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Oscillation criteria for a class of half-linear neutral conformable differential equations

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Abstract

The main aim of this note is to obtain new oscillation criteria for a certain class of half-linear neutral conformable differential equations by the method of comparison and Riccati transformation technique. A suitable example is given to illustrate our new results.

Keywords: Half-linear, neutral, oscillation, conformable differential equation.

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1. Introduction

In the last years a lot of studies on fractional differential equations have been made (see, for instance, [2, 16, 17, 21, 25, 44] and the references cited therein). In 2014, Khalil et al. [23] defined the concept of *conformable fractional derivative*. For the definition of this kind of derivative, we refer the reader to [1, 4, 9–13, 15, 19, 23, 34, 35, 41]. It is worth pointing out that conformable fractional derivatives are used in physics. Precisely, Lazo and Torres [27], where it was used to formulate an action principle for particles subjected to frictional forces. In this paper, we discuss the oscillatory behaviour of conformable neutral differential equations of the following form:

$$T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta}\right)+f(t)x^{\beta}(t)=0,\ t\geqslant t_0, \tag{1.1}$$

where $z(t)=x(t)+c(t)x(\delta(t))$ and for the definition of T_{α_i}'s $(i=1,2,3)$, see below Definition 2.1. In the sequel, we always assume that the following hypotheses hold:

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(H_1) $\delta \in C^{\alpha_1}([t_0,\infty), R)$ with $\delta(t) \leqslant t$ and $\lim_{t\to\infty} \delta(t) = \infty$ where C^{α_1} denotes a space of α_1-continuously differentiable functions;
(H_2) $c, f \in C([t_0,\infty), [0,\infty))$ with $0 \leqslant c(t) \leqslant c < 1$ and f does not vanish identically;
(H_3) β is a ratio of odd positive integers;
(H_4) $p, q \in C([t_0,\infty), (0,\infty))$ and satisfy

$$\int_{t_0}^{\infty} \frac{1}{p^{\frac{1}{\beta}}(s)} d_{\alpha_3}s = \int_{t_0}^{\infty} \frac{1}{q(s)} d_{\alpha_2}s = \infty,$$

for a definition of α-fractional integral, see below Definition 2.2.

By a solution of (1.1), we mean a nontrivial function $z(t) \in C^{\alpha_1}[t_*,\infty)$ with $t_* \geqslant t_0$, $(T_{\alpha_2}(q(t)T_{\alpha_1}z(t)))^{\beta} \in C(t_*,\infty)$, $T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}(q(t)T_{\alpha_1}z(t))\right)^{\beta}\right) \in C(t_*,\infty)$ and $x(t)$ satisfies (1.1) on $[t_*,\infty)$. We assume that equation (1.1) possesses such solutions satisfying $\sup\left\{|x(t)| : t \geqslant t'\right\} > 0$ for all $t' \geqslant t_*$. A solution $x(t)$ of (1.1) is said to be *oscillatory* in $[t_*,\infty)$ if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Oscillation phenomena take part in various models from real-world applications, we refer the reader to the papers [20, 32, 33] for models from mathematical biology where oscillation and/or delay actions may be formulated by means of cross-diffusion terms. In the last years a lot of studies related to the oscillation of ordinary differential equations have been made. See, for instance, [14, 24, 26, 36–39, 42, 43] and the references cited therein. In particular, half-linear equations have numerous applications in the study of p-Laplace equations, non-Newtonian fluid theory, porous medium, and so forth; see, for instance, the papers [6, 8] for more details and the papers [7, 8, 22, 30] and [3, 5, 6, 18, 28, 29, 31, 40] regarding the oscillation of half-linear equations and half-linear neutral equations, respectively. On the other hand, we mention that in the aforementioned works, conformable differential equations were not considered and, for this reason, this article extends the previous studies.

The results established in this paper are improvements of results in [45, 46]. In the source papers [45, 46], the results are derived in integer order differential equations. Here we extend the results in fractional order which is the conformal analogue of the main paper. The example which is given here cannot be dealt with in the integer case. This conformal is more general than the integer class.

2. Preliminaries

In this section, we describe the mathematical background that will be useful in the sequel. For the sake of brevity, for any $T \geqslant t_0$, we set:

$$P(T,t) = \int_T^t \frac{1}{p^{\frac{1}{\beta}}(s)} d_{\alpha_2}s = \int_T^t s^{1-\alpha_2}\frac{1}{p^{\frac{1}{\beta}}(s)} ds,$$
$$Q(T,t) = \int_T^t \frac{P(T,s)}{q(s)} d_{\alpha_1}s = \int_T^t s^{1-\alpha_1}\frac{P(T,s)}{q(s)} ds,$$
$$S(T,t) = \frac{t_1^{\alpha_1-1}}{q(t)} \int_T^{\infty} s^{1-\alpha_2}\left(\frac{1}{p(s)}\int_s^{\infty} u^{1-\alpha_3} f(u) du\right)^{\frac{1}{\beta}} ds,$$
$$R(T,t) = \exp\left(\int_T^t S(T,s)ds\right).$$

Definition 2.1 ([23]). Let $f : [0,\infty) \to \mathbf{R}$ and $t > 0$. Then the fractional derivative of f order of α is defined by

$$T_{\alpha}(f)(t) := \lim_{\epsilon\to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}$$

for $t > 0$ and $\alpha \in (0,1]$. Moreover, if f is α differentiable in a certain $(0,a)$, $a > 0$ and $\lim_{t\to 0+} T_{\alpha}(f)(t)$

exists, we define

$$T_\alpha(f)(0) := \lim_{t\to 0+} T_\alpha(f)(t).$$

Definition 2.2 ([44]). Let $\alpha \in (0,1]$ and $0 \leqslant a < b$. A function $f : [a,b] \to \mathbf{R}$ is α-fractional integral on $[a,b]$ if the integral

$$\int_a^b f(x)d_\alpha x = \int_a^b f(x)x^{\alpha-1}dx$$

exists and it is finite.

In order to prove the main results, we need the following technical lemmas.

Lemma 2.3 ([44]). *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leqslant 1$. Then for all $t > a$, we have*

$$I^a_\alpha T^a_\alpha(f)(t) = f(t) - f(a).$$

Lemma 2.4. *Let us assume that $x(t)$ is a positive solution of equation (1.1). Then, the corresponding function $z(t)$ satisfies one of the following two cases for all sufficiently large t:*

(I) $z(t) > 0, T_{\alpha_1}z(t) < 0,\ T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right) > 0,\ T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^\beta\right) \leqslant 0;$

(II) $z(t) > 0, T_{\alpha_1}z(t) > 0,\ T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right) > 0,\ T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^\beta\right) \leqslant 0.$

The proof of the lemma above follows from well-known results of Kiguradze and Chanturia [24].

Lemma 2.5. *Let us assume that $x(t)$ is a positive solution of equation (1.1) and let $z(t)$ satisfies Case (II) of Lemma 2.4. Then*

$$x(t) \geqslant (1 - c(t))z(\delta(t)) \tag{2.1}$$

for all sufficiently large t.

Proof. Taking into account the definition of $z(t)$, we have $z(t) \geqslant x(t)$ and

$$x(t) \geqslant z(t) - c(t)z(\delta(t)) \geqslant (1 - c(t))z(\delta(t)),$$

since z is increasing. □

Lemma 2.6. *Let us suppose that $x(t)$ is a positive solution of equation (1.1) with $z(t)$ satisfying Case (I) of Lemma 2.4 and assume that $\chi(t) = \frac{S(T,\delta(t))}{S(T,t)} - c(t) > 0$ for $t \geqslant T$. Then, $z(t)S(T,t)$ is increasing and*

$$x(t) \geqslant \chi(t)z(\delta(t)), \quad \textit{for } t \geqslant T. \tag{2.2}$$

Proof. Suppose that $x(t)$ is a positive solution of (1.1) with $z(t)$ satisfying Case (I) of Lemma 2.4 for all $t \geqslant T$, for some $T \geqslant t_0$. Then it is easy to see that $\lim_{t\to\infty} q(t)T_{\alpha_1}z(t) = 0$ and $\lim_{t\to\infty} p(t)\left(T_{\alpha_2}q(t)T_{\alpha_1}z(t)\right)^\beta = 0$. Taking I_{α_3} integration of (1.1) from t to ∞, we get

$$I_{\alpha_3}T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^\beta\right) = -\int_t^\infty f(s)x^\beta(s)d_{\alpha_3}s,$$

that is,

$$p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^\beta \leqslant \int_t^\infty f(s)z^\beta(s)d_{\alpha_3}s \leqslant z^\beta(t)\int_t^\infty f(s)d_{\alpha_3}s.$$

Taking I_{α_2} integration of the above inequality from t to ∞, we obtain

$$I_{\alpha_2}T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right) \leqslant z(t)\int_t^\infty \left(\frac{1}{p(s)}\int_s^\infty f(u)d_{\alpha_3}u\right)^\beta d_{\alpha_2}s,$$

that is,

$$q(t)T_{\alpha_1}z(t) \geqslant -z(t)\int_t^\infty \left(\frac{1}{p(s)}\int_s^\infty f(u)d_{\alpha_3}u\right)^\beta d_{\alpha_2}s,$$

hence

$$T_{\alpha_1}z(t) \geqslant -z(t)t^{1-\alpha_1}R(T,t).$$

Therefore,

$$T_{\alpha_1}\left(z(t)S(T,t)\right) = T_{\alpha_1}z(t)S(T,t) + z(t)T_{\alpha_1}S(T,t) \geqslant z(t)(T_{\alpha_1}S(T,t) - t^{1-\alpha_1}S(T,t)R(T,t)) = 0,$$

which implies that $z(t)S(T,t)$ is increasing. Using the fact that the $z(t)S(T,t)$ is increasing and definition of z, we have

$$x(t) \geqslant z(t) - c(t)z(\delta(t)) = \frac{z(t)S(T,t)}{S(T,t)} - c(t)z(\delta(t)) \geqslant \left(\frac{S(T,\delta(t))}{S(T,t)} - c(t)\right)z(\delta(t)).$$

This completes the proof. □

Lemma 2.7. *Let us suppose that $x(t)$ is a positive solution of equation (1.1) and that $z(t)$ satisfies Case (II) of Lemma 2.4 for all $t \geqslant T$. Then*

$$T_{\alpha_1}z(t) \geqslant \frac{p^{\frac{1}{\beta}}(t)}{q(t)}T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)P(T,t),$$

$$z(t) \geqslant p^{\frac{1}{\beta}}(t)T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)Q(T,t), \tag{2.3}$$

$$z(\delta(t)) \geqslant Q(T,\delta(t))\frac{q(t)T_{\alpha_1}z(t)}{P(T,t)}, \tag{2.4}$$

for all $t \geqslant T$.

Proof. Since $T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^\beta\right) \leqslant 0$, we get that $p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^\beta)$ is nondecreasing. Then

$$\begin{aligned} q(t)T_{\alpha_1}z(t) &\geqslant q(t)T_{\alpha_1}z(t) - q(t)T_{\alpha_1}z(t) \\ &= \int_T^t \frac{\left[[p(s)T_{\alpha_2}[q(s)T_{\alpha_1}z(s)]]^\beta\right]^{\frac{1}{\beta}}}{p^{\frac{1}{\beta}}(s)} d_{\alpha_2}s \geqslant p^{\frac{1}{\beta}}(t)T_{\alpha_2}[q(t)T_{\alpha_1}z(t)]P(T,t), \end{aligned}$$

that is,

$$T_{\alpha_1}z(t) \geqslant \frac{1}{q(t)}p^{\frac{1}{\beta}}(t)T_{\alpha_2}[q(t)T_{\alpha_1}z(t)]P(T,t). \tag{2.5}$$

Taking the I_{α_1} integration of the inequality (2.5) from T to t, we get

$$z(t) \geqslant p^{\frac{1}{\beta}}(t)T_{\alpha_2}[q(t)T_{\alpha_1}z(t)]\int_T^t \frac{P(T,s)}{q(s)}d_{\alpha_1}s.$$

The proof follows as in the lines from Lemma 4 in [45]. □

3. Main results

Theorem 3.1. *Let* $T_{\alpha_1}\delta(t) > 0$ *and assume that there exists a function* $\tau(t) \in C^{\alpha_1}[t_0, \infty)$ *such that*

$$T_{\alpha_1}\tau(t) \geqslant 0,\ \tau(t) > t \ \text{ and } \ \zeta(t) = \delta(\tau(\tau(t))) < t. \tag{3.1}$$

If both the first order delay differential equations

$$T_{\alpha_1}W(t) + \left[\frac{1}{q(t)}\int_t^{\tau(t)} s_2^{1-\alpha_2}\left(\frac{1}{p(s_2)}\int_{s_2}^{\tau(s_2)} s_1^{1-\alpha_3} f(s_1)\chi^{\beta}(s_1)ds_1\right) ds_2 ds_3\right] t^{2(1-\alpha_1)}W(\zeta(t)) = 0 \tag{3.2}$$

and

$$T_{\alpha_3}W(t) + f(t)(1-c(t))^{\beta}Q^{\beta}[T, \delta(t)]W(\delta(t)) = 0 \tag{3.3}$$

are oscillatory, then (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a positive solution of (1.1). Then there exists $T \geqslant t_0$ such that $x(t) > 0$ and $x(\delta(t) > 0)$ for all $t \geqslant T$. From the definition of $z(t)$, we infer that $z(t) > 0$ for all $t \geqslant T$, where t is also chosen so that Lemmas 2.4-2.7 hold for all $t \geqslant T$. Therefore, we have following two cases.

Case (I). Substituting equation (2.2) in equation (1.1), we get

$$T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta}\right) + f(t)\chi^{\beta}(t)z^{\beta}(\delta(t)) \leqslant 0.$$

Taking I_{α_3} integration from t to $\tau(t)$, we get

$$p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta} \geqslant \int_t^{\tau(t)} f(s_1)\chi^{\beta}(s_1)z^{\beta}(\delta(s_1))d_{\alpha_3}s_1 \geqslant z^{\beta}(\delta(\tau(t)))\int_t^{\tau(t)} f(s_1)\chi^{\beta}(s_1)d_{\alpha_3}s_1.$$

Hence,

$$T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right) \geqslant z(\delta(\tau(t)))\left(\frac{1}{p(t)}\int_t^{\tau(t)} f(s_1)\chi^{\beta}(s_1)d_{\alpha_3}s_1\right)^{\frac{1}{\beta}}.$$

Taking I_{α_2} integration from t to $\tau(t)$, we get

$$-q(t)T_{\alpha_1}z(t) \geqslant \int_t^{\tau(t)} z(\delta(\tau(s_2)))\left(\frac{1}{p(s_2)}\int_{s_2}^{\tau(s_2)} f(s_1)\chi^{\beta}(s_1)d_{\alpha_3}s_1\right)^{\frac{1}{\beta}} d_{\alpha_2}s_2,$$

$$-T_{\alpha_1}z(t) \geqslant z(\zeta(t))\frac{1}{q(t)}\int_t^{\tau(t)}\left(\frac{1}{p(s_2)}\int_{s_2}^{\tau(s_2)} f(s_1)\chi^{\beta}(s_1)d_{\alpha_3}s_1\right)^{\frac{1}{\beta}} d_{\alpha_2}s_2.$$

Finally, taking I_{α_1} integration from t to ∞, we get

$$z(t) \geqslant \int_t^{\infty} s_3^{1-\alpha_1}\frac{z(\zeta(s_3))}{q(s_3)}\int_{s_3}^{\tau(s_3)} s_2^{1-\alpha_2}\left(\frac{1}{p(s_2)}\int_{s_2}^{\tau(s_2)} s_1^{1-\alpha_3}f(s_1)\chi^{\beta}(s_1)ds_1\right)^{\frac{1}{\beta}} ds_2 ds_3. \tag{3.4}$$

For the sake of brevity, we denote the right side of equation (3.4) by $W(t)$. Then $W(t) > 0$ is decreasing, $W(t) < z(t)$ and it is clear that $W(t)$ is a positive solution of the following conformable differential inequality:

$$T_{\alpha_1}W(t) + \left(\frac{1}{q(t)}\int_t^{\tau(t)} s_2^{1-\alpha_2}\left(\frac{1}{p(s_2)}\int_{s_2}^{\tau(s_2)} s_1^{1-\alpha_3}f(s_1)\chi^{\beta}(s_1)ds_1\right)^{\frac{1}{\beta}} ds_2 ds_3\right) t^{2(1-\alpha_1)}W(\zeta(t)) \leqslant 0.$$

We conclude that the corresponding conformable differential equation (3.2) also has a positive solution by Theorem 1 in [36], which is a contradiction.

Case (II). Substituting (2.1) in (1.1), we get

$$T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta}\right)+f(t)\left(1-c(t)\right)^{\beta}z^{\beta}(\delta(t))\leqslant 0,\ t\geqslant T. \tag{3.5}$$

From (2.3),

$$z^{\beta}\delta(t)\geqslant p(\delta(t))\left(T_{\alpha_2}\left(q(\delta(t))T_{\alpha_1}z(\delta(t))\right)\right)^{\beta}Q^{\beta}(T,\delta(t)),\ t\geqslant T. \tag{3.6}$$

Using the inequality (3.6) in inequality (3.5), we obtain

$$T_{\alpha_3}\left(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta}\right)+f(t)\left(1-c(t)\right)^{\beta}p(\delta(t))\left(T_{\alpha_2}\left(q(\delta(t))T_{\alpha_1}z(\delta(t))\right)\right)^{\beta}Q^{\beta}(T,\delta(t))\leqslant 0,\ t\geqslant T.$$

Let

$$W(t)=(p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta}>0.$$

Then $w(t)$ is a positive solution of the inequality

$$T_{\alpha_3}W(t)+f(t)\left(1-c(t)\right)^{\beta}Q^{\beta}(T,\delta(t))W(\delta(t))\leqslant 0.$$

We conclude that the corresponding conformable differential equation (3.3) also has a positive solution by Theorem 1 in [36], which is a contradiction. This completes the proof. □

Corollary 3.2. *Let the conditions of Theorem 3.1 hold. If*

$$\liminf_{t\to\infty}\int_{\zeta(t)}^{t}\frac{1}{q(s_3)}\int_{s_3}^{\tau(s_3)}\left(\frac{1}{p(s_2)}\int_{s_2}^{\tau(s_2)}f(s_1)\chi^{\frac{1}{\beta}}(s_1)d_{\alpha_1}s_1\right)^{\frac{1}{\beta}}d_{\alpha_2}s_2d_{\alpha_3}s_3>\frac{1}{e} \tag{3.7}$$

and

$$\liminf_{t\to\infty}\int_{\delta(t)}^{t}f(s)\left(1-c(s)\right)^{\beta}Q^{\beta}(T,\delta(t))d_{\alpha_1}s_1>\frac{1}{e} \tag{3.8}$$

are fulfilled, then (1.1) is oscillatory.

Theorem 3.3. *Let $T_{\alpha_1}\delta(t)>0$ and let $\tau(t)\in C^{\alpha_1}[t_0,\infty)$ satisfy (3.1) and (3.7). If there exists a real valued nondecreasing differentiable function $\rho(t)$ such that*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\rho(s)f(s)\left(1-c(s)\right)^{\beta}\frac{Q^{\beta}(T,\delta(s))}{P^{\beta}(T,s)}-\frac{(s^{1-\alpha_2})^{\beta}p(s)(\rho'(s))^{\beta+1}}{(\beta+1)^{\beta+1}\rho^{\beta}(s)}\right]ds=\infty, \tag{3.9}$$

then (1.1) is oscillatory.

Proof. Let $x(t)$ be a positive solution of equation (1.1). Proceeding as in the proof of Theorem 3.1, we see that $z(t)$ satisfies one of cases in Lemma 2.4. Case (I) can be proved by using condition (3.7) as in the proof of Theorem 3.1. Now, let us consider Case (II). Let us define

$$W(t)=\frac{\rho(t)p(t)\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta}}{\left(q(t)T_{\alpha_1}z(t)\right)^{\beta}}\ \text{for}\ t\geqslant T, \tag{3.10}$$

then $W(t)>0$ for all $t\geqslant T$. Differentiating (3.10) and using (3.5), we obtain

$$T_{\alpha_3}W(t)\leqslant t^{1-\alpha_3}\frac{\rho'(t)}{\rho(t)}W(t)-f(t)\rho(t)(1-c(t))^{\beta}\frac{z^{\beta}(\delta(t))}{\left(q(t)T_{\alpha_1}z(t)\right)^{\beta}}$$

$$-t^{1-\alpha_3}\beta\rho(t)p(t)\frac{\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta}\left(q(t)T_{\alpha_1}z(t)\right)'}{\left(q(t)T_{\alpha_1}z(t)\right)^{\beta+1}},$$

that is,

$$t^{1-\alpha_3}W'(t) \leqslant t^{1-\alpha_3}\left(\frac{\rho'(t)}{\rho(t)}W(t) - t^{\alpha_3-1}f(t)\rho(t)(1-c(t))^{\beta}\frac{z^{\beta}(\delta(t))}{\left(q(t)T_{\alpha_1}z(t)\right)^{\beta}}\right)$$
$$-t^{1-\alpha_3}t^{\alpha_2-1}\beta\rho(t)p(t)\frac{\left(T_{\alpha_2}\left(q(t)T_{\alpha_1}z(t)\right)\right)^{\beta+1}}{\left(q(t)T_{\alpha_1}z(t)\right)^{\beta+1}},$$

which implies that

$$W'(t) \leqslant \frac{\rho'(t)}{\rho(t)}W(t) - t^{\alpha_3-1}f(t)\rho(t)(1-c(t))^{\beta}\frac{z^{\beta}(\delta(t))}{\left(q(t)T_{\alpha 1}z(t)\right)^{\beta}} - \frac{t^{\alpha_2-1}\beta W^{\frac{\beta+1}{\beta}}(t)}{\rho^{\frac{1}{\beta}}(t)p^{\frac{1}{\beta}}(t)},\ t \geqslant T. \tag{3.11}$$

Using the inequality $Av - Bv^{\frac{\beta+1}{\beta}} \leqslant \frac{\beta^{\beta}}{\beta+1^{\beta+1}}\frac{A^{\beta+1}}{B^{\beta}}$ with $A = \frac{\rho'(t)}{\rho(t)}, B = \frac{\beta t^{\alpha_2-1}}{\rho^{\frac{1}{\beta}}(t)p^{\frac{1}{\beta}}(t)}$, and $v = W(t)$ in equation (3.11), we get

$$W'(t) \leqslant -\rho(t)f(t)\left(1-c(t)\right)^{\beta} z^{\beta}\frac{z(\delta(t))}{\left(q(t)T_{\alpha_1}z(t)\right)^{\beta}} + \frac{p(t)(\rho'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\rho^{\beta}(t)(t^{\alpha_2-1})^{\beta}}. \tag{3.12}$$

At last, we use (2.4) in (3.12) and then integrating the resulting inequality from T to t, yields

$$\int_T^t\left[\rho(s)f(s)\left(1-c(s)\right)^{\beta}\frac{Q^{\beta}(T,\delta(s))}{p^{\beta}(T,s)} - \frac{p(t)(\rho'(t))^{\beta+1}}{(\beta+1)^{\beta+1}\rho^{\beta}(t)(t^{\alpha_2-1})^{\beta}}\right]ds \leqslant W(t) < \infty.$$

This contradicts (3.9) and completes the proof. □

4. Example

In this section, we present an example to illustrate the effectiveness of the main results.

Example 4.1. Let us consider the following conformable neutral delay differential equation

$$T_{\frac{1}{2}}\left(tT_{\frac{1}{2}}\left(tT_{\frac{1}{2}}\left(x(t)+cx(\mu t)\right)\right)\right) + \frac{1}{t^{\frac{5}{2}}}x(t) = 0,\ t \geqslant 1. \tag{4.1}$$

Here, we have $p(t) = q(t) = t, f(t) = \frac{1}{t^{\frac{5}{2}}}, \alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{2}, \beta = 1, \delta(t) = \mu t$ with $0 < \mu < 1$ and we take $c < \mu^2$. We can easily check that

$$\begin{aligned}
P(1,t) &= \int_1^t \frac{1}{p(s)}d_{\frac{1}{2}}(s) = \int_1^t \frac{1}{s}s^{1-\frac{1}{2}}ds = 2(\sqrt{t}-1),\\
Q(1,t) &= \int_1^t \frac{2(\sqrt{s}-1)}{s}s^{1-\frac{1}{2}}ds = 2(\sqrt{t}-1)^2,\\
R(1,t) &= \frac{t^{\frac{1}{2}}}{t}\int_t^{\infty} s^{\frac{1}{2}}\frac{1}{s}\int_s^{\infty} u^{\frac{1}{2}}\frac{1}{u^2\sqrt{u}}duds = \frac{2}{t},\\
S(1,t) &= \exp\left(\int_1^t \frac{2}{s}ds\right) = t^2,\\
\chi(t) &= \mu^2 - c > 0.
\end{aligned}$$

Choose $\tau(t) = kt$ with $k > 1$ and $\mu k^2 < 1$. Then $\zeta(t) = \mu k^2 t < t$) and (3.7) becomes

$$\begin{aligned}
&\liminf_{t\to\infty} \int_{\zeta(t)}^{t} \frac{1}{s_3} \int_{s_3}^{ks_3} \frac{1}{s_2} \int_{s_2}^{ks_2} \frac{1}{\sqrt{s_1}} \frac{1}{s_1^2} (\mu^2 - c) d_{\alpha_1} s_1 d_{\alpha_2} s_2 d_{\alpha_3} s_3 \\
&= \liminf_{t\to\infty} \int_{\zeta(t)}^{t} \frac{1}{s_3} \int_{s_3}^{ks_3} \frac{1}{s_2} (\mu^2 - c) \frac{1}{s_2} (1 - \frac{1}{k}) d_{\alpha_2} s_2 d_{\alpha_3} s_3 \\
&= \liminf_{t\to\infty} \int_{\zeta(t)}^{t} \frac{1}{s_3} (\mu^2 - c)(1 - \frac{1}{k})(1 - \frac{1}{\sqrt{k}}) \frac{2}{\sqrt{s_3}} d_{\alpha_3} s_3 \\
&= 2(\mu^2 - c)(1 - \frac{1}{k})(1 - \frac{1}{\sqrt{k}}) \ln \frac{1}{\mu k^2},
\end{aligned}$$

and

$$Q(T, \delta(t)) = \int_1^t 2 \frac{\sqrt{\mu s} - 1}{s} s^{\frac{1}{2}} ds = 2(\sqrt{\mu} t - 2\sqrt{t} - \sqrt{\mu} - 2).$$

Condition (3.8) becomes

$$\liminf_{t\to\infty} \int_{\mu t}^{t} \frac{2(1-c)}{s^2 \sqrt{s}} (\sqrt{\mu} s - 2\sqrt{s} - \sqrt{\mu} - 2) s^{\frac{1}{2}} ds = 2(1-c)\sqrt{\mu} \ln \frac{1}{\mu}.$$

Hence, by Corollary 3.2, (4.1) is oscillatory if $2(1-c)\sqrt{\mu} \ln \frac{1}{\mu} > \frac{1}{e}$ and $2(\mu^2 - c)(1 - \frac{1}{k})(1 - \frac{1}{\sqrt{k}}) \ln \frac{1}{\mu k^2} > \frac{1}{e}$.

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