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On weakly bi-ideals in ordered semigroups and their fuzzifications

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Abstract

The concept of bi-ideals in ordered semigroups is a subuniverse of ordered semigroups with certain conditions. It can be used to characterize ordered semigroups. In this paper, we extend the concept of bi-ideals to a general way, so-called weakly bi-ideals. We consider the primitive and semiprimitive of weakly bi-ideals. Some connections between prime (semiprime) weakly bi-ideals and fuzzy prime (semiprime) weakly bi-ideals is established.

Keywords: Weakly bi-ideal, fuzzy weakly bi-ideal, ordered semigroup.

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1. Introduction

A generalized notion of semigroups is the concept of ordered semigroups. To study the structural properties of ordered semigroups, the concepts of various kinds of ideals are an essential tool. One of the notions used to study ordered semigroups is the concept of fuzzy sets. Fuzzy sets were introduced by Zadeh [21] in 1965. This concept is beneficial in dealing with uncertainties in real-world problems.

Kehayopulu [4] introduced the notion of bi-ideals in ordered semigroups in 1992. It was shown by Kehayopulu et al. [8] that an ordered semigroup is an ordered group if it has no proper bi-ideals. The minimality of bi-ideals was studied and used to characterized a particular class of ordered semigroups in [3]. Mallick and Hansda [12] investigated some interesting properties of the semigroup formed by the set of all bi-ideals in ordered semigroups together with a complex product.

Bi-ideals in ordered semigroups had been extended in various ways. Bi-ideals were generalized into (m,n)-ideals by Bussaban and Changphas in [1]. Later, Tiprachot et al. [19] applied (m,n)-ideals to describe several classes of ordered semigroups. Shabir and Khan [17] initialed the notion of generalized bi-ideals in ordered semigroups. They characterized some classes of ordered semigroups by the fuzzification of generalized bi-ideals. When Linesawat et al. [10] defined the concept of anti-hybrid bi-ideals in 2021, a generalized notion of bi-ideals was introduced. They examined some of their fundamental properties.

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The concept of hybrid bi-ideals, an extension of fuzzy bi-ideals in ordered semigroups introduced in [20], is another approach to generalizing bi-ideals. Linear inequalities were utilized to categorize ordered semigroups into classes using the concept of hybrid (m, n)-ideals. Suebsung et al. [18] introduced the notion of almost bi-ideals in ordered semigroups. The authors illustrated some connections between almost bi-ideals and fuzzy almost bi-ideals.

Another concept that researchers widely study is the concept of algebraic hyperstructure. It was started by Marty, who introduced a higher level of groups, the concept of hypergroups (see [11]). Several hyperalgebraic structures were introduced after Marty introduced hypergroups, including ordered Γ-semihypergroups and ordered semihyperrings (see [2, 9, 14]). We recall some interesting results of these hyperstructures. The idea of almost Γ-hyperideals of ordered Γ-semihypergroups was presented by Rao et al. in [16]. This notion behaves differently from Γ-hyperideals in ordered Γ-semihypergroups. Indeed, the intersection of almost Γ-hyperideals need not be a Γ-hyperideal, as the authors demonstrated with an example. Additionally, they provided relationships between almost Γ-hyperideals and Γ-hyperideals. Using 2-hyperideals in ordered semihyperrings, Omidi and Davvaz [15] constructed a regular equivalence relation where the quotient of an ordered semihyperring and this relation is also an ordered semihyperring.

In this paper, ordered semigroups are the main topic. We introduce the notion of weakly bi-ideals in ordered semigroups. It turns out that this concept is a generalization of bi-ideals. Meaning that every bi-ideal is a weakly bi-ideal. An example is provided for illustrating that bi-ideals and weakly bi-ideals are distinct. We study the prime and semiprime properties of such weakly bi-ideals. Moreover, we introduce the concept of weakly bi-ideals in ordered semigroups by a fuzzification setting. The prime and semiprime properties of weakly bi-ideals are characterized by fuzzy weakly bi-ideals in ordered semigroups.

2. Preliminaries

Some basic terminologies of ordered semigroups and fuzzy sets will be recalled in this section.

A structure $\langle S; \cdot, \leqslant\rangle$ is said to be an *ordered semigroup* if $\cdot$ is associative binary operation on S and $\leqslant$ is a partial order on S such that $\leqslant$ is compatible with $\cdot$, that is, $a \leqslant b$ implies $x \cdot a \leqslant x \cdot b$ and $a \cdot x \leqslant b \cdot x$ for any $x \in S$.

We denote an ordered semigroup $\langle S; \cdot, \leqslant\rangle$ by $\mathbf{S}$ the bold letter of its universe set. Moreover, the product $x \cdot y$ is written by xy for any elements x and y of an ordered semigroup.

Let $\mathbf{S}$ be an ordered semigroup, and for nonempty subsets A and B of S. We denote $AB := \{ab \mid a \in A, b \in B\}$ and $(A] := \{t \in S \mid t \leqslant a \text{ for some } a \in A\}$. In particular, if $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$.

The well-known properties of the operator $(\cdot]$ are shown as follows. Let $\mathbf{S}$ be an ordered semigroup. For any nonempty subsets A and B of S, we have:

(i) $A \subseteq (A]$;
(ii) $((A]] = (A]$;
(iii) $A \subseteq B$ implies $(A] \subseteq (B]$;
(iv) $(A](B] \subseteq (AB]$;
(v) $(A \cup B] = (A] \cup (B]$.

Let $\mathbf{S}$ be an ordered semigroup. A nonempty subset A of S is said to be a *subsemigroup* of $\mathbf{S}$ if $AA \subseteq A$. Some particular notions of subsemigroups are recall as follows. A nonempty subset A of S is called:

(i) a *right ideal* of $\mathbf{S}$ if $AS \subseteq A$ and $(A] \subseteq A$;
(ii) a *left ideal* of $\mathbf{S}$ if $SA \subseteq A$ and $(A] \subseteq A$;
(iii) a *bi-ideal* of $\mathbf{S}$ if A is a subsemigroup of $\mathbf{S}$, $ASA \subseteq A$ and $(A] \subseteq A$.

For more detail about these ideals, the readers are referred to [4].

Remark 2.1. Let **S** be an ordered semigroup. It is not difficult to verify that any right (left) ideal of **S** is a bi-ideal of **S**.

Next, we recall the concept of fuzzy sets. A mapping $\mu\colon X \to [0,1]$, where X is a nonempty set, is called a *fuzzy set* [21] in X. Let μ and λ be any two fuzzy sets in X. Then, $\mu \subseteq \lambda, \mu \cap \lambda$ and $\mu \cup \lambda$ are fuzzy sets in X, and defined by:

(i) $\mu \subseteq \lambda$ if and only if $\mu(x) \leqslant \lambda(x)$ for all $x \in X$;
(ii) $(\mu \cap \lambda)(x) := \min\{\mu(x), \lambda(x)\}$ for all $x \in X$;
(iii) $(\mu \cup \lambda)(x) := \max\{\mu(x), \lambda(x)\}$ for all $x \in X$.

Let A be a nonempty subset of X. A fuzzy set C_A in X defined by

$$C_A(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise}, \end{cases}$$

for all $x \in X$, is called the *characteristic function* of A in X.

If **S** is an ordered semigroup, then any fuzzy set in S is called a fuzzy set of **S**.

Definition 2.2 ([13])**.** Let **S** be an ordered semigroup, and μ a fuzzy set of **S**. For any $t \in [0,1]$, the *level set* μ_t is a subset of S, and defined by $\mu_t := \{x \in S \mid \mu(x) \geqslant t\}$.

Definition 2.3 ([6])**.** Let **S** be an ordered semigroup. A fuzzy set μ of **S** is called a *fuzzy subsemigroup* of **S** if $\mu(xy) \geqslant \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$.

Definition 2.4 ([7])**.** Let **S** be an ordered semigroup. A fuzzy subsemigroup μ of **S** is called a *fuzzy bi-ideal* of **S** if:

(i) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$;
(ii) $\mu(xyz) \geqslant \min\{\mu(x), \mu(z)\}$ for all $x, y, z \in S$.

3. Weakly bi-ideals in ordered semigroups

In this section, we introduce the notion of weakly bi-ideals of ordered semigroups as a generalization of bi-ideals. The basic properties of weakly bi-ideals are also provided. Then, we define the concept of fuzzy weakly bi-ideals of ordered semigroups and investigate some relationships between weakly bi-ideals and fuzzy weakly bi-ideals in ordered semigroups.

Definition 3.1. Let **S** be an ordered semigroup. A subsemigroup A of **S** is said to be a *weakly bi-ideal* of **S** if:

(i) $\bigcup_{a \in A} aSa \subseteq A$;
(ii) $(A] \subseteq A$, that is, $s \leqslant x$ implies $s \in A$ whenever $x \in A$ and $s \in S$.

It is not difficult to see that any bi-ideal of an ordered semigroup **S** is a weakly bi-ideal of **S**. In contract, a weakly bi-ideal may not be a bi-ideal as shown by the following example.

Example 3.2. Let $S = \{a, b, c, d, e, f\}$. Define the binary operation $\cdot$ and a binary relation $\leqslant$ on S as follows:

$\cdot$	a	b	c	d	e	f
a	a	a	a	a	a	a
b	a	a	a	a	b	b
c	a	a	a	a	b	c
d	a	b	c	d	a	d
e	a	a	a	a	e	e
f	a	b	c	d	e	f

and $\leqslant := \{(a,b)\} \cup \Delta_S$, where $\Delta_S := \{(x,x) \mid x \in S\}$. Then, $\mathbf{S} := \langle S; \cdot, \leqslant \rangle$ is an ordered semigroup. Let $A = \{a,d,e\}$. It is clear that A is a subsemigroup of $\mathbf{S}$. Since $aSa = \{a\}$, $dSd = \{a,d\}$ and $eSe = \{a,e\}$, we obtain that A is a weakly bi-ideal of $\mathbf{S}$. But, $ASA = \{a,b,d,e\} \nsubseteq A$. This implies that A is not a bi-ideal of $\mathbf{S}$.

The following result illustrates the intersection of any weakly bi-ideals is also a weakly bi-ideal if it is nonempty.

Proposition 3.3. *Let $\{A_i \mid i \in \Lambda\}$ be a nonempty collection of weakly bi-ideals of an ordered semigroup $\mathbf{S}$. Then, $\bigcap_{i\in\Lambda} A_i$ is a weakly bi-ideal of $\mathbf{S}$ if it is nonempty.*

Proof. Suppose that $A := \bigcap_{i\in\Lambda} A_i$ with $A \neq \emptyset$. Since $A \subseteq A_i$ for all $i \in \Lambda$ and A_i is a subsemigroup of $\mathbf{S}$, we have that $AA \subseteq A_iA_i \subseteq A_i$ for all $i \in \Lambda$. This implies that $AA \subseteq A$. Hence, A is a subsemigroup of $\mathbf{S}$. We can see that for any $i \in \Lambda$, we have

$$\bigcup_{a\in A} aSa \subseteq \bigcup_{a\in A_i} aSa \subseteq A_i,$$

so $\bigcup_{a\in A} aSa \subseteq A$. Finally, $(A] \subseteq A_i$ for each $i \in \Lambda$ implies that $(A] \subseteq A$. Therefore, we complete that proof. □

Example 3.4. By Example 3.2, we have that $A = \{a,d,e\}$ is a weakly bi-ideal of an ordered semigroup $\mathbf{S}$. Now, we consider $B = \{a,c\}$. It is routine to verify that B is also a weakly bi-ideal of $\mathbf{S}$. However, $A \cup B = \{a,c,d,e\}$ is not a weakly bi-ideal of S since $ce = b \notin A \cup B$.

From Example 3.4, we conclude that the union of any two weakly bi-ideals need not to be a weakly bi-ideal.

Proposition 3.5. *Let $\mathbf{S}$ be an ordered semigroup, T be any subset of S and A be a weakly bi-ideal of $\mathbf{S}$. If AT (resp., TA) is a subsemigroup of $\mathbf{S}$ such that $(AT] \subseteq AT$ (resp., $(TA] \subseteq TA$), then it is a weakly bi-ideal of $\mathbf{S}$.*

Proof. We prove that AT is a weakly bi-ideal of $\mathbf{S}$. For illustrating that TA is a weakly bi-ideal of $\mathbf{S}$ can be done similarly. Assume that AT is a subsemigroup of $\mathbf{S}$ such that $(AT] \subseteq AT$. Consider,

$$\bigcup_{a\in AT} aSa = \bigcup_{b\in A, t\in T} btSbt \subseteq \bigcup_{b\in A, t\in T} bSbt \subseteq \bigcup_{t\in T} At \subseteq AT.$$

Hence, AT is a weakly bi-ideal of $\mathbf{S}$. □

Example 3.6. By Example 3.2, we have that $A = \{a,d,e\}$ is a weakly bi-ideal of $\mathbf{S}$. We put $T = \{a,f\}$. Since AT and TA are subsemigroup of $\mathbf{S}$ such that $(AT] \subseteq AT$ and $(TA] \subseteq TA$, by Proposition 3.5, we obtain that AT and TA are weakly bi-ideals of $\mathbf{S}$.

We recall the concept of prime and semiprime in ordered semigroups which were introduced in [5] by Kehayopulu.

Definition 3.7. Let $\mathbf{S}$ be an ordered semigroup, and A a nonempty subset of S. Then A is said to be:

(i) *prime* if for every $x, y \in S$, if $xy \in A$, then $x \in A$ or $y \in A$;
(ii) *semiprime* if for every $x \in S$, if $x^2 \in A$, then $x \in A$.

The following theorem provides us with a description of weakly bi-ideals being prime in terms of right and left ideals.

Theorem 3.8. *Let $\mathbf{S}$ be an ordered semigroup, and A a weakly bi-ideal of $\mathbf{S}$. Then the following statements are equivalent:*

(i) A *is prime;*

(ii) $RL \subseteq A$ *implies* $R \subseteq A$ *or* $L \subseteq A$ *for any right ideal* R *and left ideal* L *of* $\mathbf{S}$.

Proof.

(i) $\Rightarrow$ (ii). Let R and L be a right ideal and a left ideal of $\mathbf{S}$, respectively. Suppose that $RL \subseteq A$ such that $L \nsubseteq A$. Let $x \in R$. Then for any $y \in L \setminus A$, we have $xy \in RL$. By our presumption, we obtain that $xy \in A$. By the primness of a weakly bi-ideal A of $\mathbf{S}$, we have that $x \in A$.

(ii) $\Rightarrow$ (i). Suppose that $x, y \in S$ such that $xy \in A$ and $y \notin A$. It is not difficult to verify that $(xS]$ and $(Sx]$ are a right ideal and a left ideal of $\mathbf{S}$, respectively. By the ideality of A, we have that

$$(xS](Sx] \subseteq (xSx] \subseteq (A] \subseteq A.$$

This implies, by our presumption, that $(xS] \subseteq A$ or $(Sx] \subseteq A$. If $(xS] \subseteq A$, then $x^2 \in A$. It is not difficult to calculate that $(x \cup xS]$ and $(x \cup Sx]$ is a right ideal and left ideal of $\mathbf{S}$ containing x. Consider

$$(x \cup xS](x \cup Sx] \subseteq (x^2 \cup xSx].$$

Since $x^2 \in A$ and $(xS] \subseteq A$, we obtain $(x^2] \subseteq (A] \subseteq A$ and $(xSx] \subseteq (xS] \subseteq A$. By the properties of the operator $(\cdot]$, we have that $(x \cup xS](x \cup Sx] \subseteq A$. This implies that $(x \cup xS] \subseteq A$ or $(x \cup Sx] \subseteq A$. This means that $x \in A$. Similarly for the case that $(Sx] \subseteq A$, we can illustrate that $x \in A$. Thus, A is prime. □

Now, we introduce the concept of fuzzy weakly bi-ideals of ordered semigroups and consider some connections between weakly bi-ideals and fuzzy weakly bi-ideals in ordered semigroups.

Definition 3.9. Let $\mathbf{S}$ be an ordered semigroup. A fuzzy subsemigroup μ of $\mathbf{S}$ is called a *fuzzy weakly bi-ideal* of $\mathbf{S}$ if for any $x, y \in S$, we have:

(i) $x \leqslant y$ implies $\mu(x) \geqslant \mu(y)$;

(ii) $\mu(xyx) \geqslant \mu(x)$.

The following proposition provides us a connection between fuzzy bi-ideals and fuzzy weakly bi-ideals in ordered semigroups.

Proposition 3.10. *Every fuzzy bi-ideal of an ordered semigroup* $\mathbf{S}$ *is also a fuzzy weakly bi-ideal of* $\mathbf{S}$.

Proof. Let μ be a fuzzy bi-ideal of an ordered semigroup $\mathbf{S}$. It is sufficient to show that $\mu(xyx) \geqslant \mu(x)$ for all $x, y \in S$. Let $x, y \in S$. Since μ is a fuzzy bi-ideal of $\mathbf{S}$, we have that $\mu(xyx) \geqslant \min\{\mu(x), \mu(x)\} = \mu(x)$. This illustrates that μ is a fuzzy weakly bi-ideal of $\mathbf{S}$. □

The converse of Proposition 3.10 is not true in general. This can be seen by the following example.

Example 3.11. Consider an ordered semigroup $\mathbf{S}$ provided in Example 3.2. We define a fuzzy set μ of $\mathbf{S}$ by:

$$\mu(a) = 0.9, \mu(b) = 0.3, \mu(c) = 0.2, \mu(d) = 0.7, \mu(e) = 0.5 \text{ and } \mu(f) = 0.$$

By careful calculation, it follows that μ is a fuzzy weakly bi-ideal of $\mathbf{S}$. But, since there are $b, c, e \in S$ such that $\mu(dce) < \min\{\mu(d), \mu(e)\}$, we obtain that μ is not a fuzzy bi-ideal of $\mathbf{S}$.

The following theorem shows that the intersection of fuzzy weakly bi-ideals is a fuzzy weakly bi-ideal as well.

Theorem 3.12. *Let* $\mathbf{S}$ *be an ordered semigroup. If* $\{\mu_i \mid i \in \Lambda\}$ *be a family of fuzzy weakly bi-ideals of* $\mathbf{S}$*, then* $\bigcap_{i \in \Lambda} \mu_i$ *is also a fuzzy weakly bi-ideal of* $\mathbf{S}$*, where* Λ *is any index set.*

Proof. Assume that $\{\mu_i \mid i \in \Lambda\}$ is a family of fuzzy weakly bi-ideals of **S** and $\mu := \bigcap_{i\in\Lambda} \mu_i$. We recall that $\mu(x) = \bigcap_{i\in\Lambda} \mu_i(x) = \inf_{i\in\Lambda} \mu_i(x)$ for all $x \in S$. For any $x, y \in S$ such that $x \leqslant y$, we have that $\mu(x) = \inf_{i\in\Lambda} \mu_i(x) \geqslant \inf_{i\in\Lambda} \mu_i(y) = \mu(y)$. Now, we let $x, y \in S$. Then, we have

$$\mu(xy) = \inf_{i\in\Lambda} \mu_i(xy) \geqslant \inf_{i\in\Lambda} \min\{\mu_i(x), \mu_i(y)\} = \min\{\inf_{i\in\Lambda} \mu_i(x), \inf_{i\in\Lambda} \mu_i(y)\} = \min\{\mu(x), \mu(y)\}$$

and

$$\mu(xyx) = \inf_{i\in\Lambda} \mu_i(xyx) \geqslant \inf_{i\in\Lambda} \mu_i(x) = \mu(x).$$

Altogether, we have that $\bigcap_{i\in\Lambda} \mu_i$ is a fuzzy weakly bi-ideal of **S**. □

Example 3.13. According to Example 3.2, a fuzzy set ν of **S** defined by:

$$\nu(a) = 0.9, \nu(b) = 0.3, \nu(c) = 0.8, \nu(d) = 0.3, \nu(e) = 0.3, \text{ and } \nu(f) = 0.3,$$

is a fuzzy weakly bi-ideal of **S**. From Example 3.11, we have that μ is also a fuzzy weakly bi-ideal of **S**. We observe that the union of fuzzy weakly bi-ideals μ and ν is the fuzzy set $\xi := \mu \cup \nu$ of **S** defined by:

$$\xi(a) = 0.9, \xi(b) = 0.3, \xi(c) = 0.8, \xi(d) = 0.7, \xi(e) = 0.5, \text{ and } \xi(f) = 0.3.$$

However, ξ is not a fuzzy weakly bi-ideal of **S** since $\xi(ce) < \min\{\xi(c), \xi(e)\}$.

Example 3.13 illustrates the contrary of Theorem 3.12: the union of any two fuzzy weakly bi-ideals of an ordered semigroup need not to be a fuzzy weakly bi-ideal.

A relationship between fuzzy weakly bi-ideals and level sets of ordered semigroups is stated as follows.

Theorem 3.14. *Let μ be a fuzzy set of an ordered semigroup* **S**. *Then the following statements are equivalent:*

(i) *μ is a fuzzy weakly bi-ideal of* **S**;

(ii) *for every $t \in [0, 1]$, the nonempty level set μ_t is a weakly bi-ideal of* **S**.

Proof.

(i) $\Rightarrow$ (ii). Assume that μ is a fuzzy weakly bi-ideal of **S** and $t \in [0, 1]$ such that $\mu_t \neq \emptyset$. Let $x, y \in \mu_t$. Then, $\mu(xy) \geqslant \min\{\mu(x), \mu(y)\} \geqslant t$. This means that $xy \in \mu_t$. It follows that $\mu_t\mu_t \subseteq \mu_t$. Next, we let $k \in \bigcup_{a\in\mu_t} aSa$. Then, $k = asa$ for some $a \in \mu_t$ and $s \in S$. Thus, $\mu(k) = \mu(asa) \geqslant \mu(a) \geqslant t$. That is, $k \in \mu_t$. This shows that $\bigcup_{a\in\mu_t} aSa \subseteq \mu_t$. Now, let $w \in (\mu_t]$. Then, $w \leqslant y$ for some $y \in \mu_t$. Thus, $\mu(w) \geqslant \mu(y) \geqslant t$, and then $w \in \mu_t$. This means that $(\mu_t] \subseteq \mu_t$. Therefore, μ_t is a weakly bi-ideal of **S**.

(ii) $\Rightarrow$ (i). Conversely, assume that μ_t is a weakly bi-ideal of **S** for all $t \in [0, 1]$. Let $x, y \in S$ such that $x \leqslant y$. We put $t_0 := \mu(y) = t_0$. Then, $y \in \mu_{t_0}$. Since μ_{t_0} is a weakly bi-ideal of **S** and $x \leqslant y$, we have that $x \in \mu_{t_0}$. So, $\mu(x) \geqslant t_0 = \mu(y)$. Next, let $x, y \in S$. We put $s_0 := \min\{\mu(x), \mu(y)\}$. This implies that $x, y \in \mu_{s_0}$. Since $\mu_{s_0}\mu_{s_0} \subseteq \mu_{s_0}$, we have that $xy \in \mu_{s_0}$. It follows that $\mu(xy) \geqslant s_0 = \min\{\mu(x), \mu(y)\}$. Finally, for any $x, y \in S$, we let $w_0 := \mu(x)$. Then, $x \in \mu_{w_0}$. Since $\bigcup_{a\in\mu_{w_0}} aSa \subseteq \mu_{w_0}$, we obtain that $xyx \in \mu_{w_0}$. It turns out that $\mu(xyx) \geqslant w_0 = \mu(x)$. Consequently, μ is a fuzzy weakly bi-ideal of **S**. □

The following result can be directly obtained using Theorem 3.14.

Corollary 3.15. *Let* **S** *be an ordered semigroup and A be a nonempty subset of S. Then, A is a weakly bi-ideal of* **S** *if and only if C_A is a fuzzy weakly bi-ideal of* **S**.

Example 3.16. By Example 3.2, we have that $A = \{a, d, e\}$ is a weakly bi-ideal of **S**. Applying Corollary 3.15, we obtain that the fuzzy set f of **S** defined by

$$f(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise}, \end{cases}$$

for all $x \in S$, is a fuzzy weakly bi-ideal of **S**.

Now, we introduce the notion of prime fuzzy weakly bi-ideals and semiprime fuzzy weakly bi-ideals in ordered semigroups.

Definition 3.17. Let $\mathbf{S}$ be an ordered semigroup.

(i) A fuzzy weakly bi-ideal μ of $\mathbf{S}$ is called *prime* if $\mu(xy) \leqslant \max\{\mu(x), \mu(y)\}$ for all $x, y \in S$.
(ii) A fuzzy weakly bi-ideal μ of $\mathbf{S}$ is called *semiprime* if $\mu(x^2) \leqslant \mu(x)$ for all $x \in S$.

The following theorem shows the interconnection between prime weakly bi-ideals and fuzzy prime weakly bi-ideals.

Theorem 3.18. *Let $\mathbf{S}$ be an ordered semigroup and A be a nonempty subset of S. Then the following statements are equivalent:*

(i) *A is a prime weakly bi-ideal of $\mathbf{S}$;*
(ii) *C_A is a prime fuzzy weakly bi-ideal of $\mathbf{S}$.*

Proof.
(i) $\Rightarrow$ (ii). Assume that A is a prime weakly bi-ideal of $\mathbf{S}$. By Corollary 3.15, C_A is a fuzzy weakly bi-ideal of $\mathbf{S}$. Let $x, y \in S$. If $xy \in A$, then $x \in A$ or $y \in A$ by the primitive of A. It follows that

$$\max\{C_A(x), C_A(y)\} = 1 = C_A(xy).$$

If $xy \notin A$, then we obtain that

$$C_A(xy) = 0 \leqslant \max\{C_A(x), C_A(y)\}.$$

Therefore, C_A is a prime fuzzy weakly bi-ideal of $\mathbf{S}$.

(ii) $\Rightarrow$ (i). Assume that C_A is a prime fuzzy weakly bi-ideal of $\mathbf{S}$. By Corollary 3.15, A is a weakly bi-ideal of $\mathbf{S}$. Let $x, y \in S$ such that $xy \in A$. Then,

$$1 = C_A(xy) \leqslant \max\{C_A(x), C_A(y)\}.$$

This implies that $\max\{C_A(x), C_A(y)\} = 1$. This means that $x \in A$ or $y \in A$. Therefore, A is a prime weakly bi-ideal of $\mathbf{S}$. □

The next theorem shows the connection between semiprime weakly bi-ideals and fuzzy semiprime weakly bi-ideals. It can be proved similar to Theorem 3.18.

Theorem 3.19. *Let $\mathbf{S}$ be an ordered semigroup and A be a nonempty subset of S. Then the following statements are equivalent:*

(i) *A is a semiprime weakly bi-ideal of $\mathbf{S}$;*
(ii) *C_A is a semiprime fuzzy weakly bi-ideal of $\mathbf{S}$.*

4. Conclusions

The concept of weakly bi-ideals in ordered semigroups is introduced in this study. Weakly bi-ideals are discovered to be a generalization of bi-ideals. We examine some of the most fundamental aspects of such a concept, such as intersections, unions, and primness. Furthermore, we apply the concept of fuzzy sets to weakly bi-ideals. Some of this fuzzification's qualities are explored. We investigate the links between weakly bi-ideals and fuzzy weakly bi-ideals in ordered semigroups. The authors pose whether the concept of weakly bi-ideals can be described in terms of hyperalgebraic structures to the readers.

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