totally $\delta\beta$-definable set in $\mathcal{K}$. But f is not $\delta\beta$-totally rough continuous function, since $V=\{y_1\}$ is internally $\delta\beta$-definable set of Y in $\mathcal{Q}$, and $f^{-1}(V)=\{x_1\}$ is not totally definable set in $\mathcal{K}$.

**Theorem 3.16.** *Every $\delta\beta$-totally rough continuous function is $\delta\beta$-rough continuous.*

*Proof.* Let N be any internally definable set in $\mathcal{Q}$ and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a $\delta\beta$-totally rough continuous function. N is internally $\delta\beta$-definable set in $\mathcal{Q}$, according to Lemma 3.7. Because f is a $\delta\beta$-totally rough continuous function, $f^{-1}(N)$ is totally definable in $\mathcal{K}$. We have $f^{-1}(N)$ is totally $\delta\beta$-definable set in $\mathcal{K}$ from Lemma 3.7. Then, in $\mathcal{K}$, $f^{-1}(N)$ is internally $\delta\beta$-definable. As a result, the internally $\delta\beta$-definable set in $\mathcal{K}$ is the inverse image of an internally definable set in $\mathcal{Q}$. As a result, f is a $\delta\beta$-rough continuous function. □

The opposite of Theorem 3.16 does not have to be factual in general.

**Example 3.17.** Let $\mathcal{K}=(X,R_1,\tau_{\mathcal{K}})$ and $\mathcal{Q}=(Y,R_2,\tau_{\mathcal{Q}})$ be two topologized approximation spaces such that $X=\{x_1,x_2,x_3\}$, $Y=\{y_1,y_2,y_3\}$, $R_1=\{(x_1,x_1)\}$ and $R_2=\{(y_1,y_1),(y_2,y_1),(y_2,y_2)\}$. Then $x_1R_1=\{x_1\}$, $x_2R_1=\varphi$, $x_3R_1=\varphi$, $S_1=\{\varphi,\{x_1\}\}$, $B_1=\{X,\varphi,\{x_1\}\}$, $\tau_{\mathcal{K}}=\{X,\varphi,\{x_1\}\}$, $\tau^*_{\mathcal{K}}=\{X,\varphi,\{x_2,x_3\}\}$, $y_1R_2=\{y_1\}$, $y_2R_2=\{y_1,y_2\}$, $y_3R_2=\varphi$, $S_2=\{\varphi,\{y_1\},\{y_1,y_2\}\}$, $B_2=\{Y,\varphi,\{y_1\},\{y_1,y_2\}\}$ and $\tau_{\mathcal{Q}}=\{Y,\varphi,\{y_1\},\{y_1,y_2\}\}$. Hence, $\delta\beta I(\mathcal{K})=\{X,\varphi,\{x_1\},\{x_1,x_2\},\{x_1,x_3\}\}$ and $\delta\beta I(\mathcal{Q})=\{Y,\varphi,\{y_1\},\{y_1,y_2\},\{y_1,y_3\}\}$. Define a function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be such that $f(x_1)=y_1$, $f(x_2)=y_2$ and $f(x_3)=y_3$. Then, f is $\delta\beta$-rough continuous function. But f is not $\delta\beta$-totally rough continuous function, since $V=\{y_1\}$ is internally $\delta\beta$-definable set of Y in $\mathcal{Q}$, and $f^{-1}(V)=\{x_1\}$ is not totally definable set in $\mathcal{K}$.

**Theorem 3.18.** *The composition of two $\delta\beta$-totally rough continuous functions is $\delta\beta$-totally rough continuous.*

*Proof.* Let $f:\mathcal{K}\longrightarrow\mathcal{Q}$ and $g:\mathcal{Q}\longrightarrow\mathcal{G}$ be two $\delta\beta$-totally rough continuous functions, where $K=(X,R_1,\tau_K)$, $\mathcal{Q}=\left(Y,R_2,\tau_Q\right)$ and $\mathcal{G}=(Z,R_3,\tau_{\mathcal{G}})$ are three topologized approximation spaces. In $\mathcal{G}$, let N be any internally $\delta\beta$-definable set. g is a $\delta\beta$-totally rough continuous function. Then, in $\mathcal{Q}$, $g^{-1}(N)$ is totally definable set. $g^{-1}(N)$ is an internally $\delta\beta$-definable set in $\mathcal{Q}$, according to Lemma 3.7. Because f is a $\delta\beta$-totally rough continuous function, then $f^{-1}\left(g^{-1}(N)\right)=(g\circ f)^{-1}(N)$ is totally definable set in $\mathcal{K}$. As a result, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is $\delta\beta$-totally rough continuous. As a result, the composition function $g\circ f$ is a $\delta\beta$-totally rough continuous function. □

**Definition 3.19.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is said to be $\delta\beta$-rough irresolute if the inverse image of each internally $\delta\beta$-definable set in $\mathcal{Q}$ is internally $\delta\beta$-definable set in $\mathcal{K}$.

**Theorem 3.20.** *Let $K=(X,R_1,\tau_K)$, $\mathcal{Q}=\left(Y,R_2,\tau_Q\right)$, and $\mathcal{G}=(Z,R_3,\tau_{\mathcal{G}})$ be three topologized approximation spaces. If $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is $\delta\beta$-totally rough continuous and $g:\mathcal{Q}\longrightarrow\mathcal{G}$ is $\delta\beta$-rough irresolute, then, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is $\delta\beta$-totally rough continuous.*

*Proof.* In $\mathcal{G}$, let N be an internally $\delta\beta$-definable set. $g^{-1}(N)$ is an internally $\delta\beta$-definable set in $\mathcal{Q}$ because g is a $\delta\beta$-rough irresolute function. But f is $\delta\beta$-totally rough continuous, then $f^{-1}\left(g^{-1}(N)\right)=(g\circ f)^{-1}(N)$ is totally definable set in $\mathcal{K}$. Accordingly, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is $\delta\beta$-totally rough continuous. Consequently, the composition function is a $\delta\beta$-totally rough continuous function. □

**Theorem 3.21.** *Let a three topologized approximation spaces $K=(X,R_1,\tau_K)$, $\mathcal{Q}=\left(Y,R_2,\tau_Q\right)$, and $\mathcal{G}=(Z,R_3,\tau_{\mathcal{G}})$. If $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is $\delta\beta$-totally rough continuous and $g:\mathcal{Q}\longrightarrow\mathcal{G}$ is $\delta\beta$-rough continuous, then, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is totally rough continuous.*

*Proof.* Let N be a set in $\mathcal{G}$ that can be defined as internally definable. g is a $\delta\beta$-rough continuous. Then, in $\mathcal{Q}$, $g^{-1}(N)$ is internally $\delta\beta$-definable. Since f is a $\delta\beta$-totally rough continuous function, then $f^{-1}\left(g^{-1}(N)\right)=(g\circ f)^{-1}(N)$ is totally definable set in $\mathcal{K}$. Hence, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is totally rough continuous. Therefore, the composition function $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is totally rough continuous function. □

## 4. δβ-exactly rough continuous functions

In this section, will focus on $\delta\beta$-exactly rough continuous functions, and some properties and relations on it.

**Definition 4.1.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is said to be $\delta\beta$-exactly rough continuous if the inverse image of each internally $\delta\beta$-definable set in $\mathcal{Q}$ is totally $\delta\beta$-definable set in $\mathcal{K}$.

**Example 4.2** (Continued for Example 3.5**).** The family of all totally $\delta\beta$-definable sets in $\mathcal{K}$ is $\{X,\varphi,\{x_1\},\{x_2,x_3\}\}$ and the family of all internally $\delta\beta$-definable sets in $\mathcal{Q}$ is $\{Y,\varphi,\{y_1\},\{y_1,y_2\},\{y_1,y_3\}\}$. Hence, the inverse image of each internally $\delta\beta$-definable sets in $\mathcal{Q}$ is totally $\delta\beta$-definable sets in $\mathcal{K}$. Therefore, f is $\delta\beta$-exactly rough continuous.

**Theorem 4.3.** *A function* $f:\mathcal{K}\longrightarrow\mathcal{Q}$ *is* $\delta\beta$*-exactly rough continuous if and only if the inverse image of every externally* $\delta\beta$*-definable set in* $\mathcal{Q}$ *is totally* $\delta\beta$*-definable set in* $\mathcal{K}$.

*Proof.* The proof has the same manner as Theorem 3.6. □

**Theorem 4.4.** *Each strongly rough continuous function is* $\delta\beta$*-exactly rough continuous.*

*Proof.* Let $\mathcal{K}=(X, R_1, \tau_K)$ and $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$ be two topologized approximation spaces, and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a strongly rough continuous function. Allow any internally $\delta\beta$-definable set in $\mathcal{Q}$ to be N. Then, in $\mathcal{K}$, $f^{-1}(N)$ is totally definable set. This means that in $\mathcal{K}$, $f^{-1}(N)$ is totally $\delta\beta$-definable. As a result, f is a $\delta\beta$-exactly rough continuous function. □

The converse of Theorem 4.4 does not have to be true in general.

**Example 4.5.** Let $\mathcal{K}=(X, R_1, \tau_{\mathcal{K}})$ and $\mathcal{Q}=(Y, R_2, \tau_Q)$ be two topologized approximation spaces with $X=\{x_1, x_2, x_3\}$, $Y=\{y_1, y_2, y_3\}$, $R_1=\{(x_1, x_1), (x_2, x_2), (x_2, x_3)\}$ and $R_2=\{(y_1, y_1), (y_2, y_2), (y_3, y_1), (y_3, y_3)\}$. Then $x_1R_1=\{x_1\}$, $x_2R_1=\{x_2, x_3\}$, $x_3R_1=\varphi$, $S_1=\{\varphi, \{x_1\}, \{x_2, x_3\}\}$, $B_1=\{X, \varphi, \{x_1\}, \{x_2, x_3\}\}$, $\tau_{\mathcal{K}}=\{X, \varphi, \{x_1\}, \{x_2, x_3\}\}$ $=\tau^*_{\mathcal{K}}$, $y_1R_2=\{y_1\}$, $y_2R_2=\{y_2\}$, $y_3R_2=\{y_1, y_3\}$, $S_2=\{\{y_1\}, \{y_2\}, \{y_1, y_3\}\}$, $B_2=\{Y, \varphi, \{y_1\}, \{y_2\}, \{y_1, y_3\}\}$ and $\tau_{\mathcal{Q}}=\{Y, \varphi, \{y_1\}, \{y_2\}, \{y_1, y_2\}, \{y_1, y_3\}\}$. Hence, the family of all internally $\delta\beta$-definable sets in $\mathcal{Q}$ is $\{Y, \varphi, \{y_1\}, \{y_2\}, \{y_1, y_2\}, \{y_1, y_3\}\}$, the family of all totally $\delta\beta$-definable sets in $\mathcal{K}$ is $\{X, \varphi, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$ and the family of all totally definable sets in $\mathcal{K}$ is $\{X, \varphi, \{x_1\}, \{x_2, x_3\}\}$. Define $f:\mathcal{K}\longrightarrow\mathcal{Q}$ as a function with $f(x_1)=y_1$, $f(x_2)=y_2$, and $f(x_3)=y_3$. The inverse image of any internally $\delta\beta$-definable set in $\mathcal{Q}$ is a totally $\delta\beta$-definable set in $\mathcal{K}$, hence f is a $\delta\beta$-exactly rough continuous function. Because $V=\{y_3\}$ is a subset of Y in $\mathcal{Q}$ and $f^{-1}(V)=\{x_3\}$ is not a totally definable set in $\mathcal{K}$, f is not a strongly rough continuous function.

**Theorem 4.6.** *Each* $\delta\beta$*-totally rough continuous function is* $\delta\beta$*-exactly rough continuous.*

*Proof.* Let $\mathcal{K}=(X, R_1, \tau_K)$ and $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$ be two topologized approximation spaces, and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a $\delta\beta$-totally rough continuous function. Allow any internally $\delta\beta$-definable set in $\mathcal{Q}$ to be N. Then, in $\mathcal{K}$, $f^{-1}(N)$ is totally definable. This means that in $\mathcal{K}$, $f^{-1}(N)$ is totally $\delta\beta$-definable. Then any internally $\delta\beta$-definable set in $\mathcal{Q}$ has an inverse image in $\mathcal{K}$ that is totally $\delta\beta$-definable. Accordingly, f is a $\delta\beta$-exactly rough continuous function. □

The opposite of Theorem 4.6 does not have to be factual in general.

**Example 4.7** (Continued for Example 3.15)**.** Since the inverse image of every internally $\delta\beta$-definable set in $\mathcal{Q}$ is a totally $\delta\beta$-definable set in $\mathcal{K}$, f is a $\delta\beta$-exactly rough continuous function. However, because $V=\{y_1\}$ is an internally $\delta\beta$-definable set in $\mathcal{Q}$ and $f^{-1}(\{y_1\})=\{x_1\}$ is not a totally definable set in K, f is not a $\delta\beta$-totally rough continuous function.

**Theorem 4.8.** *Every* $\delta\beta$*-exactly rough continuous function is* $\delta\beta$*-rough continuous.*

*Proof.* Let $\mathcal{K}=(X, R_1, \tau_K)$ and $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$ be two topologized approximation spaces, and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a $\delta\beta$-exactly rough continuous function. Allow any internally definable set in $\mathcal{Q}$ to be N. This means that in $\mathcal{Q}$, N is an internally $\delta\beta$-definable set. Because f is a $\delta\beta$-exactly rough continuous function, the set $f^{-1}(N)$ in $\mathcal{K}$ is totally definable. That is, in $\mathcal{K}$, $f^{-1}(N)$ is an internally $\delta\beta$-definable set. As a result, every internally definable set in $\mathcal{Q}$ has an inverse image in $\mathcal{K}$ that is internally $\delta\beta$-definable. As a result, f is $\delta\beta$-rough continuous. □

The opposing of Theorem 4.8 does not have to be correct in overall.

**Example 4.9** (Continued for Example 3.17)**.** Clearly, f is $\delta\beta$-rough continuous function. But f is not $\delta\beta$-exactly rough continuous function, since for the internally $\delta\beta$-definable set $\{y_1, y_2\}$ in $\mathcal{Q}$, and $f^{-1}(\{y_1, y_2\})=\{x_1, x_2\}$ is not totally $\delta\beta$-definable set in $\mathcal{K}$.

**Definition 4.10.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is said to be strongly $\delta\beta$-rough continuous if the inverse image of each subset of Y in $\mathcal{Q}$ is totally $\delta\beta$-definable set in $\mathcal{K}$.

**Theorem 4.11.** *Every strongly $\delta\beta$-rough continuous function is $\delta\beta$-exactly rough continuous.*

*Proof.* Suppose the topologized approximation spaces $\mathcal{K}=(X, R_1, \tau_K)$ and $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$, and $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a strongly $\delta\beta$-rough continuous function. N is a subset of Y in $\mathcal{Q}$ if N is any internally $\delta\beta$-definable set in $\mathcal{Q}$. Because f is a strongly $\delta\beta$-rough continuous function, then, in $\mathcal{K}$, $f^{-1}(N)$ is totally $\delta\beta$-definable. As a result, f is a $\delta\beta$-exactly rough continuous function. □

The converse of Theorem 4.11 is not correct in the general case.

**Example 4.12** (Continued for Example 3.17)**.** f is $\delta\beta$-exactly rough continuous because the family of all internally $\delta\beta$-definable sets in $\mathcal{Q}$ is $\{Y, \varphi, \{y_1\}, \{y_2, y_3\}\}$ and all internally $\delta\beta$-definable sets in $\mathcal{K}$ is $\{X, \varphi, \{x_2\}, \{x_1, x_3\}\}$. But f is not strongly $\delta\beta$-rough continuous, since $V=\{y_2\}$ is a subset of Y in $\mathcal{Q}$, and $f^{-1}(V)=\{x_1\}$ is not totally $\delta\beta$-definable set in $\mathcal{K}$.

**Theorem 4.13.** *The composition of two $\delta\beta$-exactly rough continuous functions is $\delta\beta$-exactly rough continuous.*

*Proof.* Let Three topologized approximation spaces are $\mathcal{K}=(X, R_1, \tau_K)$, $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$, and $\mathcal{G}=(Z, R_3, \o_{\mathcal{G}})$. Allow two $\delta\beta$-exactly rough continuous functions, $f:\mathcal{K}\longrightarrow\mathcal{Q}$ and $g:\mathcal{Q}\longrightarrow\mathcal{G}$. In $\mathcal{G}$, let N be any internally $\delta\beta$-definable set. Then $g^{-1}(N)$ is totally $\delta\beta$-definable in $\mathcal{Q}$. Because g is a $\delta\beta$-exactly rough continuous function, $g^{-1}(N)$ is an internally $\delta\beta$-definable set in $\mathcal{Q}$. Then $f^{-1}\left(g^{-1}(N)\right)=(g\circ f)^{-1}(N)$ is totally $\delta\beta$-definable. Because f is a $\delta\beta$-exactly rough continuous function, $g\circ f$ is a $\delta\beta$-exactly rough continuous function. □

**Theorem 4.14.** *Let $\mathcal{K}=(X, R_1, \tau_K)$, $\mathcal{Q}\left(Y, R_2, \tau_Q\right)$, and $\mathcal{G}=(Z, R_3, \tau_{\mathcal{G}})$ are three topologized approximation spaces. If $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is $\delta\beta$-exactly rough continuous function and $g:\mathcal{Q}\longrightarrow\mathcal{G}$ is $\delta\beta$-rough irresolute, then, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is $\delta\beta$-exactly rough continuous.*

*Proof.* In $\mathcal{G}$, let N be an internally $\delta\beta$-definable set. g is $\delta\beta$-rough irresolute function. Then $g^{-1}(N)$ in $\mathcal{Q}$ is internally $\delta\beta$-definable. Because f is $\delta\beta$-exactly rough continuous, then, in $\mathcal{K}$, $f^{-1}\left(g^{-1}(N)\right)=(g\circ f)^{-1}(N)$ is totally $\delta\beta$-definable set. As a result, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ $\delta\beta$-totally rough continuous. □

**Theorem 4.15.** *Let $\mathcal{K}=(X, R_1, \o_K)$, $\mathcal{Q}=\left(Y, R_2, \o_Q\right)$, and $\mathcal{G}=(Z, R_3, \o_{\mathcal{G}})$ are three topologized approximation spaces. If $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is $\delta\beta$-exactly rough continuous function and $g:\mathcal{Q}\longrightarrow\mathcal{G}$ is $\delta\beta$-rough continuous function, then, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is totally $\delta\beta$-rough continuous.*

*Proof.* Let N be an internally definable set in $\mathcal{G}$. Since g is $\delta\beta$-rough continuous function, then, $g^{-1}(N)$ is internally $\delta\beta$-definable set in Q. Since f is $\delta\beta$-exactly rough continuous function, formerly, $f^{-1}\left(g^{-1}(N)\right)$ $=(g\circ f)^{-1}(N)$ is totally $\delta\beta$-definable set in $\mathcal{K}$. Hence, $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is totally $\delta\beta$-rough continuous. Therefore, the composition function $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is totally $\delta\beta$-rough continuous function. □

## 5. δβ-internally functions

In this section, we introduce the concepts of $\delta\beta$-internally functions, $\delta\beta$-totally functions, and $\delta\beta$-totally internally functions.

**Definition 5.1.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is supposed to be

(i) $\delta\beta$-internally function if the image of every internally $\delta\beta$-definable set in $\mathcal{K}$ is internally $\delta\beta$-definable set in $\mathcal{Q}$;

(ii) $\delta\beta$-totally function if the image of every internally $\delta\beta$-definable set in $\mathcal{K}$ is totally $\delta\beta$-definable set in $\mathcal{Q}$.

**Theorem 5.2.** *Let the bijective δβ-totally function* $f:\mathcal{K}\longrightarrow\mathcal{Q}$*. Then the image of each externally δβ-definable set in* $\mathcal{K}$ *is totally definable set in* $\mathcal{Q}$*.*

*Proof.* Let N be an externally δβ-definable set in $\mathcal{K}$. Then, in $\mathcal{K}$, $N^c$ is internally δβ-definable set. Because f is a δβ-totally function, $f(N^c)$ is totally defined in $\mathcal{Q}$. However, because f is bijective, $f(N^c)=[f(N)]^c$. As a result, $[f(N)]^c$ is a totally definable set, and so $f(N)$ is a totally definable set in $\mathcal{Q}$. □

**Theorem 5.3.** *A function* $f:\mathcal{K}\longrightarrow\mathcal{Q}$ *is δβ-totally if and only if for each subset* N *of* Y *in* $\mathcal{Q}$ *and for each externally δβ-definable set* U *in* $\mathcal{K}$ *containing* $f^{-1}(N)$*, there is a totally definable subset* V *of* Y *in* $\mathcal{Q}$ *such that* $N\subseteq V$ *and* $f^{-1}(V)\subseteq U$*.*

*Proof.* Let an internally δβ-definable set F in $\mathcal{K}$. Since $F\subseteq f^{-1}(f(F))$, then $\left[f^{-1}(f(F))\right]\subseteq F^c$. Hence, $F^c$ is an externally δβ-definable set in $\mathcal{K}$ containing $\left[f^{-1}(f(F))\right]^c=f^{-1}([f(F)]^c)$, where $[f(F)]^c$ is a subset of Y in $\mathcal{Q}$. Then there exists a totally definable subset V of Y in Q such that $[f(F)]^c\subseteq V$ and $f^{-1}(V)\subseteq F^c$. That is $V^c\subseteq f(F)$ and $F\subseteq\left[f^{-1}(V)\right]^c$. Then, $V^c\subseteq f(F)\subseteq f\left(\left[f^{-1}(V)\right]^c\right)\subseteq f\left(f^{-1}(V^c)\right)\subseteq V^c$. Thus $f(F)=V^c$. Then, $f(F)$ is totally definable set in $\mathcal{Q}$. As a consequence, f is δβ-totally function. Conversely, suppose $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a δβ-totally function and Y contains a subset N in $\mathcal{Q}$. Let an externally δβ-definable set U in $\mathcal{K}$ such that $f^{-1}(N)\subseteq U$. Since $U^c$ is an internally δβ-definable set in $\mathcal{K}$, then $f(U^c)$ is totally definable set in $\mathcal{Q}$. Now, $U^c\subseteq\left[f^{-1}(N)\right]^c=f^{-1}(N^c)$ and $f(U^c)\subseteq f\left(f^{-1}(N^c)\right)\subseteq N$. Hence, $N\subseteq[f(U^c)]^c$. Let $V=[f(U^c)]^c$, then V is totally definable set in $\mathcal{Q}$ containing N and$f^{-1}(V)=f^{-1}([f(U^c)]^c)=\left[f^{-1}(f(U^c))\right]^c$. Thus, $\left[f^{-1}(V)\right]^c=f^{-1}(f(U^c))\supseteq U^c$. Hence, $U^c\subseteq\left[f^{-1}(V)\right]^c$. Then, $f^{-1}(V)\subseteq U$. □

**Theorem 5.4.** *Let* $\mathcal{K}=(X,R_1,\tau_K)$*,* $\mathcal{Q}=\left(Y,R_2,\tau_Q\right)$ *and* $\mathcal{G}=(Z,R_3,\tau_{\mathcal{G}})$ *are three topologized approximation spaces. If* $f:\mathcal{K}\longrightarrow\mathcal{Q}$ *is δβ-internally function and* $g:\mathcal{Q}\longrightarrow\mathcal{G}$ *is δβ-totally function, then,* $f\circ g:\mathcal{K}\longrightarrow\mathcal{G}$ *is δβ-totally function.*

*Proof.* Let N be an internally δβ-definable set in $\mathcal{K}$. Since $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is δβ-internally function, then, $f(N)$ is internally δβ-definable set in $\mathcal{Q}$. But $g:\mathcal{Q}\longrightarrow\mathcal{G}$ is δβ-totally function, then, $g(f(N))$ is totally definable set in $\mathcal{G}$. But $g(f(N))=(g\circ f)(N)$, then, $(g\circ f)(N)$ is totally definable set in $\mathcal{G}$. Hence, $g\circ f$ is δβ-totally function. Therefore, the composition function $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is δβ-totally function. □

**Theorem 5.5.** *Let* $\mathcal{K}=(X,R_1,\tau_K)$*,* $\mathcal{Q}=\left(Y,R_2,\tau_Q\right)$*, and* $\mathcal{G}=(Z,R_3,\tau_{\mathcal{G}})$ *are three topologized approximation spaces. Let* $f:\mathcal{K}\longrightarrow\mathcal{Q}$ *and* $g:\mathcal{Q}\longrightarrow\mathcal{G}$ *be two functions, such that* $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ *is δβ-totally function. Then, the following hold.*

(i) *If* $f:\mathcal{K}\longrightarrow\mathcal{Q}$ *is δβ-rough irresolute and onto function, then* g *is δβ-totally function.*
(ii) *If* $g:\mathcal{Q}\longrightarrow\mathcal{G}$ *is totally rough continuous and one to one function, then* f *is δβ-totally function.*

*Proof.*

(i). suppose that $f:\mathcal{K}\longrightarrow\mathcal{Q}$ be a δβ-rough irresolute and onto function and N be any internally δβ-definable set in $\mathcal{Q}$. Formerly, $f^{-1}(N)$ is internally δβ-definable set in $\mathcal{K}$. Since $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is δβ-totally function, $(g\circ f)\left(f^{-1}(N)\right)=g(N)$ because f is onto function. Then $g(N)$ is totally definable set in $\mathcal{G}$. Therefore, g is δβ-totally function.

(ii). Since g is one to one, we have $f(N)=g^{-1}(g\circ f)(N)$ is true for every subset N of X in $\mathcal{K}$. Let U be an internally δβ-definable set in $\mathcal{K}$. Since $g\circ f:\mathcal{K}\longrightarrow\mathcal{G}$ is δβ-totally function, then $(g\circ f)(U)$ is totally definable set in $\mathcal{G}$. This main that $(g\circ f)(U)$ is internally definable set in $\mathcal{G}$. Since g is totally rough continuous, then, $g^{-1}((g\circ f)(U))=f(U)$ is totally definable set in $\mathcal{Q}$. Therefore, f is δβ-totally function. □

**Definition 5.6.** A function $f:\mathcal{K}\longrightarrow\mathcal{Q}$ is said to be δβ-totally internally if the image of every internally δβ-definable set in $\mathcal{K}$ is totally δβ-definable set in $\mathcal{Q}$.

**Theorem 5.7.** *Let two topologized approximation spaces* $\mathcal{K}=(X,R_1,\tau_K)$ *and* $\mathcal{Q}=\left(Y,R_2,\tau_Q\right)$*. When a function* $f:\mathcal{K}\longrightarrow\mathcal{Q}$ *is δβ-totally internally, then the image of each externally δβ-definable set in* $\mathcal{K}$ *is δβ-totally definable set in* $\mathcal{Q}$*.*

*Proof.* Like the proof of Theorem 5.2. □

**Theorem 5.8.** *A bijective function* $f{:}\mathcal{K}\longrightarrow\mathcal{Q}$ *is* $\delta\beta$*-totally internally if and only if for each subset* $B$ *of* $Y$ *in* $\mathcal{Q}$ *and for each externally* $\delta\beta$*-definable set* $U$ *in* $\mathcal{K}$ *containing* $f^{-1}(B)$*, there is a totally* $\delta\beta$*-definable subset* $V$ *of* $Y$ *in* $\mathcal{Q}$ *such that* $B\subseteq V$ *and* $f^{-1}(V)\subseteq U$*.*

*Proof.* Like the proof of Theorem 5.3. □

**Theorem 5.9.** *Let* $f{:}\mathcal{K}\longrightarrow\mathcal{Q}$ *be bijective function. Then, the inverse function of* $f$ *is* $\delta\beta$*-exactly rough continuous if and only if* $f$ *is* $\delta\beta$*-totally internally function.*

*Proof.* Let $N$ be an internally $\delta\beta$-definable set in $\mathcal{K}$. Since the inverse function of $f$ is $\delta\beta$-exactly rough continuous, then $\left(f^{-1}\right)^{-1}(N)=f(N)$ is totally $\delta\beta$-definable set in $\mathcal{Q}$. Therefore, $f$ is $\delta\beta$-exactly internally function. Conversely, let $f$ be a $\delta\beta$-exactly internally function and $M$ be any internally $\delta\beta$-definable set in $\mathcal{K}$. Then $f(M)$ is totally $\delta\beta$-definable set in $\mathcal{Q}$. That is $\left(f^{-1}\right)^{-1}(M)=f(M)$ is totally $\delta\beta$-definable set in $\mathcal{Q}$. Therefore $f^{-1}$ is $\delta\beta$-exactly rough continuous. □

**Theorem 5.10.** *The composition of two* $\delta\beta$*-totally internally functions is* $\delta\beta$*-totally internally function.*

*Proof.* Let $\mathcal{K}=(X, R_1, \tau_K)$, $\mathcal{Q}=\left(Y, R_2, \tau_Q\right)$, and $\mathcal{G}=(Z, R_3, \tau_{\mathcal{G}})$ be three topologized approximation spaces and let $f{:}\mathcal{K}\longrightarrow\mathcal{Q}$ and $g{:}\mathcal{Q}\longrightarrow\mathcal{G}$ be two $\delta\beta$-exactly internally functions. Let $N$ be an internally $\delta\beta$-definable set in $\mathcal{K}$. Since $f$ is $\delta\beta$-exactly internally function, then $f(N)$ is totally $\delta\beta$-definable set in $\mathcal{Q}$. That is $f(N)$ is internally $\delta\beta$-definable set in $\mathcal{Q}$. Since $g$ is $\delta\beta$-exactly internally function, then $g(f(V))=(g\circ f)(V)$ is totally $\delta\beta$-definable set in $\mathcal{G}$. Hence, $g\circ f$ is $\delta\beta$-exactly internally function. □

## 6. Conclusions

Developments of topology in the making of convinced rough functions will serve as a connection for numerous applications [15, 31, 35–37] and will disclose concealed data families. Topological allowances of the impression of rough functions deliver the way for rough continuity to be connected to the ground of near continuous functions. Applications of topological rough functions of information systems exposed the door to many alterations between many classes of evidence schemes, such as multi-valued and single-valued information systems.

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