J. Math. Computer Sci., 30 (2023), 173–189

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Strong convergence theorems for a solution of split common fixed point problem involving λ-strict quasi-G_f-pseudocontractive mappings in Banach spaces

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Abstract

The purpose of this paper is to propose a method for approximating the solution of the split common fixed point problem involving λ-strict quasi-G_f-pseudocontractive mappings in the setting of two Banach spaces using $G_f(.,.)$ functional. We prove that the proposed method converges strongly to a solution of the split common fixed point problem. In addition, we provide some applications of our method and provide numerical results to demonstrate the applicability of the proposed method.

Keywords: Split common fixed point problem, λ-strict quasi-G_f-pseudocontractive mappings.

2020 MSC: 47H09, 47J25.

1. Introduction

Let C and D be nonempty subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is given by the following:

$$\text{find } x \in C \text{ such that } Ax \in D,$$

where $A : H_1 \to H_2$ is a bounded linear map with the adjoint operator A^*.

This problem has attracted the attention of many authors due to its application in different disciplines such as image restoration, computer tomography, radiation therapy treatment planning, antenna design, sensor arrays, data communication and data compression (see, for instance, [3–5, 7–9, 11, 20, 33, 45]). Different methods of solving this problem have been studied by several authors on the bases of its applications (see, for instance, [4, 5, 42, 45–47]).

Several generalized problems related to the SFP have also been studied. In this connection, it is worth to mention, for instance, the multiple-set SFP (MSSFP) (see [9, 23]), the split common fixed point problem

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doi: 10.22436/jmcs.030.02.08

Received: 2022-09-27 *Revised:* 2022-11-07 *Accepted:* 2022-11-29

(SCFPP) (see [12, 25]), the split variational inequality problem (SVIP) (see [10]), and the split common null point problem (SCNPP) (see [6, 30–32]).

Given two real Hilbert spaces H_1 and H_2, let $S : H_1 \longrightarrow H_1$ and $T : H_2 \longrightarrow H_2$ be nonlinear mappings and let $A : H_1 \longrightarrow H_2$ be a bounded linear operator. We usually denote the set of fixed points of T by $\mathcal{F}(T)$ and that of S by $\mathcal{F}(S)$. The *split common fixed point problem* (SCFPP) is given by the following:

$$x^\star \in \mathcal{F}(S) \text{ such that } Ax^\star \in \mathcal{F}(T). \tag{1.1}$$

This problem is originated from Censor and Segal [12] and was studied by employing the following algorithm:

$$x_{n+1} = T(x_n - \tau_n A^*(I - S)Ax_n), n \geqslant 1,$$

where S and T are quasi-firmly nonexpansive mappings and $\tau_n \in (0, \frac{2}{||A||^2})$ for all $n \geqslant 1$. The generated sequence has been shown to converge weakly towards a solution of (1.1) in the setting of Hilbert spaces. Motivated by the result of Censor and Segal [12], Wang [41] proved a weak convergence theorem for a solution of SCFPP using the following algorithm:

$$x_{n+1} = x_n - \tau_n[(I - T)x_n + A^*(I - S)Ax_n], \tag{1.2}$$

where S and T are firmly nonexpansive mappings and $\{\tau_n\} \subset (0, +\infty)$ satisfying some mild conditions. We observe that the step size τ_n in algorithm (1.2) is independent of the norm of the bounded linear operator A.

In [25] and [24], Moudafi extended this algorithm to the case of quasi-nonexpansive mappings and the case of demicontractive mappings, respectively, in Hilbert spaces. Since then, there has been growing interests in the split common fixed point problem in the setting of Hilbert spaces. More specifically, in [48], Yao et al. remarked that problem (1.1) can be viewed as solving the following fixed-point equation

$$x^\star = Sx^\star - A^\star(I - T)Ax^\star.$$

Consequently, they proposed a new algorithm for solving the problem in the setting of Hilbert spaces and they obtained a weak convergence result under some mild assumptions.

In an attempt to solve the SCFPP (1.1) in the setting of Banach spaces, several authors (see, for instance, [13, 21, 22, 37, 39]) have introduced and studied algorithms for solving the problem in a p-uniformly smooth Banach space for $1 < p < +\infty$.

In 2015, Takahashi and Yao [36] considered and investigated the SCNPP in the setting of one Hilbert space and one Banach space by using a hybrid projection method. Their method is described as follows. Let H be a Hilbert space and let F be a uniformly convex Banach space whose norm is Frechet differentiable. Let J_F be the duality mapping on F and let $A : H \to 2^H$ and $B : F \to 2^{F^*}$ be maximal monotone operators. Let J_λ be the resolvent of A for $\lambda > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$. Let $T : H \longrightarrow F$ be a bounded linear operator such that $T \neq 0$ and let $T^\star$ be the adjoint operator of T. Suppose that $A^{-1}0 \cap T^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \longrightarrow u$. Let $x_1 \in H$, $C_1 = H$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^\star J_F(Tx_n - Q_u Tx_n)), \\ C_{n+1} = \{z \in H : ||z_n - z|| \leqslant ||x_n - z||\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \ \forall n \in \mathbb{N}, \end{cases}$$

where $0 \leqslant c \leqslant \lambda_n ||T||^2 \leqslant 2$ and $\lambda_n \in (0, +\infty)$ for some $c \in \mathbb{R}$. They proved that the sequence $\{x_n\}$ converges strongly to a point $z_0 \in A^{-1}0 \cap T^{-1}(B^{-1}0)$, where $z_0 = P_{A^{-1}0 \cap T^{-1}(B^{-1}0)} u$.

This result has been extended to the case of two Banach spaces in [35].

Recently, Liu et al. [19] constructed an iterative method for solving the SCFPP for λ-strict quasi-ϕ-pseudocontractive mappings in the setting of two Banach spaces. They proved that the sequence generated by the proposed iterative algorithm converges strongly to a solution of the SCFPP.

Motivated and inspired by the work of Liu et al. [19], we propose an algorithm for approximating solutions of SCFPP involving λ-strict quasi-G_f-pseudocontractive mappings in the setting of two Banach spaces using functional $G(.,.)$. In addition, we prove that the sequence generated by the proposed algorithm converges strongly to a solution of the SCFPP. Finally, we provide some applications of our method and give a numerical experiment to explain the results.

2. Preliminaries

Let $E^\star$ be the dual space of a real Banach space E with norm $||.||$. We denote the value of $f^\star \in E^\star$ at $x \in E$ by $\langle x, f^\star\rangle$. For a sequence $\{x_n\}$ in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and its weak convergence by $x_n \rightharpoonup x$.

A Banach space E is said to be strictly convex if $\frac{||x+y||}{2} < 1$ for all $x, y \in S_E = \{z \in E : ||z|| = 1\}$ with $x \neq y$. The modulus of convexity δ_E of E is defined by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{||x+y||}{2} : ||x|| \leqslant 1, ||y|| \leqslant 1, ||x-y|| \geqslant \epsilon\right\}$$

for all $\epsilon \in [0, 2]$. If $\delta_E(\epsilon) > 0$ for all $0 < \epsilon \leqslant 2$, then E is said to be uniformly convex. It is well known that the Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E if $\lim_{n\to+\infty} ||x_n|| = \lim\limits_{n\to+\infty} ||y_n|| = 1$, and $\lim\limits_{n\to+\infty} ||x_n + y_n|| = 2$, then

$$\lim_{n\to+\infty} ||x_n - y_n|| = 0.$$

It is also known that if a Banach space is uniformly convex, then it is strictly convex and reflexive (see, for instance, [14, 29]).

The modulus of smoothness of E is a map $\rho_E : [0, +\infty) \longrightarrow [0, +\infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(||x+y|| + ||x-y||) - 1 : x \in S_E, ||y|| \leqslant t\right\}.$$

If $\rho_E(t)/t \to 0$ as $t \to 0$, then the Banach space E is said to be uniformly smooth. A typical example of uniformly smooth Banach space is the L_p space, where $1 < p < +\infty$. The normalized duality mapping $J : E \to 2^{E^\star}$ is defined by

$$J(x) = \left\{f^\star \in E^\star : \langle x, f^\star\rangle = ||x||^2 = ||f^\star||^2\right\}, \forall x \in E.$$

From the Hahn-Banach theorem, it follows that $J(x)$ is nonempty (see [34]). If E is smooth, then J is single-valued and hemi-continuous, that is, J is continuous from the strong topology of E to the weak star topology of E^*. It is also known that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is smooth, strictly convex, and reflexive, then J is a single-valued bijective map. In this case, the inverse mapping J^{-1} coincides with the normalized duality mapping $J^\star$ on $E^\star$ (see, for instance, [14, 15, 26, 29, 34]). If $E = H$ is a Hilbert space, then $J = I$, where I is the identity mapping.

A Banach space E is said to have *Kadec-Klee property* if for any sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, we have $x_n \to x$. Note that every uniformly convex Banach space is strictly convex, reflexive and has Kadec-Klee property.

Proposition 2.1 ([44]). *Let $s > 0$ and let E be a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, +\infty) \longrightarrow [0, +\infty)$, $g(0) = 0$, such that*

$$||x+y||^2 \geqslant ||x||^2 + 2\langle y, j\rangle + g(||y||),$$

for all $x, y \in \{z \in E : ||z|| \leqslant s\}$ and $j \in Jx$.

Let E be a smooth real Banach space. For any fixed $\rho > 0$, let $G_f : E \times E \longrightarrow \mathbb{R}$ be functional defined as follows:

$$G_f(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 + 2\rho f(x), \ \forall x, y \in E, \tag{2.1}$$

where $f : E \longrightarrow \mathbb{R}$ is convex, lower semicontinuous and bounded from below. From the definition of G_f and property of f it is easy to see that $G_f(x, y)$ is convex and lower semicontinuous with respect to x when y is fixed. In (2.1) if we consider $f = 0$ (a zero function), then G_f reduces to

$$G_0(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 = \phi(x, y), \ \forall x, y \in E, \tag{2.2}$$

where $\phi(x, y)$ is a Lyapunov function introduced by Alber [1]. Now, we consider the generalized f-projection operator in a Banach space which is analogous to the generalized f-projection operator discussed in Wu et al. ([43]).

Definition 2.2. Let C be a nonempty, closed and convex subset of a real smooth Banach space E. We say that $\Pi^f_C : E \longrightarrow 2^C$ is a generalized f-projection operator if

$$\Pi^f_C x = \left\{ u \in C : G_f(u, x) = \inf_{\xi \in C} G_f(\xi, x) \right\}, \forall x \in E.$$

Remark 2.3. The generalized projection operator defined by Alber [1, 2] and Li [17] is the special case of the generalized f-projection operator when $f(x) = 0$ for all $x \in E$.

Following the results in Li et al. [18], we obtain the following lemma.

Lemma 2.4. *Let C be a nonempty closed convex subset of a real reflexive and smooth Banach space E. The following statements hold:*

(i) *$\Pi^f_C x$ is a nonempty closed convex subset of C for all $x \in E$;*
(ii) *for all $x \in E$, $x^\star \in \Pi^f_C x$ if and only if*

$$\langle x^\star - y, Jx - Jx^\star\rangle + \rho f(y) - \rho f(x^\star) \geqslant 0, \ \forall y \in C;$$

(iii) *if E is strictly convex, then Π^f_C is a single valued mapping.*

It is easy to verify that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 + 2\rho f(x) \leqslant G_f(x, y) \leqslant (\|x\| + \|y\|)^2 + 2\rho f(x), \tag{2.3}$$
$$2\langle x - y, Jz - Jw\rangle = G_f(x, w) + G_f(y, z) - G_f(x, z) - G_f(y, w),$$
$$G_f(x, y) = 2\rho f(x) \Longleftrightarrow x = y. \tag{2.4}$$

It is also easy to observe that, in a Hilbert space H, $G_f(x, y) = \|x - y\|^2 + 2\rho f(x)$ for all $x, y \in H$.

Lemma 2.5. *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $G_f(y_n, z_n) - 2\rho f(y_n) \to 0$ as $n \to +\infty$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $\|y_n - z_n\| \to 0$ as $n \to +\infty$.*

Proof. Since $G_f(y_n, z_n) - 2\rho f(y_n) \to 0$ as $n \to +\infty$, the sequence $\{G_f(y_n, z_n) - 2\rho f(y_n)\}$ is bounded. Then it follows from (2.3) that if one of the sequences $\{y_n\}$ and $\{z_n\}$ is bounded, so is the other. Therefore, since E is uniformly convex by Proposition 2.1, there exists a continuous, strictly increasing, and convex function $g : [0, +\infty) \longrightarrow [0, +\infty)$, $g(0) = 0$, such that

$$\begin{aligned} g(\|y_n - z_n\|) &\leqslant \|z_n + (y_n - z_n)\|^2 - \|z_n\|^2 - 2\langle y_n - z_n, Jz_n\rangle \\ &= \|y_n\|^2 - \|z_n\|^2 - 2\langle y_n, Jz_n\rangle + 2\|z_n\|^2 = G_f(y_n, z_n) - 2\rho f(y_n). \end{aligned}$$

Then it follows from $G_f(y_n, z_n) - 2\rho f(y_n) \to 0$ that $g(\|y_n - z_n\|) \longrightarrow 0$. Then the continuity of g and $g(0) = 0$ yield that $\|y_n - z_n\| \to 0$ as $n \to +\infty$. □

Lemma 2.6. *Let* C *be a nonempty closed convex subset of a reflexive and smooth Banach space* E, *and let* $x \in E$. *Then*

$$G_f(y, \Pi^f_C x) + G_f(\Pi^f_C x, x) - 2\rho f(y) \leqslant G_f(y, x) \text{ for all } y \in C.$$

Proof. By definition, we have

$$\begin{aligned} &G_f(y,x) - G_f(\Pi^f_C x, x) - G_f(y, \Pi^f_C x) + 2\rho f\left(\Pi^f_C x\right) \\ &\quad = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2 - \|\Pi^f_C x\|^2 + 2\langle \Pi^f_C x, Jx\rangle - \|x\|^2 - \|y\|^2 + 2\langle y, J\Pi^f_C x\rangle - \|\Pi^f_C x\|^2 \\ &\quad = -2\langle y, Jx\rangle + 2\langle \Pi^f_C x, Jx\rangle + 2\langle y, J\Pi^f_C x\rangle - 2\|\Pi^f_C x\|^2 \\ &\quad = 2\left[\langle y, J\Pi^f_C x - Jx\rangle + \langle \Pi^f_C x, Jx\rangle - \langle \Pi^f_C x, J\Pi^f_C x\rangle\right] \\ &\quad = 2\left[\langle y, J\Pi^f_C x - Jx\rangle - \langle \Pi^f_C x, J\Pi^f_C x - Jx\rangle\right] \\ &\quad = 2\langle y - \Pi^f_C x, J\Pi^f_C x - Jx\rangle. \end{aligned} \tag{2.5}$$

Adding $2\rho f(y) - 2\rho f\left(\Pi^f_C x\right)$ to both sides of (2.5) and using Lemma 2.4, we get

$$G_f(y,x) - G_f(\Pi^f_C x, x) - G_f(y, \Pi^f_C x) + 2\rho f(y) = 2\langle y - \Pi^f_C x, J\Pi^f_C x - Jx\rangle + 2\rho f(y) - 2\rho f\left(\Pi^f_C x\right) \geqslant 0, \forall y \in C.$$

This completes the proof. □

Recall that a point $p \in C$ is called a fixed point of a mapping $T : C \to C$ if $Tp = p$. The set of fixed points of T is denoted by $\mathcal{F}(T)$.

Definition 2.7. Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E. Let $\phi : E \times E \longrightarrow [0, +\infty)$ be a functional given by (2.2). A mapping $T : C \longrightarrow C$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in C$;
(ii) quasi-nonexpansive if $\mathcal{F}(T) \neq \emptyset$ and $\|Tx - x^\star\| \leqslant \|x - x^\star\|$, $\forall x \in C$, $x^\star \in \mathcal{F}(T)$;
(iii) quasi-ϕ-firmly nonexpansive (see [16]) if $\mathcal{F}(T) \neq \emptyset$ and

$$\phi(x^\star, Tx) + \phi(Tx, x) \leqslant \phi(x^\star, x), \ \forall x^\star \in \mathcal{F}(T), x \in C;$$

(iv) quasi-ϕ-nonexpansisve (see [27, 49]) if $\mathcal{F}(T) \neq \emptyset$ and

$$\phi(x^\star, Tx) \leqslant \phi(x^\star, x), \ \forall x^\star \in \mathcal{F}(T), x \in C;$$

(v) λ-strict quasi-ϕ-pseudocontractive (see [28]) if $\mathcal{F}(T) \neq \emptyset$, and there exists a constant $\lambda \in [0, 1)$ such that

$$\phi(x^\star, Tx) \leqslant \phi(x^\star, x) + \lambda\phi(x, Tx), \ \forall x \in C, x^\star \in \mathcal{F}(T);$$

(vi) λ-strict quasi-G_f-pseudocontractive if $\mathcal{F}(T) \neq \emptyset$ and there exists $\lambda \in [0, 1)$ such that

$$G_f(x^\star, Tx) \leqslant G_f(x^\star, x) + \lambda(G_f(x, Tx) - 2\rho f(x^\star)) \tag{2.6}$$

or equivalently

$$\langle x - x^\star, Jx - JTx\rangle + \rho f(x) - \lambda\rho f(x^\star) \geqslant \frac{1-\lambda}{2} G_f(x, Tx), \ \ \forall x \in C, x^\star \in \mathcal{F}(T), \tag{2.7}$$

where G_f is as defined in (2.1). We note that an analogous definition is assumed in [40];
(vii) closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$ as $n \to +\infty$, then $Tx = y$.

From the above definitions, we observe that if T is a λ-strict quasi-ϕ-pseudocontractive, then T is a λ-strict quasi-G_f-pseudocontractive with $f = 0$. An example of λ-strict quasi-G_f-pseudocontractive mapping which is not λ-strict quasi-ϕ-pseudocontractive mapping is given below.

Example 2.8. Let $T : \mathbb{R} \longrightarrow \mathbb{R}$ be given by

$$T(x) = 3x, \ \forall x \in \mathbb{R}.$$

Then, we show that T is not λ-strict quasi-ϕ-pseudocontractive but it is λ-strict quasi-G_f-pseudocontractive mapping. Suppose there exists $\lambda \in [0, 1)$ such that

$$\phi(x^\star, Tx) \leqslant \phi(x^\star, x) + \lambda\phi(x, Tx), \ \forall x \in H = \mathbb{R}, x^\star \in \mathcal{F}(T).$$

This implies that

$$|0 - Tx|^2 \leqslant |0 - x|^2 + \lambda|x - Tx|^2,$$

and hence

$$9x^2 \leqslant x^2 + 4\lambda x^2, \forall x \neq 0 \text{ if and only if } \lambda \geqslant 2,$$

which is a contradiction to our assumption. Therefore, T is not λ-strict quasi-ϕ-pseudocontractive mapping.

Next, we show that T is λ-strict quasi-G_f-pseudocontractive mapping.

It is easy to see that for any $\lambda \in (0, 1)$, we have $\lambda\phi(x, Tx) \geqslant 0$. Now, for any $\rho > 0$, take

$$f(x) = \frac{8}{\rho}x^2, \text{ and } \lambda = \tfrac{1}{2}.$$

Then, we have

$$\begin{aligned} G_f(0, Tx) &= \|0\|^2 - 2\langle 0, JTx\rangle + \|Tx\|^2 + 2\rho f(0) \\ &= 9x^2 = x^2 + 8x^2 \leqslant x^2 + 8x^2 + \frac{1}{2}\phi(x, Tx) = G_f(0, x) - 2\rho f(0) + \frac{1}{2}(\phi(x, Tx) + 2\rho f(x)). \end{aligned} \tag{2.8}$$

Thus, from (2.8), we obtain that

$$G_f(0, Tx) \leqslant G_f(0, x) + \lambda(G_f(x, Tx) - 2\rho f(0)), \ \forall x \in C,$$

where $\lambda = \frac{1}{2}$ and $f(x) = \frac{8}{\rho}x^2$. Therefore, T is λ-strict quasi-G_f-pseudocontractive mapping.

Lemma 2.9. *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E. If $T : C \longrightarrow C$ is a closed and λ-strict quasi-G_f-pseudocontractive, then $\mathcal{F}(T)$ is closed and convex.*

Proof. First, we show that $\mathcal{F}(T)$ is closed. Let $\{x_n\}$ be a sequence in $\mathcal{F}(T)$ such that $x_n \to x$ as $n \to +\infty$. We need to show that $x \in \mathcal{F}(T)$. From the definition of T and the lower semicontinuity of f, we have

$$\begin{aligned} \frac{1-\lambda}{2}\phi(x, Tx) + (1-\lambda)\rho f(x) &= \frac{1-\lambda}{2}G_f(x, Tx) \\ &\leqslant \liminf_{n \longrightarrow +\infty} \Big(\langle x - x_n, Jx - JTx\rangle + \rho f(x) - \lambda\rho f(x_n)\Big) \\ &\leqslant \limsup_{n \longrightarrow +\infty} \Big(\langle x - x_n, Jx - JTx\rangle + \rho f(x) - \lambda\rho f(x_n)\Big) \\ &\leqslant \langle x - x, Jx - JTx\rangle + \rho f(x) - \lambda\rho \liminf_{n \longrightarrow +\infty} f(x_n) \\ &\leqslant \rho f(x) - \lambda\rho f(x), \end{aligned}$$

which implies that $\phi(x, Tx) = 0$. Hence, we get

$$G_f(x, Tx) - 2\rho f(x) = 0.$$

Thus, from (2.4), we obtain that $x = Tx$ and hence $\mathcal{F}(T)$ is closed. Next, we show that $\mathcal{F}(T)$ is convex. Let $z_1, z_2 \in \mathcal{F}(T)$ and $z = tz_1 + (1-t)z_2$, where $t \in (0,1)$. We show that $z = Tz$. From the definition of T we have

$$\frac{1-\lambda}{2} G_f(z, Tz) \leqslant \langle z - z_1, Jz - JTz \rangle + \rho f(z) - \lambda \rho f(z_1), \tag{2.9}$$

and

$$\frac{1-\lambda}{2} G_f(z, Tz) \leqslant \langle z - z_2, Jz - JTz \rangle + \rho f(z) - \lambda \rho f(z_2). \tag{2.10}$$

Multiplying inequalities (2.9) and (2.10) by t and $(1-t)$, respectively, yields that

$$\frac{t(1-\lambda)}{2} G_f(z, Tz) \leqslant t\langle z - z_1, Jz - JTz \rangle + t\rho f(z) - t\lambda \rho f(z_1), \tag{2.11}$$

and

$$\frac{(1-t)(1-\lambda)}{2} G_f(z, Tz) \leqslant (1-t)\langle z - z_2, Jz - JTz \rangle + (1-t)\rho f(z) - (1-t)\lambda \rho f(z_2). \tag{2.12}$$

Adding inequalities (2.11) and (2.12) and using the convexity of f, we get

$$\begin{aligned}
\frac{1-\lambda}{2}\phi(z, Tz) + (1-\lambda)\rho f(z) &= \frac{1-\lambda}{2} G_f(z, Tz) \\
&\leqslant t\langle z - z_1, Jz - JTz \rangle + (1-t)\langle z - z_2, Jz - JTz \rangle + \rho f(z) - \lambda\rho t f(z_1) \\
&\quad - \lambda\rho(1-t) f(z_2) \\
&\leqslant \langle z - (tz_1 + (1-t)z_2, Jz - JTz \rangle + \rho f(z) - \lambda \rho f(tz_1 + (1-t)z_2) \\
&= \langle z - z, Jz - JTz \rangle + \rho f(z) - \lambda \rho f(z) = (1-\lambda)\rho f(z),
\end{aligned}$$

which implies that

$$G_f(z, Tz) - 2\rho f(z) = 0.$$

Thus, from (2.4), we get $z = Tz$ and hence $\mathcal{F}(T)$ is convex. □

3. Main result

In this section, we present precise statement of our algorithm for the approximation of the split common fixed point problem (1.1). We will make use of the following assumptions for the convergence of the proposed algorithm.

Conditions:

C1. Let E and F be smooth and uniformly convex real Banach spaces. Let J_E and J_F be the normalized duality mappings on E and F, respectively.

C2. Let $f : E \longrightarrow \mathbb{R}$ and $h : F \longrightarrow \mathbb{R}$ be convex and lower semicontinuous functionals that are bounded from below.

C3. Let $S : E \longrightarrow E$ be a closed λ_1-strict quasi-G_f-pseudocontractive mapping with $\mathcal{F}(S) \neq \emptyset$, and $T : F \longrightarrow F$ be a closed λ_2-strict quasi-G_h-pseudocontractive mapping with $\mathcal{F}(T) \neq \emptyset$.

C4. Let $A : E \longrightarrow F$ be a bounded linear operator with adjoint $A^\star$.

C5. Let $\Omega := \{x^* : x^* \in \mathcal{F}(S) \text{ and } Ax^* \in \mathcal{F}(T)\}$ be nonempty.

C6. Let ρ_1 and ρ_2 be positive real numbers.

Theorem 3.1. *Assume that conditions* C1-C6 *hold. Let* $C_0 = E$ *and for any given* $x_0 \in C_0$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}[J_E S x_n - A^\star(J_F A x_n - J_F T A x_n)], \\ C_{n+1} = \Big\{ z \in C_n : \langle x_n - z, J_E x_n - J_E y_n \rangle + \rho_1 f(x_n) - \lambda_1 \rho_1 f(z) + \rho_2 h(Ax_n) - \lambda_2 \rho_2 h(Az) \\ \qquad\qquad \geqslant \frac{1-\lambda_1}{2} G_f(x_n, Sx_n) + \frac{1-\lambda_2}{2} G_h(Ax_n, TAx_n) \Big\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \forall n \geqslant 0, \end{cases}$$

where $\Pi^f_{C_{n+1}}$ *is the generalized* f*-projection of* E *onto* C_{n+1} *for all* $n \geqslant 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \Omega$, *where* $x^\star = \Pi^f_\Omega x_0$.

Proof. We split the proof into six steps.

Step 1. We show that Ω is closed and convex. Since $\mathcal{F}(S)$ and $\mathcal{F}(T)$ are closed and convex, and A is a bounded linear operator, Ω is closed and convex.

Step 2. We show that C_n is convex for each $n \geqslant 0$. It is obvious that $C_0 = E$ is convex. Suppose that C_k is convex for some $k \geqslant 0$. We show that C_{k+1} is convex for the same k. Let $z_1, z_2 \in C_{k+1}$ and $z = tz_1 + (1-t)z_2$, where $t \in (0, 1)$. It follows that

$$\begin{aligned} \langle x_k - z_1, J_E x_k - J_E y_k \rangle + \rho_1 f(x_k) - \lambda_1 \rho_1 f(z_1) + \rho_2 h(Ax_k) - \lambda_2 \rho_2 h(Az_1) \\ \geqslant \frac{1-\lambda_1}{2} G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2} G_h(Ax_k, TAx_k), \end{aligned}$$

and

$$\begin{aligned} \langle x_k - z_2, J_E x_k - J_E y_k \rangle + \rho_1 f(x_k) - \lambda_1 \rho_1 f(z_2) + \rho_2 h(Ax_k) - \lambda_2 \rho_2 h(Az_2) \\ \geqslant \frac{1-\lambda_1}{2} G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2} G_h(Ax_k, TAx_k). \end{aligned}$$

These two inequalities imply that

$$\begin{aligned} \langle x_k - z_1, J_E x_k - J_E y_k \rangle \geqslant & \frac{1-\lambda_1}{2} G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2} G_h(Ax_k, TAx_k), \\ & - \rho_1 f(x_k) + \lambda_1 \rho_1 f(z_1) - \rho_2 h(Ax_k) + \lambda_2 \rho_2 h(Az_1) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \langle x_k - z_2, J_E x_k - J_E y_k \rangle \geqslant & \frac{1-\lambda_1}{2} G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2} G_h(Ax_k, TAx_k) \\ & - \rho_1 f(x_k) + \lambda_1 \rho_1 f(z_2) - \rho_2 h(Ax_k) + \lambda_2 \rho_2 h(Az_2). \end{aligned} \tag{3.2}$$

Now, from inequalities (3.1) and (3.2) and the convexity of f and h, we have

$$\langle x_k - z, J_E x_k - J_E y_k \rangle + \rho_1 f(x_k) - \lambda_1 \rho_1 f(z) + \rho_2 h(Ax_k) - \lambda_2 \rho_2 h(Az)$$

$$
\begin{aligned}
&= \langle tx_k + (1-t)x_k - (tz_1 + (1-t)z_2), J_E x_k - J_E y_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&= t\langle x_k - z_1, J_E x_k - J_E y_k\rangle + (1-t)\langle x_k - z_2, J_E x_k - J_E y_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&\geqslant t\left(\frac{1-\lambda_1}{2}G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k) - \rho_1 f(x_k) + \lambda_1\rho_1 f(z_1) - \rho_2 h(Ax_k) + \lambda_2\rho_2 h(Az_1)\right)\\
&\quad + (1-t)\left(\frac{1-\lambda_1}{2}G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k)\right.\\
&\quad \left. - \rho_1 f(x_k) + \lambda_1\rho_1 f(z_2) - \rho_2 h(Ax_k) + \lambda_2\rho_2 h(Az_2)\right) + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&= \frac{1-\lambda_1}{2}G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k)\\
&\quad + \lambda_1\rho_1(tf(z_1) + (1-t)f(z_2)) + \lambda_2\rho_2(th(Az_1) + (1-t)h(Az_2)) - \lambda_1\rho_1 f(z) - \lambda_2\rho_2 h(Az)\\
&\geqslant \frac{1-\lambda_1}{2}G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k) + \lambda_1\rho_1 f(z) + \lambda_2\rho_2 h(Az) - \lambda_1\rho_1 f(z) - \lambda_2\rho_2 h(Az)\\
&= \frac{1-\lambda_1}{2}G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\langle x_k - z, J_E x_k - J_E y_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&\quad \geqslant \frac{1-\lambda_1}{2}G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k), \forall z \in C_k.
\end{aligned}
$$

Then we obtain that $z \in C_{k+1}$. Hence, C_{k+1} is convex. Thus, by induction, C_n is convex for all $n \geqslant 0$.

Step 3. We show that C_n is closed for each $n \geqslant 0$. We proceed by induction. Since $C_0 = E$, C_0 is closed. Suppose that C_k is closed for some $k \geqslant 0$. We show that C_{k+1} is also closed for the same k. Let $\{z_m\}$ be a sequence in C_{k+1} such that $z_m \to z$ as $m \to +\infty$. Since $C_{k+1} \subset C_k$ and C_k is closed, $z \in C_k$. We show that $z \in C_{k+1}$. Note that $z_m \in C_{k+1}$ implies that

$$
\begin{aligned}
&\langle x_k - z_m, J_E x_k - J_E y_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z_m) + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az_m)\\
&\quad \geqslant \frac{1-\lambda_1}{2}G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k).
\end{aligned} \tag{3.3}
$$

On the other hand, since A is continuous and f and h are lower semicontinuous, we have

$$
-f(z) \geqslant -\liminf_{m\longrightarrow+\infty} f(z_m) = \limsup_{m\to\infty} -f(z_m)
$$

and

$$
-h(Az) \geqslant -\liminf_{m\to+\infty} h(Az_m) = \limsup_{m\to\infty} -h(Az_m).
$$

Thus, taking lim sup as $m \to +\infty$ on both sides of (3.3), we get $z \in C_{k+1}$. Therefore, C_n is closed for all $n \geqslant 0$. Moreover, the fact that C_n is also convex implies that it is weakly closed for all $n \geqslant 0$.

Step 4. We show that $\Omega \subset C_n$ for each $n \geqslant 0$. It is obvious that $\Omega \subset C_0 = E$. Suppose that $\Omega \subset C_k$ for some $k \geqslant 0$. For any $z \in \Omega \subset C_k$, the definition of S and T imply that

$$
\langle x_k - z, J_E x_k - J_E Sx_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) \geqslant \frac{1-\lambda_1}{2}G_f(x_k, Sx_k), \tag{3.4}
$$

and

$$
\langle Ax_k - Az, J_F Ax_k - J_F TAx_k\rangle + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az) \geqslant \frac{1-\lambda_2}{2}G_h(Ax_k, TAx_k). \tag{3.5}
$$

Furthermore, from (3.4) and (3.5), we have

$$\begin{aligned}
&\langle x_k - z, J_E x_k - J_E y_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&= \langle x_k - z, J_E x_k - J_E Sx_k + A^\star(J_F Ax_k - J_F TAx_k)\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&= \langle x_k - z, J_E x_k - J_E Sx_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \langle x_k - z, A^\star(J_F Ax_k - J_F TAx_k)\rangle + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&= \langle x_k - z, J_E x_k - J_E Sx_k\rangle + \rho_1 f(x_k) - \lambda_1\rho_1 f(z) + \langle Ax_k - Az, J_F Ax_k - J_F TAx_k\rangle + \rho_2 h(Ax_k) - \lambda_2\rho_2 h(Az)\\
&\geqslant \frac{1-\lambda_1}{2} G_f(x_k, Sx_k) + \frac{1-\lambda_2}{2} G_h(Ax_k, TAx_k),
\end{aligned}$$

which implies that $z \in C_{k+1}$. This shows that $\Omega \subset C_{k+1}$ and hence $\Omega \subset C_n$ for each $n \geqslant 0$.

Step 5. We show that the sequence $\{x_n\}$ is bounded and $\|x_{n+1} - x_n\| \to 0$ as $n \to +\infty$. Since $x_n = \Pi^f_{C_n} x_0$ and $\Omega \subset C_n$, for all $y \in \Omega$, it follows from Lemma 2.6 that

$$\begin{aligned}
G_f(x_n, x_0) = G_f(\Pi^f_{C_n} x_0, x_0) &\leqslant G_f(y, x_0) - G_f(y, x_n) + 2\rho_1 f(y)\\
&= G_f(y, x_0) - \phi(y, x_n) \leqslant G_f(y, x_0), \ \forall y \in C_n.
\end{aligned}$$

This implies that the sequence $\{G_f(x_n, x_0)\}$ is bounded. Then, it follows from (2.3) that the sequence $\{x_n\}$ is also bounded. In addition, from Lemma 2.6, we obtain that

$$G_f(x_n, x_0) - 2\rho_1 f(x_{n+1}) \leqslant G_f(x_{n+1}, x_0) - G_f(x_{n+1}, x_n) = G_f(x_{n+1}, x_0) - \phi(x_{n+1}, x_n) - 2\rho_1 f(x_{n+1}),$$

which implies that

$$G_f(x_n, x_0) \leqslant G_f(x_{n+1}, x_0), \ \forall n \geqslant 0.$$

Hence, $\{G_f(x_n, x_0)\}$ is increasing and so $\lim_{n\to+\infty} G_f(x_n, x_0)$ exists.

Furthermore, from Lemma 2.6, we get

$$G_f(x_{n+1}, x_n) = G_f(x_{n+1}, \Pi_{C_n} x_0) \leqslant G_f(x_{n+1}, x_0) - G_f(x_n, x_0) + 2\rho_1 f(x_{n+1}),$$

which implies that

$$G_f(x_{n+1}, x_n) - 2\rho_1 f(x_{n+1}) \leqslant G_f(x_{n+1}, x_0) - G_f(x_n, x_0) \to 0, \ \text{as } n \to +\infty.$$

Therefore, by Lemma 2.5, we get that

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to +\infty. \tag{3.6}$$

Step 6. We show that $x_n \to x^\star$ as $n \to +\infty$, where $x^\star = \Pi^f_\Omega x_0$. In fact, since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \bar{x}$ as $i \to +\infty$. Since C_{n_i} is weakly closed, we get $\bar{x} \in C_{n_i}$ for all $i \geqslant 1$. It follows from Lemma 2.6 that

$$G_f(x_{n_i}, x_0) = G_f(\Pi^f_{C_{n_i}} x_0, x_0) \leqslant G_f(\bar{x}, x_0) - G_f(\bar{x}, x_{n_i}) + 2\rho_1 f(\bar{x}) = G_f(\bar{x}, x_0) - \phi(\bar{x}, x_{n_i}) \leqslant G_f(\bar{x}, x_0). \tag{3.7}$$

On the other hand, from the weak lower semicontinuity of the norm and f together with inequality (3.7), we have

$$\begin{aligned}
\phi(\bar{x}, x_0) &= \|\bar{x}\|^2 - 2\langle \bar{x}, J_E x_0\rangle + \|x_0\|^2\\
&\leqslant \liminf_{i\to+\infty} (\|x_{n_i}\|^2 - 2\langle x_{n_i}, J_E x_0\rangle + \|x_0\|^2)\\
&= \liminf_{i\to+\infty} \phi(x_{n_i}, x_0) \leqslant \limsup_{i\to+\infty} \phi(x_{n_i}, x_0)\\
&= \limsup_{i\to+\infty} (G_f(x_{n_i}, x_0) - 2\rho_1 f(x_{n_i}))
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \limsup_{i\to+\infty}(G_f(x_{n_i},x_0)+\limsup_{i\to+\infty}(-2\rho_1 f(x_{n_i}))\\
&\leqslant G_f(\overline{x},x_0)-2\rho_1\liminf_{i\to+\infty} f(x_{n_i})\\
&\leqslant G_f(\overline{x},x_0)-2\rho_1 f(\overline{x})=\phi(\overline{x},x_0),
\end{aligned}
$$

which implies that

$$\lim_{i\to+\infty}\phi(x_{n_i},x_0)=\phi(\overline{x},x_0). \tag{3.8}$$

Hence, we have $\|x_{n_i}\|\to\|\overline{x}\|$ as $i\to+\infty$. Then by the Kadec-Klee property of E, we obtain $x_{n_i}\to\overline{x}$ as $i\to+\infty$.

Moreover, the weak lower semicontinuity of f together with (3.7) and (3.8) yield

$$
\begin{aligned}
2\rho_1 f(\overline{x})\leqslant \liminf_{i\to+\infty}2\rho_1 f(x_{n_i})&\leqslant \limsup_{i\to+\infty}2\rho_1 f(x_{n_i})=\limsup_{i\to+\infty}\left(G_f(x_n,x_0)-\phi(x_n,x_0)\right)\\
&\leqslant \limsup_{i\to+\infty}G_f(x_{n_i},x_0)+\limsup_{i\to+\infty}\left(-\phi(x_{n_i},x_0)\right)\\
&=\limsup_{i\to+\infty}G_f(x_{n_i},x_0)-\liminf_{i\to+\infty}\phi(x_{n_i},x_0)\\
&\leqslant G_f(\overline{x},x_0)-\phi(\overline{x},x_0)=2\rho_1 f(\overline{x}).
\end{aligned}
$$

Hence, we obtain

$$f(x_{n_i})\to f(\overline{x}) \text{ as } i\to+\infty. \tag{3.9}$$

Since, $Ax_{n_i}\to A\overline{x}$ as $i\to+\infty$, one can also easily show that

$$h(Ax_{n_i})\to h(A\overline{x}) \text{ as } i\to+\infty. \tag{3.10}$$

Furthermore, the fact that $x_{n_i+1}=\Pi^f_{C_{n_i+1}}x_0\in C_{n_i+1}$ implies

$$
\begin{aligned}
&\frac{1-\lambda_1}{2}\left[G_f(x_{n_i},Sx_{n_i})-2\rho_1 f(x_{n_i})\right]+\frac{1-\lambda_2}{2}\left[G_h(Ax_{n_i},TAx_{n_i})-2\rho_2 h(Ax_{n_i})\right]\\
&=\frac{1-\lambda_1}{2}G_f(x_{n_i},Sx_{n_i})+\frac{1-\lambda_2}{2}G_h(Ax_{n_i},TAx_{n_i})-(1-\lambda_1)\rho_1 f(x_{n_i})-(1-\lambda_2)\rho_2 h(Ax_{n_i})\\
&\leqslant\langle x_{n_i}-x_{n_i+1},J_Ex_{n_i}-J_Ey_{n_i}\rangle+\rho_1 f(x_{n_i})-\lambda_1\rho_1 f(x_{n_i+1})+\rho_2 h(Ax_{n_i})-\lambda_2\rho_2 h(Ax_{n_i+1})\\
&\quad-(1-\lambda_1)\rho_1 f(x_{n_i})-(1-\lambda_2)\rho_2 h(Ax_{n_i})\\
&=\langle x_{n_i}-x_{n_i+1},J_Ex_{n_i}-J_Ey_{n_i}\rangle+\lambda_1\rho_1 f(x_{n_i})-\lambda_1\rho_1 f(x_{n_i+1})+\lambda_2\rho_2 h(Ax_{n_i})-\lambda_2\rho_2 h(Ax_{n_i+1}).
\end{aligned}
$$

This together with (3.6), (3.9), and (3.10) imply that

$$G_f(x_{n_i},Sx_{n_i})-2\rho_1 f(x_{n_i})\to 0 \text{ and } G_h(Ax_{n_i},TAx_{n_i})-2\rho_2 h(Ax_{n_i})\to 0 \text{ as } i\to+\infty.$$

In view of Lemma 2.5, we arrive at

$$\|Sx_{n_i}-x_{n_i}\|\to 0 \text{ and } \|TAx_{n_i}-Ax_{n_i}\|\to 0. \tag{3.11}$$

Now, the fact that $x_{n_i}\to\overline{x}$ and $Ax_{n_i}\to A\overline{x}$ as $i\to+\infty$, and

$$\|Sx_{n_i}-\overline{x}\|\leqslant\|Sx_{n_i}-x_{n_i}\|+\|x_{n_i}-\overline{x}\|,\quad \text{and}\quad \|TAx_{n_i}-A\overline{x}\|\leqslant\|TAx_{n_i}-Ax_{n_i}\|+\|Ax_{n_i}-A\overline{x}\|,$$

with inequality (3.11), we obtain that

$$Sx_{n_i}\to\overline{x} \text{ and } TAx_{n_i}\to A\overline{x} \text{ as } i\to+\infty.$$

Thus, from the closedness of S and T, we get $\bar{x} = S\bar{x}$, that is, $\bar{x} \in \mathcal{F}(S)$ and $A\bar{x} = TA\bar{x}$ and hence $A\bar{x} \in \mathcal{F}(T)$. Therefore, $\bar{x} \in \Omega$. Finally, we show that $\bar{x} = x^* = \Pi^f_\Omega x_0$. Indeed, we obtain from $x_{n_i} = \Pi^f_{C_{n_i}} x_0$ that

$$\langle x_{n_i} - y, J_E x_0 - J_E x_{n_i}\rangle + \rho_1 f(y) - \rho_1 f(x_{n_i}) \geqslant 0, \ \forall y \in C_{n_i}.$$

In particular, we have

$$\langle x_{n_i} - \omega, J_E x_0 - J_E x_{n_i}\rangle + \rho_1 f(\omega) - \rho_1 f(x_{n_i}) \geqslant 0, \ \forall \omega \in \Omega. \tag{3.12}$$

Taking the limit as $i \to +\infty$ in (3.12), we obtain that

$$\langle \bar{x} - \omega, J_E x_0 - J_E \bar{x}\rangle + \rho_1 f(\omega) - \rho_1 f(\bar{x}) \geqslant 0, \ \forall \omega \in \Omega.$$

Hence, we obtain from Lemma 2.4 that $\bar{x} = \Pi^f_\Omega x_0$. Since $\{x_{n_i}\}$ is arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to \bar{x} = x^* = \Pi^f_\Omega x_0$. This completes the proof. □

Corollary 3.2. *Assume that conditions* C1, C4, *and* C5 *hold. Let* $h : F \longrightarrow \mathbb{R}$ *be convex, lower semicontinuous functional bounded from below. Let* $S : E \longrightarrow E$ *be a closed* λ_1*-strict quasi-*ϕ*-pseudocontractive mapping, and* $T : F \longrightarrow F$ *be a closed* λ_2*-strict quasi-*G_h*-pseudocontractive mapping. Let* $\rho > 0$ *and* $C_0 = E$*. For any given* $x_0 \in C_0$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}[J_E S x_n - A^\star(J_F A x_n - J_F T A x_n)], \\ C_{n+1} = \Big\{ z \in C_n : \langle x_n - z, J_E x_n - J_E y_n\rangle + \rho h(Ax_n) - \lambda_2 \rho h(Az) \\ \qquad \geqslant \frac{1-\lambda_1}{2}\phi(x_n, Sx_n) + \frac{1-\lambda_2}{2} G_h(Ax_n, TAx_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geqslant 0, \end{cases}$$

where $\Pi_{C_{n+1}}$ *is the generalized projection of* E *onto* C_{n+1} *for all* $n \geqslant 0$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \Omega$*, where* $x^\star = \Pi_\Omega x_0$*.*

Proof. The conclusion follows from Theorem 3.1 with $f = 0$. □

In Theorem 3.1, if we take $h = f = 0$, we obtain the following corollary.

Corollary 3.3. *Assume that conditions* C1, C4, *and* C5 *hold. Let* $S : E \longrightarrow E$ *be a closed* λ_1*-strict quasi-*ϕ*-pseudocontractive mapping and* $T : F \longrightarrow F$ *be a closed* λ_2*-strict quasi-*ϕ*-pseudocontractive mapping. Let* $C_0 = E$ *and for any given* $x_0 \in C_0$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}[J_E S x_n - A^\star(J_F A x_n - J_F T A x_n)], \\ C_{n+1} = \Big\{ z \in C_n : \langle x_n - z, J_E x_n - J_E y_n\rangle \geqslant \frac{1-\lambda_1}{2}\phi(x_n, Sx_n) + \frac{1-\lambda_2}{2} \phi(Ax_n, TAx_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geqslant 0, \end{cases}$$

where $\Pi_{C_{n+1}}$ *is the generalized projection of* E *onto* C_{n+1} *for all* $n \geqslant 0$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \Omega$*, where* $x^\star = \Pi_\Omega x_0$*.*

Corollary 3.4. *Assume that conditions* C1, C2, *and* C4-C6 *hold. Let* $S : E \longrightarrow E$ *and* $T : F \longrightarrow F$ *be closed 0-strict quasi-*G_f*-pseudocontractive and 0-strict quasi-*G_h*-pseudocontractive mappings, respectively. Let* $C_0 = E$ *and for any given* $x_0 \in E$*, let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}[J_E S x_n - A^\star(J_F A x_n - J_F T A x_n)], \\ C_{n+1} = \Big\{ z \in C_n : \langle x_n - z, J_E x_n - J_E y_n\rangle + \rho_1 f(x_n) + \rho_2 h(Ax_n) \geqslant \frac{1}{2} G_f(x_n, Sx_n) + \frac{1}{2} G_h(Ax_n, TAx_n) \Big\}, \\ x_{n+1} = \Pi^f_{C_{n+1}} x_0, \forall n \geqslant 0, \end{cases}$$

where $\Pi^f_{C_{n+1}}$ *is the generalized* f*-projection of* E *onto* C_{n+1} *for all* $n \geqslant 0$*. Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \Omega$*, where* $x^\star = \Pi^f_\Omega x_0$*.*

Proof. The conclusion follows from Theorem 3.1 with $\lambda_1 = \lambda_2 = 0$. □

In Corollary 3.4, if we take $h = f = 0$, we obtain the following corollary.

Corollary 3.5. *Assume that conditions* C1, C_4, *and* C5 *hold. Let* $S : E \longrightarrow E$ *and* $T : F \longrightarrow F$ *be closed quasi-*ϕ*-nonexpansive mappings. Let* $C_0 = E$ *and for any given* $x_0 \in E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}[J_E S x_n - A^\star(J_F A x_n - J_F T A x_n)], \\ C_{n+1} = \left\{ z \in C_n : \langle x_n - z, J_E x_n - J_E y_n \rangle \geqslant \frac{1}{2}\phi(x_n, Sx_n) + \frac{1}{2}\phi(Ax_n, TAx_n) \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geqslant 0, \end{cases}$$

where $\Pi_{C_{n+1}}$ *is the generalized projection of* E *onto* C_{n+1} *for all* $n \geqslant 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \Omega$, *where* $x^\star = \Pi_\Omega x_0$.

We observe that in the definition of λ-strict quasi-G_f-pseudocontractive mapping, if we assume $f(x^\star) = 0$ for all $x^\star \in \mathcal{F}(T)$, then (2.6) reduces to

$$\phi(x^\star, Tx) \leqslant \phi(x^\star, x) + \lambda G_f(x, Tx)$$

and (2.7) reduces to

$$\langle x - x^\star, Jx - JTx \rangle + \rho f(x) \geqslant \frac{1-\lambda}{2} G_f(x, Tx), \quad \forall x \in C, x^\star \in \mathcal{F}(T).$$

4. Applications

4.1. Application to the split common null point problem

Let E be a reflexive, strictly convex, and smooth Banach space. Let $\mathcal{M} : E \longrightarrow 2^{E^\star}$ be a maximal monotone mapping such that $\mathcal{M}^{-1}(0)$ is nonempty. For any $r > 0$, the generalized resolvent of $\mathcal{M}$ (see [16, 35]) is defined by

$$J_r x = (J_E + r\mathcal{M})^{-1} J_E x, \ \forall x \in E.$$

We remark that J_r is closed and quasi-ϕ-nonexpansive from E onto dom($\mathcal{M}$) with $\mathcal{F}(J_r) = \mathcal{M}^{-1}(0) \neq \emptyset$, where dom($\mathcal{M}$) is the domain of $\mathcal{M}$ (see, for example, [16]). Notice that quasi-ϕ-nonexpansive mapping is a special case of λ-strict quasi-G_f-pseudocontractive mappings with $\lambda = 0$ and $f = 0$ (the zero function).

Let E and F be two real Banach spaces, and $\mathcal{M} : E \to 2^{E^*}$ and $\mathcal{N} : F \to 2^{F^*}$ be two maximal monotone mappings such that $\mathcal{M}^{-1}(0) \neq \emptyset$ and $\mathcal{N}^{-1}(0) \neq \emptyset$, respectively. Let $A : E \longrightarrow F$ be a bounded linear operator. The so-called split common null point problem is to

$$\text{find } x^* \in E \text{ such that } 0 \in \mathcal{M}x^\star \text{ and } 0 \in \mathcal{N}(Ax^\star). \tag{4.1}$$

The following theorem approximates the solution of common null point problem given in (4.1).

Theorem 4.1. *Let* E *and* F *be smooth and uniformly convex real Banach spaces. Let* J_E *and* J_F *be the normalized duality mappings on* E *and* F, *respectively. Let* $\mathcal{M} : E \to 2^{E^*}$ *and* $\mathcal{N} : F \to 2^{F^*}$ *be two maximal monotone mappings. Let* J_r *and* Q_t *be the generalized resolvents of* $\mathcal{M}$ *for* $r > 0$ *and* $\mathcal{N}$ *for* $t > 0$, *respectively. Let* $A : E \longrightarrow F$ *be a bounded linear operator with the adjoint operator* $A^\star$. *Suppose that* $\Omega_2 = \{x^\star \in E : x^\star \in \mathcal{M}^{-1}(0) \text{ and } Ax^\star \in \mathcal{N}^{-1}(0)\} \neq \emptyset$. *Let* $C_0 = E$ *and for any given* $x_0 \in E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}[J_E J_r x_n - A^\star(J_F A x_n - J_F Q_t A x_n)], \\ C_{n+1} = \left\{ z \in C_n : \langle x_n - z, J_E x_n - J_E y_n \rangle \geqslant \frac{1}{2}\phi(x_n, J_r x_n) + \frac{1}{2}\phi(Ax_n, Q_t A x_n) \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geqslant 0, \end{cases}$$

where $\Pi_{C_{n+1}}$ *is the generalized projection of* E *onto* C_{n+1} *for all* $n \geqslant 0$. *Then, the sequence* $\{x_n\}$ *converges strongly to* $x^\star \in \Omega_2$, *where* $x^\star = \Pi_{\Omega_2} x_0$.

Proof. Since J_r and Q_t are both closed and 0-strict quasi-G_0 pseudocontractive mappings and $\mathcal{F}(J_r) = \mathcal{M}^{-1}(0)$, $\mathcal{F}(Q_t) = \mathcal{N}^{-1}(0)$, the conclusion follows from Theorem 3.1. □

4.2. Application to the split quasi-inclusion problem

Let E be a reflexive real Banach space with E^* as its dual space. Let $\mathcal{M} : E \to 2^{E^*}$ be a maximal monotone mapping. Then the resolvent $Res_{\lambda\mathcal{M}} : E \to E$ of $\mathcal{M}$ for $\lambda > 0$, is defined by

$$Res_{\lambda\mathcal{M}} = (J_E + \lambda\mathcal{M})^{-1} \circ J_E,$$

where J_E denotes the normalized duality mapping on E. The mapping $\mathcal{M}$ is called Bregman inverse strongly monotone if for any $x, y \in E$ and each $u \in \mathcal{M}x, v \in \mathcal{M}y$, we have

$$\langle u - v, J_E^*(J_E(x) - u) - J_E^*(J_E(y) - v)\rangle \geqslant 0.$$

For any operator $\mathcal{M} : E \to E^*$, the anti-resolvent $\mathcal{M}_\lambda : E \to E$ of $\mathcal{M}$, for $\lambda > 0$, is defined by

$$\mathcal{M}_\lambda = J_E^* \circ (J_E - \lambda\mathcal{M}).$$

We have the following lemma due to Tang et al. [38].

Lemma 4.2 ([38]). *Let E be smooth, strictly convex and reflexive real Banach spaces. Let $\mathcal{M}_1 : E \to E^*$ be a Bregman inverse strongly monotone mapping and $\mathcal{M}_2 : E \to 2^{E^*}$ be a maximal monotone mapping. Define a mapping T_λ by $T_\lambda x := Res_{\lambda\mathcal{M}_2} \circ \mathcal{M}_{1\lambda}(x)$ for $x \in E$ and $\lambda > 0$ such that $\mathcal{F}(T_\lambda)$ is nonempty. Then, the following holds:*

(i) $\mathcal{F}(T_\lambda) = (\mathcal{M}_1 + \mathcal{M}_2)^{-1}(0)$;
(ii) T_λ *is quasi ϕ-nonexpansive mapping.*

Let E be smooth, strictly convex and reflexive real Banach spaces, and J_E and J_F are the duality mappings on E and F, respectively. Let $\mathcal{M}_1 : E \longrightarrow 2^{E^*}$ and $\mathcal{M}_2 : F \longrightarrow 2^{F^*}$ be maximal monotone mappings with the resolvent mappings $Res_{r\mathcal{M}_1}$ for $r > 0$ and $Res_{t\mathcal{M}_2}$ for $t > 0$, respectively. Let $B_1 : E \longrightarrow E^*$ and $B_2 : F \longrightarrow F^*$ be Bregman inverse strongly monotone mappings, $A : E \longrightarrow F$ be a bounded linear mapping with the adjoint operator $A^\star$. The so-called split quasi-inclusion problem is to find a point $x^\star \in E$ such that

$$0 \in \mathcal{M}_1x^\star + B_1x^\star \text{ and } 0 \in \mathcal{M}_2Ax^\star + B_2Ax^\star. \tag{4.2}$$

The following theorem approximates the solution of split quasi-inclusion problem given in (4.2).

Theorem 4.3. *Let E and F be smooth and uniformly convex real Banach spaces. Let J_E and J_F be the normalized duality mappings on E and F, respectively. Let $\mathcal{M}_1 : E \longrightarrow 2^{E^*}$ and $\mathcal{M}_2 : F \longrightarrow 2^{F^*}$ be maximal monotone mappings. Let $B_1 : E \longrightarrow E^*$ and $B_2 : F \longrightarrow F^*$ be Bregman inverse strongly monotone mappings, $A : E \longrightarrow F$ be a bounded linear operator with adjoint $A^\star$. Let $T_\lambda x := Res_{\lambda\mathcal{M}_1} \circ J_E^*(J_E - \lambda B_1)(x)$ for $x \in E$ and $\lambda > 0$ and $S_r x := Res_{r\mathcal{M}_2} \circ J_F^*(J_F - rB_2)(x)$ for $x \in F$ and $r > 0$. Suppose that $\Omega_3 = \{x^\star \in E : x^\star \in (\mathcal{M}_1 + B_1)^{-1}(0)$ and $Ax^\star \in (\mathcal{M}_2 + B_2)^{-1}(0)\} \neq \emptyset$. Let $C_0 = E$ and for any given $x_0 \in E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = J_E^{-1}[J_E T_\lambda x_n - A^\star(J_F Ax_n - J_F S_r Ax_n)], \\ C_{n+1} = \left\{ z \in C_n : \langle x_n - z, J_E x_n - J_E y_n\rangle \geqslant \frac{1}{2}\phi(x_n, T_\lambda x_n) + \frac{1}{2}\phi(Ax_n, S_r Ax_n) \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \forall n \geqslant 0, \end{cases}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1} for all $n \geqslant 0$. Then the sequence $\{x_n\}$ converges strongly to $x^\star \in \Omega_3$, where $x^\star = \Pi_{\Omega_3} x_0$.

Proof. Note that by Lemma 4.2, we have

$$\mathcal{F}(T_\lambda) = (\mathcal{M}_1 + B_1)^{-1}(0) \text{ and } \mathcal{F}(S_r) = (\mathcal{M}_2 + B_2)^{-1}(0).$$

Moreover, we observe that T_λ and S_r are quasi-ϕ-nonexpansive mappings and hence they are 0-strict quasi-G_0-pseudocontractive mappings. Therefore, the conclusion follows from Theorem 3.1. □

5. Numerical example

Example 5.1. Let $H_1 = \mathbb{R}$ and $H_2 = \mathbb{R}^2$. Let $S : H_1 \longrightarrow H_1$, $f : H_1 \longrightarrow [0, +\infty)$, $T : H_2 \longrightarrow H_2$ and $h : H_2 \longrightarrow [0, +\infty)$ be defined by

$$S(x_1) = 3x_1, f(x_1) = \frac{8}{\rho_1}x_1^2, T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 0 \end{pmatrix}, h\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{8}{\rho_2}x_1^2.$$

Let $A : \mathbb{R} \to \mathbb{R}^2$ be a bounded linear mapping defined by

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then, it is easy to see that $A^\star = \begin{pmatrix} 1 & 1 \end{pmatrix}$, S and T are $\frac{1}{2}$-quasi-G_f-pseudocontractive and $\frac{1}{2}$-quasi-G_h-pseudocontractive mappings, respectively. It is also easy to observe that $0 \in \Omega = \{x^\star \in \mathcal{F}(S) : Ax^\star \in \mathcal{F}(T)\}$.

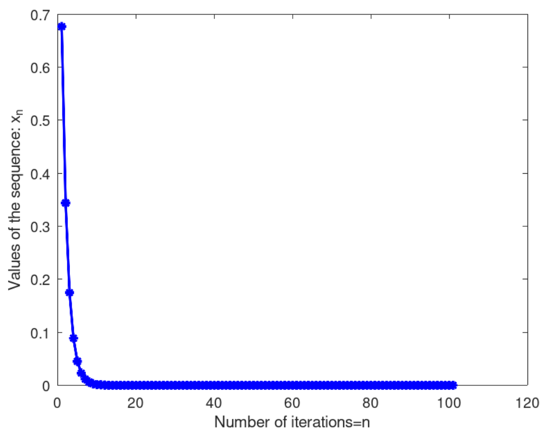
For any given $x_0 \in C_0 = \mathbb{R}$, the sequence $\{x_n\}$ generated by (3.1) reduces to:

$$y_n = 4x_n, \quad C_{n+1} = \left\{z \in C_n : 3|z||x_n| + 13x_n^2 - 8z^2 \geqslant \frac{41}{4}x_n^2\right\}, \quad x_{n+1} = P^f_{C_{n+1}}x_0, \forall n \geqslant 0.$$

The numerical experiments were carried out using MATLAB R2020a version and we obtain the following table and graph.

Table 1: Some values of x_n in the experiment.

Number of iteration=n	Values of x_n
0	rand(1)=x_0
10	$4.2251e-04$
20	$1.3675e-07$
30	$3.1706e-10$
40	$8.6405e-13$
50	$4.9055e-16$
60	$3.3583e-19$
70	$1.1166e-21$
80	$1.1535e-25$
90	$3.0344e-27$
100	$2.7840e-30$



6. Conclusion

In this paper, an iterative algorithm for approximating the solution of the split common fixed point problem involving λ-strict quasi-G_f-pseudocontractive mappings is constructed in the setting of two Banach spaces using functional $G_f(.,.)$. Strong convergence results of the proposed algorithm to a solution of the split common fixed point problem is established. Furthermore, some applications of the proposed method and numerical experiments are presented. The results obtained in this paper extend, unify and complement many of the results in the literature. For instance, the results in this paper enhances and generalizes the work of Takahashi and Yao [36] to Banach spaces more general than Hilbert spaces. In addition, the results in this paper improve and generalize the work of Liu et al. [19] in the sense that the results in this paper are valid for the class of λ-strict quasi-G_f-pseudocontractive mappings which properly includes the class of λ-strict quasi-ϕ-pseudocontractive mappings.

Acknowledgment

The first author is supported by the Simons Foundation (ID: 267269FY17) based at Botswana International University of Science and Technology (BIUST).

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