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On some extensions of dynamic Hardy-type inequalities on time scales

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Abstract

The objective of this paper is to establish a new class of dynamic inequalities of the Hardy type which generalize and improve some recent results given in the literature, and we derive some new weighted Hardy type integral inequalities on the time scale.

Keywords: Delta derivative, Hardy's inequality, Hölder's inequality, time scales.

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1. Introduction

In [5], Hardy proved the following result.

Theorem 1.1. *If* $\{\Omega(z)\}_{z=1}^{\infty}$ *is a sequence of non-negative real numbers and* $\omega > 1$, *then*

$$\sum_{z=1}^{\infty} \frac{1}{z^{\omega}} \left(\sum_{\kappa=1}^{z} \Omega(\kappa) \right)^{\omega} \leqslant \left(\frac{\omega}{\omega - 1} \right)^{\omega} \sum_{z=1}^{\infty} \Omega^{\omega}(z). \tag{1.1}$$

In [6], Hardy obtained the continuous of inequality (1.1) in the next theorem.

Theorem 1.2. *For* $\phi \geqslant 0$ *is continuous function on* $[0, \infty)$, *if* $\omega > 1$, *then*

$$\int_0^{\infty} \frac{1}{z^{\omega}} \left(\int_0^{z} \phi(\zeta) d\zeta \right)^{\omega} dz \leqslant \left(\frac{\omega}{\omega - 1} \right)^{\omega} \int_0^{\infty} \phi^{\omega}(z) dz, \tag{1.2}$$

where the constant $(\omega/(\omega-1))^{\omega}$ *can't be found smaller than it.*

In [8], Hardy and Littlewood proved the following theorem.

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Theorem 1.3. *If* $\varphi \geqslant 0$ *is continuous function on* $[0,\infty)$ *and* $0<\omega<1$, *then*

$$\int_0^\infty \frac{1}{z^\omega}\left(\int_z^\infty \varphi(\zeta)d\zeta\right)^\omega dz \leqslant \left(\frac{\omega}{1-\omega}\right)^\omega \int_0^\infty \varphi^\omega(z)dz. \tag{1.3}$$

In [7], Hardy generalized (1.2) and (1.3) in the following result.

Theorem 1.4. *If* $\varphi \geqslant 0$ *is continuous function on* $[0,\infty)$, *then*

$$\int_0^\infty \frac{1}{z^\kappa}\left(\int_0^z \varphi(\zeta)d\zeta\right)^\omega dz \leqslant \left(\frac{\omega}{\kappa-1}\right)^\omega \int_0^\infty z^{\omega-\kappa}\varphi^\omega(z)dz, \quad \textit{for} \quad \omega \geqslant \kappa > 1, \tag{1.4}$$

and

$$\int_0^\infty \frac{1}{z^\kappa}\left(\int_z^\infty \varphi(\zeta)d\zeta\right)^\omega dz \leqslant \left(\frac{\omega}{1-\kappa}\right)^\omega \int_0^\infty z^{\omega-\kappa}\varphi^\omega(z)dz, \quad \textit{for} \quad \omega > 1 > \kappa > 0. \tag{1.5}$$

Remark 1.5 ([1]). We note that the inequalities (1.4) and (1.5) are equivalent to that

$$\int_0^\infty \frac{1}{z^{\omega-\delta}}\left(\int_0^z \varphi(\zeta)d\zeta\right)^\omega dz \leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^\omega \int_0^\infty z^{\delta}\varphi^\omega(z)dz, \quad \text{for} \quad \delta < \omega-1, \tag{1.6}$$

and

$$\int_0^\infty \frac{1}{z^{\omega-\delta}}\left(\int_z^\infty \varphi(\zeta)d\zeta\right)^\omega dz \leqslant \left(\frac{\omega}{1+\delta-\omega}\right)^\omega \int_0^\infty z^{\delta}\varphi^\omega(z)dz, \quad \text{for} \quad \delta > \omega-1. \tag{1.7}$$

In [2], Zeki et al. extended the classical Hardy inequality in the next results.

Theorem 1.6. *Let* $\varphi, \theta, \vartheta$ *be non-negative functions on* $(0,\infty), 0<r<t<\infty, 1<\omega\leqslant\gamma<\infty$ *and* $\kappa>1$. *If*

$$\Phi_1(z) = \frac{1}{\Omega(z)}\int_r^z \vartheta(\zeta)\theta(\varphi(\zeta))d\zeta \quad \textit{and} \quad \lambda \geqslant \frac{\kappa-1}{\omega+\kappa-1},$$

then

$$\int_r^t \frac{\vartheta(z)}{\Omega^\kappa(z)}\Phi_1^\omega(z)dz \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^\omega \left(\int_r^t \vartheta(z)dz\right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \frac{\vartheta(z)}{\Omega^{\frac{\kappa\gamma}{\omega}}(z)}\theta^\gamma(\varphi(z))dz\right)^{\frac{\omega}{\gamma}}, \tag{1.8}$$

where

$$\Omega(z) = \int_0^z \vartheta(\zeta)d\zeta.$$

Theorem 1.7. *Let* $\varphi, \theta, \vartheta$ *be non-negative functions on* $(0,\infty), 0<r<t<\infty, 1<\omega\leqslant\gamma<\infty$ *and* $\omega+\kappa<1$. *If*

$$\tilde{\Phi}_1(z) = \frac{1}{\Omega(z)}\int_z^t \vartheta(\zeta)\theta(\varphi(\zeta))d\zeta \quad \textit{and} \quad \lambda \geqslant \frac{1-\kappa}{1-\kappa-\omega} > 0,$$

then

$$\int_r^t \frac{\vartheta(z)}{\Omega^\kappa(z)}\tilde{\Phi}_1^\omega(z)dz \leqslant \left(\frac{\lambda\omega}{1-\kappa}\right)^\omega \left(\int_r^t \vartheta(z)dz\right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \frac{\vartheta(z)}{\Omega^{\frac{\kappa\gamma}{\omega}}(z)}\theta^\gamma(\varphi(z))dz\right)^{\frac{\omega}{\gamma}}, \tag{1.9}$$

where

$$\Omega(z) = \int_0^z \vartheta(\zeta)d\zeta.$$

In 1988 Stefan Hilger [1] gave a new definition for time scales $\mathbb{T}$ to unify continuous and discrete analysis. In the last decades, many researchers studied the dynamic inequalities on time scales. For example, in [10], Řehak proved the time scales versions of (1.1) and (1.2) as follows.

Theorem 1.8. *Let $\mathbb{T}$ be a time scale. If $r \in \mathbb{T}$ and $\omega > 1$, then*

$$\int_r^\infty \left(\frac{\Omega^\sigma(z)}{\sigma(z)-r}\right)^\omega \triangle z \leqslant \left(\frac{\omega}{\omega-1}\right)^\omega \int_r^\infty \phi^\omega(z) \triangle z,$$

where

$$\Omega(z) = \int_r^z \phi(\zeta) \triangle \zeta, \quad \textit{for} \quad z \in [r, \infty)_{\mathbb{T}},$$

unless $\phi \equiv 0$. In addition, the constant $(\omega/(\omega-1))^\omega$ is the best constant if $\mu(z)/z \to 0$ as $z \to \infty$.

In [9], Saker et al. proved a new dynamic Hardy-type inequalities that can be considered as extension of (1.4) and (1.5), respectively, as follows.

Theorem 1.9. *Let $\mathbb{T}$ be a time scale. If $r \in \mathbb{T}$ and $\omega \geqslant \kappa > 1$, then*

$$\int_r^\infty \frac{(\Omega^\sigma(z))^\omega}{(\sigma(z)-r)^\kappa} \triangle z \leqslant \left(\frac{\omega}{\kappa-1}\right)^\omega \int_r^\infty \frac{(\sigma(z)-r)^{\kappa(\omega-1)}}{(z-r)^{(\kappa-1)\omega}} \phi^\omega(z) \triangle z,$$

where

$$\Omega(z) = \int_r^z \phi(\zeta) \triangle \zeta, \quad \textit{for} \quad z \in [r, \infty)_{\mathbb{T}}.$$

If $r \in \mathbb{T}$ and $\omega > 1 > \kappa > 0$, then

$$\int_r^\infty \frac{\Omega^\omega(z)}{\sigma^\kappa(z)} \triangle z \leqslant \left(\frac{\omega}{1-\kappa}\right)^\omega \int_r^\infty \sigma^{\omega-\kappa}(z) \phi^\omega(z) \triangle z,$$

where

$$\Omega(z) = \int_z^\infty \phi(\zeta) \triangle \zeta, \quad \textit{for} \quad z \in [r, \infty)_{\mathbb{T}}.$$

The general idea of this paper is to prove the dynamic inequalities for (1.8) and (1.9) on time scales and have a continuous inequality and discrete inequality when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, respectively. The paper is organized in the following way. In second section, we give some basic concepts about the delta calculus. In third section, we state and prove the main results.

2. Preliminaries

In this section, we will introduce some basic definitions and properties about delta calculus on time scales. For more details, we recommend reader reading books [1, 3].

Definition 2.1 ([1])**.** A time scale $\mathbb{T}$ is a non-empty closed subset of real numbers $\mathbb{R}$.

Definition 2.2 ([1])**.** The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(\zeta) = \inf \{s \in \mathbb{T} : s > \zeta\},$$

and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(\zeta) = \sup \{s \in \mathbb{T} : s < \zeta\}.$$

A point $\zeta \in \mathbb{T}$ is right-dense if $\sigma(\zeta) = \zeta$, is right–scattered if $\sigma(\zeta) > \zeta$, is left-dense if $\rho(\zeta) = \zeta$ and is left–scattered if $\rho(\zeta) < \zeta$. If $\mathbb{T}$ has left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$.

Definition 2.3 ([1]). A function $\phi : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be rd-continuous provided ϕ is continuous at right-dense points and left sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$.

Theorem 2.4 ([1]). *Let $\phi, \theta : \mathbb{T} \to \mathbb{R}$ be differentiable at $\zeta \in \mathbb{T}^k$. Then*

$$(\phi\theta)^{\triangle} = \phi^{\triangle}\theta + \phi^{\sigma}\theta^{\triangle} = \phi^{\triangle}\theta^{\sigma} + \phi\theta^{\triangle}, \ \textit{where} \ \theta^{\sigma} = \theta \circ \sigma$$

and

$$\left(\frac{\phi}{\theta}\right)^{\triangle} = \frac{\theta\phi^{\triangle} - \phi\theta^{\triangle}}{\theta\theta^{\sigma}}, \ \textit{where} \ \theta\theta^{\sigma} \neq 0.$$

Theorem 2.5 (Integration by parts [1]). *Let $r, t \in \mathbb{T}$ and $u, v \in C_{rd}(\mathbb{T})$. Then*

$$\int_r^t u^{\triangle}(z)v(z) \triangle z = u(z)v(z)|_r^t - \int_r^t u^{\sigma}(z)v^{\triangle}(z) \triangle z. \tag{2.1}$$

Theorem 2.6 (Chain rule [1]). *For $\theta : \mathbb{T} \to \mathbb{R}$ is continuous and delta differentiable on $\mathbb{T}^k$, if $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then*

$$(\phi \circ \theta)^{\triangle}(\zeta) = \phi^{'}(\theta(s))\theta^{\triangle}(\zeta), \ \textit{where} \quad s \in [\zeta, \sigma(\zeta)]_{\mathbb{R}}.$$

Theorem 2.7 (Leibniz integral [4]). *Assume $h, u : \mathbb{T} \longrightarrow \mathbb{T}$ are delta differentiable. If ϕ and $\phi^{\triangle}$ are continuous, then*

$$\left[\int_{u(\zeta)}^{h(\zeta)} \phi(\zeta, z) \triangle z\right]^{\triangle} = \int_{u(\zeta)}^{h(\zeta)} \phi^{\triangle}(\zeta, z) \triangle z + h^{\triangle}(\zeta)\phi(\sigma(\zeta), h(\zeta)) - u^{\triangle}(\zeta)\phi(\sigma(\zeta), u(\zeta)),$$

where $\phi^{\triangle}(\zeta, z)$ is the delta derivative with respect to ζ.

Lemma 2.8 (Hölder inequality [1]). *If $\phi, \theta \in C_{rd}(\mathbb{T})$ and $r, t \in \mathbb{T}$, then*

$$\int_r^t |\phi(z)\theta(z)| \triangle z \leqslant \left(\int_r^t |\phi(z)|^{\omega} \triangle z\right)^{\frac{1}{\omega}} \left(\int_r^t |\theta(z)|^{\gamma} \triangle z\right)^{\frac{1}{\gamma}}, \tag{2.2}$$

where $\omega > 1$ and $1/\omega + 1/\gamma = 1$.

Lemma 2.9 ([1]). *Let $\phi, \theta \in C_{rd}(\mathbb{T}), \phi \geqslant 0, \theta \geqslant 0$ and $\vartheta \geqslant 0$. If $1 \leqslant \omega$ and $1/\omega + 1/\gamma = 1$, then*

$$\int_r^t \phi(z)\theta(z)\vartheta(z) \triangle z \leqslant \left(\int_r^t \phi^{\omega}(z)\vartheta(z) \triangle z\right)^{\frac{1}{\omega}} \left(\int_r^t \theta^{\omega}(z)\vartheta(z) \triangle z\right)^{\frac{1}{\gamma}}.$$

Lemma 2.10. *Let $\phi, \theta \in C_{rd}(\mathbb{T}), \phi \geqslant 0, \theta \geqslant 0$ and $\vartheta \geqslant 0$. If $0 < \omega \leqslant \gamma < \infty$, then*

$$\int_r^t \phi^{\omega}(z)\vartheta(z) \triangle z \leqslant \left(\int_r^t \vartheta(z) \triangle z\right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \phi^{\gamma}(z)\vartheta(z) \triangle z\right)^{\frac{\omega}{\gamma}}.$$

Proof. Let $0 < \omega \leqslant \gamma < \infty$. Applying Lemma 2.9 with $\theta = 1$ and using $\gamma/\omega \geqslant 1$, we obtain

$$\int_r^t \phi^{\omega}(z)\vartheta(z) \triangle z \leqslant \left(\int_r^t \phi^{\gamma}(z)\vartheta(z) \triangle z\right)^{\frac{\omega}{\gamma}} \left(\int_r^t \vartheta(z) \triangle z\right)^{1-\frac{\omega}{\gamma}}.$$

□

Lemma 2.11. *Let* $\phi, \theta \in C_{rd}(\mathbb{T}), \phi \geqslant 0, \theta \geqslant 0$ *and* $\vartheta \geqslant 0$. *If* $0 < \omega \leqslant \gamma < \infty, \kappa \neq 1$ *and*

$$\Omega(z) = \int_0^z \vartheta(\zeta) \triangle \zeta,$$

then

$$\int_r^t \frac{\vartheta(z)}{\Omega^\kappa(z)} \theta^\omega(\phi(z)) \triangle z \leqslant \left(\int_r^t \vartheta(z) \triangle z \right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \frac{\vartheta(z)}{\Omega^{\frac{\kappa\gamma}{\omega}}(z)} \theta^\gamma(\phi(z)) \triangle z \right)^{\frac{\omega}{\gamma}}. \tag{2.3}$$

Proof. By applying Lemma 2.10, we get

$$\begin{aligned} \int_r^t \frac{\vartheta(z)}{\Omega^\kappa(z)} \theta^\omega(\phi(z)) \triangle z &= \int_r^t \left(\frac{\theta(\phi(z))}{\Omega^{\frac{\kappa}{\omega}}(z)} \right)^\omega \vartheta(z) \triangle z \\ &\leqslant \left(\int_r^t \vartheta(z) \triangle z \right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \left(\frac{\theta(\phi(z))}{\Omega^{\frac{\kappa}{\omega}}(z)} \right)^\gamma \vartheta(z) \triangle z \right)^{\frac{\omega}{\gamma}} \\ &= \left(\int_r^t \vartheta(z) \triangle z \right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \frac{\vartheta(z)}{\Omega^{\kappa\frac{\gamma}{\omega}}(z)} \theta^\gamma(\phi(z)) \triangle z \right)^{\frac{\omega}{\gamma}}. \end{aligned}$$

□

Remark 2.12. If we put $\kappa = \omega - \delta$, then

$$\int_r^t \frac{\vartheta(z)}{\Omega^{\omega-\delta}(z)} \theta^\omega(\phi(z)) \triangle z \leqslant \left(\int_r^t \vartheta(z) \triangle z \right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \frac{\vartheta(z)}{\Omega^{\gamma-\frac{\delta\gamma}{\omega}}(z)} \theta^\gamma(\phi(z)) \triangle z \right)^{\frac{\omega}{\gamma}}.$$

3. Main results

In this section, we will assume that $r, t \in \mathbb{T}, 0 < r < t < +\infty$ with functions ϕ, θ, ϑ are rd-continous and non-negative functions. We define a nondecreasing function

$$\Omega(z) = \int_0^z \vartheta(\zeta) \triangle \zeta.$$

Theorem 3.1. *Let* $1 < \omega \leqslant \gamma < \infty, \kappa > 1$ *and* $\Omega(\infty) = \infty$. *If there exists constant* $\lambda > 0$ *such that*

$$\lambda \geqslant \frac{\kappa - 1}{\omega + \kappa - 1},$$

then

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa} [\Phi_1^\sigma(z)]^\omega \triangle z \leqslant \left(\frac{\lambda\omega}{\kappa - 1} \right)^\omega \left[\int_r^t \vartheta(z) \triangle z \right]^{1-\frac{\omega}{\gamma}} \left[\int_r^t \frac{\vartheta(z) [\Omega^\sigma(z)]^{\frac{\kappa\gamma(\omega-1)}{\omega}}}{\Omega^{\kappa\gamma}(z)} \theta^\gamma(\phi(z)) \triangle z \right]^{\frac{\omega}{\gamma}}, \tag{3.1}$$

where

$$\Phi_1(z) = \frac{1}{\Omega(z)} \int_r^z \vartheta(\zeta) \theta(\phi(\zeta)) \triangle \zeta, \tag{3.2}$$

and $\int_0^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa} \triangle \zeta < \infty$.

Proof. By applying (2.1) on the left hand side of (3.1) with

$$u^{\Delta}(z)=\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}} \quad \text{and} \quad \nu^{\sigma}(z)=(\Phi_1^{\sigma}(z))^{\omega},$$

we have

$$\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}[\Phi_1^{\sigma}(z)]^{\omega}\triangle z=[u(z)\nu(z)]_r^t+\int_r^t(-u(z))\left(\Phi_1^{\omega}(z)\right)^{\triangle}\triangle z, \tag{3.3}$$

where

$$u(z)=-\int_z^{\infty}\frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\kappa}}\triangle\zeta.$$

Applying Theorem 2.6 with $\Omega^{\triangle}(z)\geqslant 0$ and $s\in[z,\sigma(z)]_{\mathbb{R}}$, to obtain

$$\left(\Omega^{1-\kappa}(z)\right)^{\triangle}=(1-\kappa)\Omega^{-\kappa}(s)\Omega^{\triangle}(z)=(1-\kappa)\frac{\Omega^{\triangle}(z)}{\Omega^{\kappa}(s)}\leqslant(1-\kappa)\frac{\Omega^{\triangle}(z)}{[\Omega^{\sigma}(z)]^{\kappa}}. \tag{3.4}$$

Therefore, integrating (3.4) from z to ∞ with respect to ζ, we get

$$-u(z)\leqslant\frac{1}{\kappa-1}\Omega^{1-\kappa}(z), \tag{3.5}$$

from (3.3) and (3.5), we see that

$$\begin{aligned}\int_r^t\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}[\Phi_1^{\sigma}(z)]^{\omega}\triangle z&\leqslant\left[-\Phi_1^{\omega}(z)\int_z^{\infty}\frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\kappa}}\triangle\zeta\right]_r^t+\frac{1}{\kappa-1}\int_r^t\Omega^{1-\kappa}(z)\left(\Phi_1^{\omega}(z)\right)^{\triangle}\triangle z\\&=\left[-\Phi_1^{\omega}(t)\int_t^{\infty}\frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\kappa}}\triangle\zeta\right]+\frac{1}{\kappa-1}\int_r^t\Omega^{1-\kappa}(z)\left(\Phi_1^{\omega}(z)\right)^{\triangle}\triangle z\\&\leqslant\frac{1}{\kappa-1}\int_r^t\Omega^{1-\kappa}(z)\left(\Phi_1^{\omega}(z)\right)^{\triangle}\triangle z.\end{aligned} \tag{3.6}$$

Now, from (3.2), we have

$$\Phi_1^{\triangle}(z)=\frac{1}{\Omega(z)}\vartheta(z)\theta(\phi(z))-\frac{\vartheta(z)}{\Omega(z)\Omega^{\sigma}(z)}\int_r^{\sigma(z)}\vartheta(\zeta)\theta(\phi(\zeta))\triangle\zeta=\frac{\vartheta(z)\theta(\phi(z))}{\Omega(z)}-\frac{\vartheta(z)}{\Omega(z)}\Phi_1^{\sigma}(z).$$

Applying Theorem 2.6, we have

$$\begin{aligned}\left(\Phi_1^{\omega}\right)^{\triangle}(z)=\omega\Phi_1^{\omega-1}(s)\Phi_1^{\triangle}(z)&\leqslant\omega\left[\Phi_1^{\sigma}(z)\right]^{\omega-1}\Phi_1^{\triangle}(z)\\&=\omega\left[\Phi_1^{\sigma}(z)\right]^{\omega-1}\left[\frac{\vartheta(z)\theta(\phi(z))}{\Omega(z)}-\frac{\vartheta(z)}{\Omega(z)}\Phi_1^{\sigma}(z)\right].\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.6), we get

$$\begin{aligned}\int_r^t\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}[\Phi_1^{\sigma}(z)]^{\omega}\triangle z&\leqslant\frac{\omega}{\kappa-1}\int_r^t\Omega^{1-\kappa}(z)\left[\Phi_1^{\sigma}(z)\right]^{\omega-1}\left[\frac{\vartheta(z)\theta(\phi(z))}{\Omega(z)}-\frac{\vartheta(z)}{\Omega(z)}\Phi_1^{\sigma}(z)\right]\triangle z\\&=\frac{\omega}{\kappa-1}\int_r^t\frac{\vartheta(z)\theta(\phi(z))}{\Omega^{\kappa}(z)}[\Phi_1^{\sigma}(z)]^{\omega-1}\triangle z-\frac{\omega}{\kappa-1}\int_r^t\frac{\vartheta(z)}{\Omega^{\kappa}(z)}[\Phi_1^{\sigma}(z)]^{\omega}\triangle z\\&\leqslant\frac{\omega}{\kappa-1}\int_r^t\frac{\vartheta(z)\theta(\phi(z))}{\Omega^{\kappa}(z)}[\Phi_1^{\sigma}(z)]^{\omega-1}\triangle z-\frac{\omega}{\kappa-1}\int_r^t\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}[\Phi_1^{\sigma}(z)]^{\omega}\triangle z.\end{aligned}$$

Hence,

$$\frac{\omega+\kappa-1}{\kappa-1}\int_r^t\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}[\Phi_1^{\sigma}(z)]^{\omega}\triangle z\leqslant\frac{\omega}{\kappa-1}\int_r^t\frac{\vartheta(z)\theta(\phi(z))}{\Omega^{\kappa}(z)}[\Phi_1^{\sigma}(z)]^{\omega-1}\triangle z.$$

This implies that

$$\begin{aligned}\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa}[\Phi_1^\sigma(z)]^\omega \triangle z &\leqslant \frac{\lambda\omega}{\kappa-1}\int_r^t \frac{\vartheta(z)\theta(\phi(z))}{\Omega^\kappa(z)}[\Phi_1^\sigma(z)]^{\omega-1}\triangle z\\ &= \frac{\lambda\omega}{\kappa-1}\int_r^t\left(\frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa}[\Phi_1^\sigma(z)]^\omega\right)^{\frac{\omega-1}{\omega}}\frac{\vartheta(z)^{\frac{1}{\omega}}[\Omega^\sigma(z)]^{\frac{\kappa(\omega-1)}{\omega}}}{\Omega^\kappa(z)}\theta(\phi(z))\triangle z.\end{aligned}$$

By applying (2.2) with ω and $\omega/(\omega-1)$, we have

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa}[\Phi_1^\sigma(z)]^\omega \triangle z \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^\omega \int_r^t \vartheta(z)\left(\frac{(\Omega^\sigma(z))^{(\omega-1)}}{\Omega^\omega(z)}\right)^\kappa \theta^\omega(\phi(z))\triangle z. \tag{3.8}$$

Applying (2.3) on the right hand side of (3.8), we obtain

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa}[\Phi_1^\sigma(z)]^\omega \triangle z \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^\omega \left[\int_r^t \vartheta(z)\triangle z\right]^{1-\frac{\omega}{\gamma}}\left[\int_r^t \frac{\vartheta(z)\,[\Omega^\sigma(z)]^{\kappa\frac{\gamma(\omega-1)}{\omega}}}{\Omega^{\kappa\gamma}(z)}\theta^\gamma(\phi(z))\triangle z\right]^{\frac{\omega}{\gamma}}.$$

□

In Theorem 3.1 if $\mathbb{T}=\mathbb{N}$, then we have that $\sigma(n)=n+1$ and we obtain the following corollary.

Corollary 3.2. *Let* $\{\vartheta(\zeta)\}_{\zeta=1}^\infty$ *and* $\{\theta(\phi(\zeta))\}_{\zeta=1}^\infty$ *be increasing and non-negative sequences. For any* $1<\omega\leqslant\gamma<\infty$ *and* $\kappa>1$*, if there exists constant* $\lambda>0$ *such that*

$$\lambda \geqslant \frac{\kappa-1}{\omega+\kappa-1},$$

then

$$\sum_{\zeta=r}^{t-1}\frac{\vartheta(\zeta)}{\Omega^\kappa(\zeta+1)}\Phi_1^\omega(\zeta+1) \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^\omega\left(\sum_{\zeta=r}^{t-1}\vartheta(\zeta)\right)^{1-\frac{\omega}{\gamma}}\left(\sum_{\zeta=r}^{t-1}\frac{\vartheta(\zeta)\Omega^{\kappa\frac{\gamma(\omega-1)}{\omega}}(\zeta+1)}{\Omega^{\kappa\gamma}(\zeta)}\theta^\gamma(\phi(\zeta))\right)^{\frac{\omega}{\gamma}},$$

where

$$\Phi_1(\zeta)=\frac{1}{\Omega(\zeta)}\sum_{i=r}^{\zeta-1}\vartheta(i)\theta(\phi(i)) \quad \textit{and} \quad \Omega(\infty)=\infty.$$

Remark 3.3. For $\mathbb{T}=\mathbb{R}$ Theorem 3.1 reduces to Theorem 1.6.

The next result follows from Theorem 3.1 by choosing $\theta(\phi(z))=\phi(z), \vartheta(z)=1$ and $\gamma=\omega$.

Corollary 3.4. *Let* $\omega>1$ *and* $\kappa>1$*. If there exists* $\lambda>0$ *such that*

$$\lambda \geqslant \frac{\kappa-1}{\omega+\kappa-1},$$

then

$$\int_r^t \frac{1}{\sigma^{\kappa+\omega}(z)}\left(\int_r^{\sigma(z)}\phi(\zeta)\triangle\zeta\right)^\omega \triangle z \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^\omega\int_r^t\frac{\sigma^{\kappa(\omega-1)}(z)}{z^{\kappa\omega}}\phi^\omega(z)\triangle z.$$

Remark 3.5. For $\mathbb{T}=\mathbb{R}$, Corollary 3.4 reduces to [2, Corollary 1].

Theorem 3.6. *Let* $1 < \omega \leqslant \gamma < \infty$ *and* $\kappa + \omega < 1$. *If there exists constant* $\lambda > 0$ *such that*

$$\lambda \geqslant \frac{1-\kappa}{1-\kappa-\omega},$$

then

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa}\tilde{\Phi}_1^\omega(z)\triangle z \leqslant \left(\frac{\lambda\omega}{1-\kappa}\right)^\omega \left[\int_r^t \vartheta(z)\triangle z\right]^{1-\frac{\omega}{\gamma}} \left[\int_r^t \frac{\vartheta(z)\theta^\gamma(\phi(z))}{[\Omega^\sigma(z)]^{\kappa\frac{\gamma}{\omega}}}\triangle z\right]^{\frac{\omega}{\gamma}}, \tag{3.9}$$

where

$$\tilde{\Phi}_1(z) = \frac{1}{\Omega(z)}\int_z^t \vartheta(\zeta)\theta(\phi(\zeta))\triangle \zeta, \tag{3.10}$$

and $\int_0^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa}\triangle\zeta < \infty$.

Proof. By applying (2.1) on the left hand side of (3.9) with

$$u^\triangle(z) = \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa} \quad \text{and} \quad v(z) = \tilde{\Phi}_1^\omega(z),$$

we have

$$\begin{aligned}\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa}\tilde{\Phi}_1^\omega(z)\triangle z &= [u(z)v(z)]_r^t + \int_r^t u^\sigma(z)\left(-\tilde{\Phi}_1^\omega\right)^\triangle(z)\triangle z \\ &= \left[-v(z)\int_z^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa}\triangle\zeta\right]_r^t + \int_r^t u^\sigma(z)\left(-\tilde{\Phi}_1^\omega\right)^\triangle(z)\triangle z \\ &= \left[-v(r)\int_r^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa}\triangle\zeta\right] + \int_r^t u^\sigma(z)\left(-\tilde{\Phi}_1^\omega\right)^\triangle(z)\triangle z \\ &\leqslant \int_r^t u^\sigma(z)\left(-\tilde{\Phi}_1^\omega\right)^\triangle(z)\triangle z,\end{aligned} \tag{3.11}$$

where

$$u(z) = -\int_z^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa}\triangle\zeta.$$

By applying Theorem 2.6 with $\Omega^\triangle(z) \geqslant 0$ and $s \in [z, \sigma(z)]_{\mathbb{R}}$, we obtain

$$\left(\Omega^{1-\kappa}(z)\right)^\triangle = (1-\kappa)\Omega^{-\kappa}(s)\Omega^\triangle(z) = (1-\kappa)\frac{\Omega^\triangle(z)}{\Omega^\kappa(s)} \leqslant (1-\kappa)\frac{\Omega^\triangle(z)}{[\Omega^\sigma(z)]^\kappa}. \tag{3.12}$$

Therefore, integrating (3.12) from z to ∞ with respect to ζ, we obtain

$$u(z) \leqslant \frac{1}{1-\kappa}\Omega^{1-\kappa}(z), \tag{3.13}$$

substituting (3.13) into (3.11), we see that

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa}\tilde{\Phi}_1^\omega(z)\triangle z \leqslant \frac{1}{1-\kappa}\int_r^t [\Omega^\sigma(z)]^{1-\kappa}\left(-\tilde{\Phi}_1^\omega(z)\right)^\triangle \triangle z. \tag{3.14}$$

Now, from (3.10), we have

$$\tilde{\Phi}_1^\triangle(z) = \frac{-1}{\Omega^\sigma(z)}\vartheta(z)\theta(\phi(z)) - \frac{\vartheta(z)}{\Omega(z)\Omega^\sigma(z)}\int_z^t \vartheta(\zeta)\theta(\phi(\zeta))\triangle\zeta = -\left[\frac{\vartheta(z)\theta(\phi(z))}{\Omega^\sigma(z)} + \frac{\vartheta(z)}{\Omega^\sigma(z)}\tilde{\Phi}_1(z)\right].$$

Applying Theorem 2.6 with $\tilde{\Phi}_1^{\triangle}(z) \leqslant 0$, we have

$$\left(\tilde{\Phi}_1^{\omega}(z)\right)^{\triangle} = \omega\tilde{\Phi}_1^{\omega-1}(s)\tilde{\Phi}_1^{\triangle}(z) \geqslant \omega\tilde{\Phi}_1^{\omega-1}(z)\tilde{\Phi}_1^{\triangle}(z) = -\omega\tilde{\Phi}_1^{\omega-1}(z)\left[\frac{\vartheta(z)\theta(\phi(z))}{\Omega^{\sigma}(z)} + \frac{\vartheta(z)}{\Omega^{\sigma}(z)}\tilde{\Phi}_1(z)\right], \quad (3.15)$$

substituting (3.15) into (3.14), it yields

$$\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega}(z)\triangle z \leqslant \frac{\omega}{1-\kappa}\int_r^t \frac{\vartheta(z)\theta(\phi(z))}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega-1}(z)\triangle z + \frac{\omega}{1-\kappa}\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega}(z)\triangle z.$$

Hence,

$$\frac{1-\kappa-\omega}{1-\kappa}\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega}(z)\Delta z \leqslant \frac{\omega}{1-\kappa}\int_r^t \frac{\vartheta(z)\theta(\phi(z))}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega-1}(z)\triangle z.$$

This implies that

$$\begin{aligned}\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega}(z)\triangle z &\leqslant \frac{\lambda\omega}{1-\kappa}\int_r^t \frac{\vartheta(z)\theta(\phi(z))}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega-1}(z)\triangle z\\ &= \frac{\lambda\omega}{1-\kappa}\int_r^t \left(\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega}(z)\right)^{\frac{\omega-1}{\omega}}\vartheta^{\frac{1}{\omega}}(z)(\Omega^{\sigma}(z))^{-\frac{\kappa}{\omega}}\theta(\phi(z))\triangle z.\end{aligned}$$

By applying (2.2) with ω and $\omega/(\omega-1)$, we have

$$\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega}(z)\triangle z \leqslant \left(\frac{\lambda\omega}{1-\kappa}\right)^{\omega}\int_r^t \frac{\vartheta(z)\theta^{\omega}(\phi(z))}{(\Omega^{\sigma}(z))^{\kappa}}\triangle z. \quad (3.16)$$

Applying (2.3) on the right hand side of (3.16), we obtain

$$\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\tilde{\Phi}_1^{\omega}(z)\triangle z \leqslant \left(\frac{\lambda\omega}{1-\kappa}\right)^{\omega}\left(\int_r^t \vartheta(z)\triangle z\right)^{1-\frac{\omega}{\gamma}}\left(\int_r^t \frac{\vartheta(z)\theta^{\gamma}(\phi(z))}{[\Omega^{\sigma}(z)]^{\kappa\frac{\gamma}{\omega}}}\triangle z\right)^{\frac{\omega}{\gamma}}.$$

□

In Theorem 3.6 if $\mathbb{T} = \mathbb{N}$, then we obtain the following corollary.

Corollary 3.7. *Let* $\{\vartheta(\zeta)\}_{\zeta=1}^{\infty}$ *and* $\{\theta(\phi(\zeta))\}_{\zeta=1}^{\infty}$ *be increasing and non-negative sequences. For any* $1 < \omega \leqslant \gamma < \infty$ *and* $\kappa + \omega < 1$, *if there exists constant* $\lambda > 0$ *such that*

$$\lambda \geqslant \frac{1-\kappa}{1-\kappa-\omega},$$

then

$$\sum_{\zeta=r}^{t-1}\frac{\vartheta(\zeta)}{\Omega^{\kappa}(\zeta+1)}\tilde{\Phi}_1^{\omega}(\zeta) \leqslant \left(\frac{\lambda\omega}{1-\kappa}\right)^{\omega}\left(\sum_{\zeta=r}^{t-1}\vartheta(\zeta)\right)^{1-\frac{\omega}{\gamma}}\left(\sum_{\zeta=r}^{t-1}\frac{\vartheta(\zeta)}{\Omega^{\kappa\frac{\gamma}{\omega}}(\zeta+1)}\theta^{\gamma}(\phi(\zeta))\right)^{\frac{\omega}{\gamma}},$$

where

$$\tilde{\Phi}_1(\zeta) = \frac{1}{\Omega(\zeta)}\sum_{i=\zeta}^{t-1}\vartheta(i)\theta(\phi(i)).$$

Remark 3.8. For $\mathbb{T} = \mathbb{R}$, Theorem 3.6 reduces to Theorem 1.7.

The next result follows from Theorem 3.6 by choosing $\theta(\phi(z)) = \phi(z), \vartheta(z) = 1$ and $\gamma = \omega$.

Corollary 3.9. *Let $1 < \omega \leqslant \gamma < \infty$ and $\kappa + \omega < 1$. If there exists $\lambda > 0$ such that*

$$\lambda \geqslant \frac{1-\kappa}{1-\kappa-\omega},$$

then

$$\int_r^t \frac{1}{\sigma^\kappa(z)} \left(\frac{1}{z}\int_z^t \phi(\zeta) \triangle \zeta\right)^\omega \triangle z \leqslant \left(\frac{\lambda\omega}{1-\kappa}\right)^\omega \int_r^t \frac{1}{\sigma^\kappa(z)} \phi^\omega(z) \triangle z.$$

Remark 3.10. For $\mathbb{T} = \mathbb{R}$, Corollary 3.9 reduces to [2, Corollary 2].

Theorem 3.11. *Let $\frac{z}{2}, \frac{\sigma(z)}{2} \in \mathbb{T}, \omega > 1$ and $\kappa > 1$. If there exists constant $\lambda > 0$ such that*

$$\lambda \geqslant \frac{\kappa-1}{\omega+\kappa-1},$$

then

$$\int_0^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa} [\Phi_3^\sigma(z)]^\omega \triangle z \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^\omega \int_0^t \frac{[\Omega^\sigma(z)]^{\kappa(\omega-1)}}{\Omega^{\kappa\omega}(z)} \vartheta(z) |\psi(z)|^\omega \triangle z, \tag{3.17}$$

where

$$\Phi_3(z) = \frac{1}{\Omega(z)} \int_{\frac{z}{2}}^z \vartheta(\zeta)\theta(\phi(\zeta)) \triangle \zeta, \quad \psi(z) = \theta(\phi(z)) - \frac{\vartheta(\frac{z}{2})\theta(\phi(\frac{z}{2}))}{2\vartheta(z)}, \quad \Omega(\infty) = \infty, \tag{3.18}$$

and $\int_0^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa} \triangle \zeta < \infty$.

Proof. By applying (2.1) on the left hand side of (3.17) with

$$u^\triangle(z) = \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa} \text{ and } v^\sigma(z) = (\Phi_3^\sigma(z))^\omega,$$

we have

$$\begin{aligned} \int_0^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa} [\Phi_3^\sigma(z)]^\omega \triangle z &= [u(z)v(z)]_0^t + \int_0^t (-u(z)) (\Phi_3^\omega(z))^\triangle \triangle z \\ &= \left[-\Phi_3^\omega(z) \int_z^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa} \triangle \zeta\right]_0^t + \int_0^t (-u(z)) (\Phi_3^\omega(z))^\triangle \triangle z \\ &\leqslant \int_0^t (-u(z)) (\Phi_3^\omega(z))^\triangle \triangle z, \end{aligned} \tag{3.19}$$

where

$$u(z) = -\int_z^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^\kappa} \triangle \zeta.$$

By applying Theorem 2.6 with $\Omega^\triangle(z) \geqslant 0$ and $s \in [z, \sigma(z)]_\mathbb{R}$, we obtain

$$\left(\Omega^{1-\kappa}(z)\right)^\triangle = (1-\kappa)\Omega^{-\kappa}(s)\Omega^\triangle(z) = (1-\kappa)\frac{\Omega^\triangle(z)}{\Omega^\kappa(s)} \leqslant (1-\kappa)\frac{\Omega^\triangle(z)}{[\Omega^\sigma(z)]^\kappa}, \tag{3.20}$$

therefore, integrating (3.20) from z to ∞ with respect to ζ, we obtain

$$-u(z) \leqslant \frac{1}{\kappa-1}\Omega^{1-\kappa}(z), \tag{3.21}$$

substituting (3.21) into (3.19), we see that

$$\int_0^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^\kappa} [\Phi_3^\sigma(z)]^\omega \triangle z \leqslant \frac{1}{\kappa-1} \int_0^t \Omega^{1-\kappa}(z) (\Phi_3^\omega(z))^\triangle \triangle z. \tag{3.22}$$

Now, from (3.18), we have

$$\Phi_3^{\triangle}(z) = \frac{\vartheta(z)}{\Omega(z)}\psi(z) - \frac{\vartheta(z)}{\Omega(z)}\Phi_3^{\sigma}(z).$$

Applying Theorem 2.6, we have

$$\begin{aligned}(\Phi_3^{\omega}(z))^{\triangle} &= \omega\Phi_3^{\omega-1}(s)\Phi_3^{\triangle}(z) \leqslant \omega\left(\Phi_3^{\sigma}(z)\right)^{\omega-1}\Phi_3^{\triangle}(z)\\ &= \omega\left(\Phi_3^{\sigma}(z)\right)^{\omega-1}\left[\frac{\vartheta(z)}{\Omega(z)}\psi(z) - \frac{\vartheta(z)}{\Omega(z)}\Phi_3^{\sigma}(z)\right].\end{aligned} \tag{3.23}$$

Substituting (3.23) into (3.22), yields

$$\begin{aligned}\int_0^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\left[\Phi_3^{\sigma}(z)\right]^{\omega}\triangle z &\leqslant \frac{\omega}{\kappa-1}\int_0^t \frac{\vartheta(z)\left|\psi(z)\right|}{\Omega^{\kappa}(z)}\left[\Phi_3^{\sigma}(z)\right]^{\omega-1}\triangle z - \frac{\omega}{\kappa-1}\int_0^t \frac{\vartheta(z)}{\Omega^{\kappa}(z)}\left[\Phi_3^{\sigma}(z)\right]^{\omega}\triangle z\\ &\leqslant \frac{\omega}{\kappa-1}\int_0^t \frac{\vartheta(z)\left|\psi(z)\right|}{\Omega^{\kappa}(z)}\left[\Phi_3^{\sigma}(z)\right]^{\omega-1}\triangle z - \frac{\omega}{\kappa-1}\int_0^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\left[\Phi_3^{\sigma}(z)\right]^{\omega}\triangle z.\end{aligned}$$

Hence,

$$\frac{\omega+\kappa-1}{\kappa-1}\int_0^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\left[\Phi_3^{\sigma}(z)\right]\triangle z \leqslant \frac{\omega}{\kappa-1}\int_0^t \frac{\vartheta(z)\left|\psi(z)\right|}{\Omega^{\kappa}(z)}\left[\Phi_3^{\sigma}(z)\right]^{\omega-1}\triangle z.$$

This implies that

$$\begin{aligned}\int_0^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\left[\Phi_3^{\sigma}(z)\right]^{\omega}\triangle z &\leqslant \frac{\lambda\omega}{\kappa-1}\int_0^t \frac{\vartheta(z)\left|\psi(z)\right|}{\Omega^{\kappa}(z)}\left[\Phi_3^{\sigma}(z)\right]^{\omega-1}\triangle z\\ &= \frac{\lambda\omega}{\kappa-1}\int_0^t \left(\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\left[\Phi_3^{\sigma}(z)\right]^{\omega}\right)^{\frac{\omega-1}{\omega}} \frac{\vartheta^{\frac{1}{\omega}}(z)\left|\psi(z)\right|\left[\Omega^{\sigma}(z)\right]^{\kappa\left(\frac{\omega-1}{\omega}\right)}}{\Omega^{\kappa}(z)}\triangle z.\end{aligned}$$

Now, applying Hölder's inequality (2.2) with ω and $\omega/(\omega-1)$, we have

$$\int_0^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\kappa}}\left[\Phi_3^{\sigma}(z)\right]^{\omega}\triangle z \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^{\omega}\int_0^t \frac{[\Omega^{\sigma}(z)]^{\kappa(\omega-1)}}{\Omega^{\kappa\omega}(z)}\vartheta(z)\left|\psi(z)\right|^{\omega}\triangle z.$$

□

In Theorem 3.11 if $\mathbb{T}=\mathbb{N}$, then we obtain the following corollary.

Corollary 3.12. *Let $\{\vartheta(\zeta)\}_{\zeta=1}^{\infty}$ and $\{\theta(\phi(\zeta))\}_{\zeta=1}^{\infty}$ be increasing and non-negative sequences. For any $\omega>1$ and $\kappa>1$, if there exists constant $\lambda>0$ such that*

$$\lambda \geqslant \frac{\kappa-1}{\omega+\kappa-1},$$

then

$$\sum_{z=0}^{t-1}\frac{\vartheta(z)}{\Omega^{\kappa}(z+1)}\Phi_3^{\omega}(z+1) \leqslant \left(\frac{\lambda\omega}{\kappa-1}\right)^{\omega}\sum_{z=0}^{t-1}\frac{\Omega^{\kappa(\omega-1)}(z+1)}{\Omega^{\kappa\omega}(z)}\vartheta(z)\left|\psi(z)\right|^{\omega},$$

where

$$\Phi_3(z) = \frac{1}{\Omega(z)}\sum_{i=\frac{z}{2}}^{z-1}\vartheta(i)\theta(\phi(i)), \quad \textit{for}\ \frac{z}{2},\frac{z+1}{2}\in\mathbb{N}, \quad \psi(z)=\theta(\phi(z)) - \frac{\vartheta(\frac{z}{2})\theta(\phi(\frac{z}{2}))}{2\vartheta(z)} \quad \textit{and} \quad \Omega(\infty)=\infty.$$

Remark 3.13. For $\mathbb{T}=\mathbb{R}$, Theorem 3.11 reduces to [2, Theorem 3].

The next result follows from Theorem 3.11 by choosing $\theta(\phi(z)) = \phi(z), \vartheta(z) = 1$ and $\gamma = \omega$.

Corollary 3.14. *Let $\omega > 1$ and $\kappa > 1$. If there exists $\lambda > 0$ such that*

$$\lambda \geqslant \frac{1-\kappa}{1-\kappa-\omega},$$

then

$$\int_0^t \frac{1}{\sigma^{\kappa+\omega}(z)} \left(\int_{\frac{\sigma(z)}{2}}^{\sigma(z)} \phi(\zeta) \triangle \zeta \right)^{\omega} \triangle z \leqslant \left(\frac{\lambda\omega}{\kappa-1} \right)^{\omega} \int_0^t \frac{\sigma^{\kappa(\omega-1)}(z)}{z^{\kappa\omega}} \left| \phi(z) - \frac{1}{2}\phi(\frac{z}{2}) \right|^{\omega} \triangle z.$$

Remark 3.15. For $\mathbb{T} = \mathbb{R}$, Corollary 3.14 reduces to [2, Corollary 3].

Theorem 3.16. *If $\delta < \omega - 1$ and $1 < \omega \leqslant \gamma < \infty$, then*

$$\begin{aligned} &\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} [\Phi_2^{\sigma}(z)]^{\omega} \triangle z \\ &\leqslant \left(\frac{\omega}{\omega-\delta-1} \right)^{\omega} \left(\int_r^t \vartheta(z) \triangle z \right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \left(\frac{[\Omega^{\sigma}(z)]^{\omega-1}}{\Omega^{\omega}(z)} \right)^{\gamma-\frac{\delta\gamma}{\omega}} \vartheta(z)\theta^{\gamma}(\phi(z)) \triangle z \right)^{\frac{\omega}{\gamma}}, \end{aligned} \tag{3.24}$$

where

$$\Phi_2(z) = \int_r^z \frac{\vartheta(\zeta)}{\Omega(\zeta)} \theta(\phi(\zeta)) \triangle \zeta, \quad \Omega(\infty) = \infty,$$

and $\int_0^{\infty} \frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\omega-\delta}} \triangle \zeta < \infty$.

Proof. By applying (2.1) on the left hand side of (3.24) with

$$u^{\Delta}(z) = \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} \quad \text{and} \quad v^{\sigma}(z) = (\Phi_2^{\sigma}(z))^{\omega},$$

we have

$$\begin{aligned} \int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} [\Phi_2^{\sigma}(z)]^{\omega} \Delta z &= [u(z)v(z)]_r^t + \int_r^t (-u(z)) (\Phi_2^{\omega})^{\triangle}(z) \triangle z \\ &= \left[-\Phi_2^{\omega}(z) \int_z^{\infty} \frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\omega-\delta}} \triangle \zeta \right]_r^t + \int_r^t (-u(z)) (\Phi_2^{\omega})^{\triangle}(z) \triangle z \\ &\leqslant \int_r^t (-u(z)) (\Phi_2^{\omega})^{\triangle}(z) \triangle z, \end{aligned} \tag{3.25}$$

where

$$u(z) = -\int_z^{\infty} \frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\omega-\delta}} \triangle \zeta.$$

By applying Theorem 2.6 with $\Omega^{\triangle}(z) \geqslant 0$ and $s \in [z, \sigma(z)]_{\mathbb{R}}$, we obtain

$$\begin{aligned} \left(\Omega^{1-(\omega-\delta)}(z) \right)^{\triangle} &= (1-(\omega-\delta))\Omega^{-(\omega-\delta)}(s)\Omega^{\triangle}(z) \\ &= (1-(\omega-\delta)) \frac{\Omega^{\triangle}(z)}{\Omega^{\omega-\delta}(s)} \leqslant (1-(\omega-\delta)) \frac{\Omega^{\triangle}(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}. \end{aligned} \tag{3.26}$$

Therefore, integrating (3.26) from z to ∞ with respect to ζ, we get

$$-u(z) \leqslant \frac{1}{\omega-\delta-1} \Omega^{1-(\omega-\delta)}(z). \tag{3.27}$$

Substituting (3.27) into (3.25), we see that

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\left[\Phi_2^\sigma(z)\right]^\omega \Delta z \leqslant \frac{1}{\omega-\delta-1}\int_r^t \Omega^{1-(\omega-\delta)}(z)\left(\Phi_2^\omega\right)^\triangle(z)\triangle z. \tag{3.28}$$

Applying Theorem 2.6, we have

$$\left(\Phi_2^\omega\right)^\triangle(z) = \omega\,\Phi_2^{\omega-1}(s)\Phi_2^\triangle(z) \leqslant \omega\left[\Phi_2^\sigma(z)\right]^{\omega-1}\Phi_2^\triangle(z) = \omega\frac{\vartheta(z)\theta(\phi(z))}{\Omega(z)}\left[\Phi_2^\sigma(z)\right]^{\omega-1}. \tag{3.29}$$

Substituting (3.29) into (3.28), we have

$$\begin{aligned}
&\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\left(\Phi_2^\sigma(z)\right)^\omega \Delta z\\
&\quad\leqslant \frac{\omega}{\omega-\delta-1}\int_r^t \Omega^{1-(\omega-\delta)}(z)\frac{\vartheta(z)}{\Omega(z)}\theta(\phi(z))\left(\Phi_2^\sigma(z)\right)^{\omega-1}\triangle z\\
&\quad= \frac{\omega}{\omega-\delta-1}\int_r^t \frac{\vartheta(z)\theta(\phi(z))}{\Omega^{\omega-\delta}(z)}\left(\Phi_2^\sigma(z)\right)^{\omega-1}\triangle z\\
&\quad= \frac{\omega}{\omega-\delta-1}\int_r^t \left(\frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\left(\Phi_2^\sigma(z)\right)^\omega\right)^{\frac{\omega-1}{\omega}} \frac{\vartheta^{\frac{1}{\omega}}(z)\theta(\phi(z))}{\Omega^{\omega-\delta}(z)}\left[\Omega^\sigma(z)\right]^{(\omega-\delta)\frac{\omega-1}{\omega}}\triangle z.
\end{aligned}$$

Now, applying Hölder's inequality (2.2) with ω and $\omega/(\omega-1)$, we have

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\left[\Phi_2^\sigma(z)\right]^\omega \Delta z \leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^\omega \int_r^t \left(\frac{[\Omega^\sigma(z)]^{(\omega-1)}}{\Omega^\omega(z)}\right)^{\omega-\delta}\vartheta(z)\theta^\omega(\phi(z))\triangle z. \tag{3.30}$$

Applying (2.3) to the right hand side of (3.30), we have

$$\begin{aligned}
&\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\left[\Phi_2^\sigma(z)\right]^\omega \Delta z\\
&\quad\leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^\omega \left(\int_r^t \vartheta(z)\triangle z\right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \left(\frac{[\Omega^\sigma(z)]^{\omega-1}}{\Omega^\omega(z)}\right)^{\gamma-\frac{\delta\gamma}{\omega}}\vartheta(z)\theta^\gamma(\phi(z))\triangle z\right)^{\frac{\omega}{\gamma}}.
\end{aligned}$$

□

In Theorem 3.16 if $\mathbb{T}=\mathbb{N}$, then we obtain the following corollary.

Corollary 3.17. *Let* $\{\vartheta(z)\}_{z=1}^\infty$ *and* $\{\theta(\phi(z))\}_{z=1}^\infty$ *be increasing and non-negative sequences. If* $\delta < \omega-1$ *and* $1<\omega\leqslant\gamma<\infty$*, then*

$$\begin{aligned}
&\sum_{z=r}^{t-1}\frac{\vartheta(z)}{\Omega^{\omega-\delta}(z+1)}\Phi_2^\omega(z+1)\\
&\quad\leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^\omega \left(\sum_{r=z}^{t-1}\vartheta(z)\right)^{1-\frac{\omega}{\gamma}} \left(\sum_{z=r}^{t-1}\left(\frac{\Omega^{\omega-1}(z+1)}{\Omega^\omega(z)}\right)^{\gamma-\frac{\delta\gamma}{\omega}}\vartheta(z)\theta^\gamma(\phi(z))\right)^{\frac{\omega}{\gamma}},
\end{aligned}$$

where

$$\Phi_2(z) = \sum_{i=r}^{z-1}\frac{\vartheta(i)}{\Omega(i)}\theta(\phi(i)) \quad \textit{and} \quad \Omega(\infty)=\infty.$$

Remark 3.18. For $\mathbb{T} = \mathbb{R}$, Theorem 3.16 reduces to [2, Theorem 4].

The next result follows from Theorem 3.16 by choosing $\theta(\phi(z)) = z\phi(z), \vartheta(z) = 1$ and $\gamma = \omega$.

Corollary 3.19. *If $\delta < \omega - 1$ and $1 < \omega < \infty$, then*

$$\int_r^t \frac{1}{\sigma^{\omega-\delta}(z)} \left(\int_r^{\sigma(z)} \phi(\zeta) \triangle \zeta \right)^{\omega} \triangle z \leqslant \left(\frac{\omega}{\omega - \delta - 1} \right)^{\omega} \int_r^t \left(\frac{\sigma^{\omega-1}(z)}{z^{\omega}} \right)^{\omega-\delta} (z\phi(z))^{\omega} \triangle z. \tag{3.31}$$

Remark 3.20. In Corollary 3.19, if $\mathbb{T} = \mathbb{R}, r = 0$ and $t \to \infty$, then (3.31) reduce to (1.6).

Theorem 3.21. *If $\delta > \omega - 1$ and $1 < \omega \leqslant \gamma < \infty$, then*

$$\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} \tilde{\Phi}_2^{\omega}(z) \triangle z \leqslant \left(\frac{\omega}{\delta - \omega + 1} \right)^{\omega} \left(\int_r^t \vartheta(z) \triangle z \right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\gamma - \frac{\delta\gamma}{\omega}}} \theta^{\gamma}(\phi(z)) \triangle z \right)^{\frac{\omega}{\gamma}}, \tag{3.32}$$

where

$$\tilde{\Phi}_2(z) = \int_z^t \frac{\vartheta(\zeta)}{\Omega^{\sigma}(\zeta)} \theta(\phi(\zeta)) \triangle \zeta.$$

Proof. By applying (2.1) on the left hand side of (3.32) with

$$u^{\Delta}(z) = \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} \quad \text{and} \quad v(z) = \tilde{\Phi}_2^{\omega}(z),$$

we have

$$\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} \tilde{\Phi}_2^{\omega}(z) \Delta z = [u(z)v(z)]_r^t + \int_r^t u^{\sigma}(z) \left(-\tilde{\Phi}_2^{\omega}\right)^{\triangle}(z) \triangle z = \int_r^t u^{\sigma}(z) \left(-\tilde{\Phi}_2^{\omega}\right)^{\triangle}(z) \triangle z, \tag{3.33}$$

where

$$u(z) = \int_r^z \frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\omega-\delta}} \triangle \zeta.$$

By applying Theorem 2.6 with $\Omega^{\triangle}(z) \geqslant 0$ and $s \in [z, \sigma(z)]_{\mathbb{R}}$, we obtain

$$\begin{aligned} \left(\Omega^{1-(\omega-\delta)}(z)\right)^{\triangle} &= (1 - (\omega - \delta))\Omega^{-(\omega-\delta)}(s)\Omega^{\triangle}(z) \\ &= (1 - (\omega - \delta)) \frac{\Omega^{\triangle}(z)}{\Omega^{\omega-\delta}(s)} \geqslant (1 - (\omega - \delta)) \frac{\Omega^{\triangle}(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}. \end{aligned} \tag{3.34}$$

Therefore, integrating (3.34) from r to z with respect to ζ, we have

$$u(z) \leqslant \frac{1}{1 - (\omega - \delta)} \Omega^{1-(\omega-\delta)}(z). \tag{3.35}$$

Now, from (3.35) and (3.33), we obtain

$$\int_r^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} \tilde{\Phi}_2^{\omega}(z) \triangle z \leqslant \frac{1}{1 - (\omega - \delta)} \int_r^t [\Omega^{\sigma}(z)]^{1-(\omega-\delta)} \left(-\tilde{\Phi}_2^{\omega}\right)^{\triangle}(z) \triangle z, \tag{3.36}$$

applying Theorem 2.6 with $\tilde{\Phi}_2^{\triangle}(z) \leqslant 0$, we have

$$\left(\tilde{\Phi}_2^{\omega}\right)^{\triangle}(z) = \omega \tilde{\Phi}_2^{\omega-1}(s) \tilde{\Phi}_2^{\triangle}(z) \geqslant \omega \tilde{\Phi}_2^{\omega-1}(z) \tilde{\Phi}_2^{\triangle}(z) = -\omega \tilde{\Phi}_2^{\omega-1}(z) \frac{\vartheta(z)}{\Omega^{\sigma}(z)} \theta(\phi(z)). \tag{3.37}$$

Substituting (3.37) into (3.36), we obtain

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\tilde{\Phi}_2^\omega(z)\Delta z \leqslant \frac{\omega}{\delta-\omega+1}\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\theta(\phi(z))\tilde{\Phi}_2^{\omega-1}(z)\triangle z$$
$$= \frac{\omega}{\delta-\omega+1}\int_r^t \left(\frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\tilde{\Phi}_2^\omega(z)\right)^{\frac{\omega-1}{\omega}} \frac{\vartheta^{\frac{1}{\omega}}(z)}{[\Omega^\sigma(z)]^{\frac{\omega-\delta}{\omega}}}\theta(\phi(z))\triangle z.$$

Now, applying Hölder's inequality (2.2) with ω and $\omega/(\omega-1)$, we have

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\tilde{\Phi}_2^\omega(z)\Delta z \leqslant \left(\frac{\omega}{\delta-\omega+1}\right)^\omega \int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\theta^\omega(\phi(z))\triangle z, \tag{3.38}$$

by applying (2.3) on the right hand side of (3.38), we get

$$\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}\tilde{\Phi}_2^\omega(z)\Delta z \leqslant \left(\frac{\omega}{\delta-\omega+1}\right)^\omega \left(\int_r^t \vartheta(z)\triangle z\right)^{1-\frac{\omega}{\gamma}} \left(\int_r^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\gamma-\frac{\delta\gamma}{\omega}}}\theta^\gamma(\phi(z))\triangle z\right)^{\frac{\omega}{\gamma}}.$$

□

In Theorem 3.21 if $\mathbb{T}=\mathbb{N}$, then we obtain the following corollary.

Corollary 3.22. *Let $\{\vartheta(z)\}_{z=1}^\infty$ and $\{\theta(\phi(z))\}_{z=1}^\infty$ be increasing and non-negative sequences. If $\delta > \omega-1$ and $1<\omega\leqslant\gamma<\infty$, then*

$$\sum_{z=r}^{t-1} \frac{\vartheta(z)}{\Omega^{\omega-\delta}(z+1)}\tilde{\Phi}_2^\omega(z) \leqslant \left(\frac{\omega}{\delta-\omega+1}\right)^\omega \left(\sum_{z=r}^{t-1}\vartheta(z)\right)^{1-\frac{\omega}{\gamma}} \left(\sum_{z=r}^{t-1} \frac{\vartheta(z)}{\Omega^{\gamma-\frac{\delta\gamma}{\omega}}(z+1)}\theta^\gamma(\phi(z))\right)^{\frac{\omega}{\gamma}},$$

where

$$\tilde{\Phi}_2(z) = \sum_{i=z}^{t-1}\frac{\vartheta(i)}{\Omega(i+1)}\theta(\phi(i)).$$

Remark 3.23. For $\mathbb{T}=\mathbb{R}$, Theorem 3.21 reduces to [2, Theorem 5].

The next result follows from Theorem 3.21 by choosing $\theta(\phi(z)) = z\phi(z), \vartheta(z)=1$ and $\gamma=\omega$.

Corollary 3.24. *If $\delta>\omega-1$ and $1<\omega<\infty$, then*

$$\int_r^t \frac{1}{\sigma^{\omega-\delta}(z)}\left(\int_z^t \phi(\zeta)\triangle\zeta\right)^\omega \Delta z \leqslant \left(\frac{\omega}{\delta-\omega+1}\right)^\omega \int_r^t \frac{z^\omega}{\sigma^{1-\delta}(z)}\phi^\omega(z)\triangle z. \tag{3.39}$$

Remark 3.25. In Corollary 3.24, if $\mathbb{T}=\mathbb{R}, r=0$ and $t\to\infty$, then (3.39) reduces to (1.7).

Theorem 3.26. *If $\delta<\omega-1$ and $1<\omega<\infty$, then*

$$\int_0^t \frac{\vartheta(z)}{[\Omega^\sigma(z)]^{\omega-\delta}}[\Phi_4^\sigma(z)]^\omega\triangle z \leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^\omega \int_0^t \frac{[\Omega^\sigma(z)]^{(\omega-1)(\omega-\delta)}}{\Omega^{\omega(\omega-\delta)}(z)}\vartheta(z)\,|\psi_2(z)|^\omega\triangle z, \tag{3.40}$$

where

$$\Phi_4(z) = \int_{\frac{z}{2}}^z \frac{\vartheta(\zeta)}{\Omega(\zeta)}\theta(\phi(\zeta))\triangle\zeta, \quad \psi_2(z)=\theta(\phi(z)) - \frac{\vartheta(\frac{z}{2})\Omega(z)}{2\vartheta(z)\Omega(\frac{z}{2})}\theta(\phi(\frac{z}{2})), \ \textit{and} \quad \Omega(\infty)=\infty, \tag{3.41}$$

and $\int_0^\infty \frac{\vartheta(\zeta)}{[\Omega^\sigma(\zeta)]^{\delta-\omega}}\triangle\zeta<\infty$.

Proof. By applying (2.1) on the left hand side of (3.40) with

$$u^{\Delta}(z)=\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}} \quad \text{and} \quad \nu^{\sigma}(z)=(\Phi_4^{\sigma}(z))^{\omega},$$

we have

$$\begin{aligned}\int_0^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}\left(\Phi_4^{\sigma}(z)\right)^{\omega}\triangle z&=[u(z)\nu(z)]_0^t+\int_0^t(-u(z))\left(\Phi_4^{\omega}\right)^{\triangle}(z)\triangle z\\&=\left[-\Phi_4^{\omega}(z)\int_z^{\infty}\frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\omega-\delta}}\triangle\zeta\right]_0^t+\int_0^t(-u(z))\left(\Phi_4^{\omega}\right)^{\triangle}(z)\triangle z\\&=\left[-\Phi_4^{\omega}(t)\int_t^{\infty}\frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\omega-\delta}}\triangle\zeta\right]+\int_0^t(-u(z))\left(\Phi_4^{\omega}\right)^{\triangle}(z)\triangle z\\&\leqslant\int_0^t(-u(z))\left(\Phi_4^{\omega}\right)^{\triangle}(z)\triangle z,\end{aligned}\tag{3.42}$$

where

$$u(z)=-\int_z^{\infty}\frac{\vartheta(\zeta)}{[\Omega^{\sigma}(\zeta)]^{\omega-\delta}}\triangle\zeta.$$

By applying Theorem 2.6 with $\Omega^{\triangle}(z)\geqslant 0$ and $s\in[z,\sigma(z)]_{\mathbb{R}}$, we obtain

$$\begin{aligned}\left(\Omega^{1-(\omega-\delta)}(z)\right)^{\triangle}&=(1-(\omega-\delta))\,\Omega^{-(\omega-\delta)}(s)\Omega^{\triangle}(z)\\&=(1-(\omega-\delta))\frac{\Omega^{\triangle}(z)}{\Omega^{\omega-\delta}(s)}\leqslant(1-(\omega-\delta))\frac{\Omega^{\triangle}(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}.\end{aligned}\tag{3.43}$$

Therefore, integrating (3.43) from z to ∞ with respect to ζ, we have

$$-u(z)\leqslant\frac{1}{\omega-\delta-1}\Omega^{1-(\omega-\delta)}(z),\tag{3.44}$$

substituting (3.44) into (3.42), we see that

$$\int_0^t\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}\left(\Phi_4^{\sigma}(z)\right)^{\omega}\Delta z\leqslant\frac{1}{\omega-\delta-1}\int_0^t\Omega^{1-(\omega-\delta)}(z)\left(\Phi_4^{\omega}\right)^{\triangle}(z)\triangle z.\tag{3.45}$$

Now, from (3.41), we obtain

$$\Phi_4^{\triangle}(z)=\frac{\vartheta(z)}{\Omega(z)}\theta(\phi(z))-\frac{\vartheta(\frac{z}{2})}{2\Omega(\frac{z}{2})}\theta(\phi(\frac{z}{2})).$$

Applying Theorem 2.6, we have

$$\begin{aligned}\left(\Phi_4^{\omega}\right)^{\triangle}(z)=\omega\Phi_4^{\omega-1}(s)\Phi_4^{\triangle}(z)&\leqslant\omega\left(\Phi_4^{\sigma}(z)\right)^{\omega-1}\Phi_4^{\triangle}(z)\\&=\omega\left(\Phi_4^{\sigma}(z)\right)^{\omega-1}\left[\frac{\vartheta(z)}{\Omega(z)}\theta(\phi(z))-\frac{\vartheta(\frac{z}{2})}{2\Omega(\frac{z}{2})}\theta(\phi(\frac{z}{2}))\right]\\&=\omega\frac{\vartheta(z)\psi_2(z)}{\Omega(z)}\left(\Phi_4^{\sigma}(z)\right)^{\omega-1}.\end{aligned}\tag{3.46}$$

Substituting (3.46) into (3.45), it yields

$$\int_0^t\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}\left[\Phi_4^{\sigma}(z)\right]^{\omega}\triangle z$$

$$\leqslant \frac{\omega}{\omega-\delta-1}\int_0^t \frac{\vartheta(z)\psi_2(z)}{\Omega^{\omega-\delta}(z)}[\Phi_4^{\sigma}(z)]^{\omega-1}\triangle z$$
$$= \frac{\omega}{\omega-\delta-1}\int_0^t \left(\frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}[\Phi_4^{\sigma}(z)]^{\omega}\right)^{\frac{\omega-1}{\omega}} \frac{[\Omega^{\sigma}(z)]^{(\omega-\delta)(\frac{\omega-1}{\omega})}}{\Omega^{\omega-\delta}(z)}\vartheta^{\frac{1}{\omega}}(z)\psi_2(z)\triangle z.$$

Now, applying Hölder's inequality (2.2) with ω and $\omega/(\omega-1)$, we have

$$\int_0^t \frac{\vartheta(z)}{[\Omega^{\sigma}(z)]^{\omega-\delta}}(\Phi_4^{\sigma}(z))^{\omega}\triangle z \leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^{\omega}\int_0^t \frac{[\Omega^{\sigma}(z)]^{(\omega-\delta)(\omega-1)}}{\Omega^{\omega(\omega-\delta)}(z)}\vartheta(z)\psi_2^{\omega}(z)\triangle z.$$

□

In Theorem 3.26 if $\mathbb{T}=\mathbb{N}$, then we obtain the following corollary.

Corollary 3.27. *Let* $\{\vartheta(\zeta)\}_{\zeta=1}^{\infty}$ *and* $\{\theta(\phi(\zeta))\}_{\zeta=1}^{\infty}$ *be increasing and non-negative sequences. If* $\delta<\omega-1$ *and* $1<\omega$, *then*

$$\sum_{z=0}^{t-1}\frac{\vartheta(z)}{\Omega^{\omega-\delta}(z+1)}\Phi_4^{\omega}(z+1) \leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^{\omega}\sum_{z=0}^{t-1}\frac{\Omega^{(\omega-\delta)(\omega-1)}(z+1)}{\Omega^{\omega(\omega-\delta)}(z)}\vartheta(z)\left|\psi_2(z)\right|^{\omega},$$

where

$$\Phi_4(z)=\sum_{i=\frac{z}{2}}^{z-1}\frac{\vartheta(i)}{\Omega(i)}\theta(\phi(i)),\quad \textit{for}\ \ \frac{z}{2},\frac{z+1}{2}\in\mathbb{N};\ \ \psi_2(z)=\theta(\phi(z))-\frac{\vartheta(\frac{z}{2})\Omega(z)}{2\vartheta(z)\Omega(\frac{z}{2})}\theta(\phi(\frac{z}{2})),\quad \textit{and}\quad \Omega(\infty)=\infty.$$

Remark 3.28. For $\mathbb{T}=\mathbb{R}$, Theorem 3.26 reduces to [2, Theorem 6].

The next result follows from Theorem 3.26 by choosing $\theta(\phi(z))=z\phi(z)$ and $\vartheta(z)=1$.

Corollary 3.29. *If* $\delta<\omega-1$ *and* $1<\omega$, *then*

$$\int_0^t \frac{1}{\sigma^{\omega-\delta}(z)}\left(\int_{\frac{\sigma(z)}{2}}^{\sigma(z)}\phi(\zeta)\triangle\zeta\right)^{\omega}\triangle z \leqslant \left(\frac{\omega}{\omega-\delta-1}\right)^{\omega}\int_0^t \frac{\sigma^{(\omega-\delta)(\omega-1)}(z)}{z^{\omega(\omega-\delta)}}\left|z\phi(z)-\frac{z}{2}\phi(\frac{z}{2})\right|^{\omega}\triangle z.$$

Remark 3.30. For $\mathbb{T}=\mathbb{R}$, Corollary 3.29 reduces to [2, Corollary 4].

4. Conclusion

In this paper, we obtained some new types of the dynamic Hardy inequalities on time scales by using weighted mean operators $\Phi_1 := (\Phi_1)_{\theta}^{\vartheta}$ and $\Phi_2 := (\Phi_2)_{\theta}^{\vartheta}$ are defined as in Theorems 3.1 and 3.21, where θ, ϑ are rd-continous and non-negative functions. Our results are the variants and extension of the previous result of [2] in the case of $\mathbb{T}=\mathbb{R}$. In additional, we obtained some new inequalities in the case of $\mathbb{T}=\mathbb{N}$ which are essentially new. In the future work, we will continue to generalize more fractional dynamic inequalities on time scales and it will also be very enjoyable to introduce such inequalities in quantum calculus.

Author's contributions

Resources and methodology, A.A.E.-D.; data curation, A.A.E.-D.; writing-original draft preparation, K.A.-B.; conceptualization, writing-review and editing, H.M.R.; administration, project, A.A.E.-D. All authors read and approved the final manuscript.

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