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Continuity and separation axioms via infra-topological spaces

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Abstract

In order to investigate a particular topic in mathematics, more specifically, general topology, it is always desirable to find a weaker condition. This work is planned to study a weak (topological) structure named infra-topological space. An infra-topological space is the collection of subsets of a universe that includes the empty set and is closed under finite intersections. The continuity, openness, and homeomorphism of mappings between infra-topological spaces are explored. Through the use of some examples, analogous properties and characterizations of ordinary mappings cannot be hopped on infra-topological structures. Then, the concepts of product and coproduct of infra-topological spaces are analyzed. Furthermore, the notion of infra-quotient topologies, which are inspired by infra-continuity, is introduced. The essential properties indicate that infra-quotient topologies and ordinary quotient topologies act in parallel. The final part of this paper is devoted to the investigation of infra separation axioms (infra T_i-spaces, $i = 0, 1, \dots, 4$). The behaviour of ordinary separation axioms cannot be translated to an infra-topological structure. More precisely, infra-T_3 and infra-T_4-spaces are independent, and singletons need not be infra-closed in infra-T_1-spaces.

Keywords: Infra-topology, infra-continuity, infra-open, infra separation axioms.

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1. Introduction and preliminaries

The area of topology that deals with the fundamental set-theoretic notions and constructions used in topology is known as general topology. Most other areas of topology, such as differential topology, geometric topology, and algebraic topology, are built on it. The continuous supply of classes of topological spaces, examples, and their features and linkages have aided the development of topology until reached its peak. Researchers then considered broadening the concept of topological space.

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Mashhour et al. [31] established the concept of a supra topological space that is free from being closed under finite intersections in 1983, which was the first invention. Many authors from all across the world have contributed to this research; see, [4, 5, 13, 23, 28, 37].

Császár [20] began a systematic study of families that are closed exclusively by arbitrary unions under the term of a generalized topological space in the following decade. In computer science and its applications, this field of research is critical. Soldano [38] considered the generalized topological space, under the name of extensional abstraction, in formal concept analysis and data clustering. In [27, 34], it also appeared as an application in Banach game theory and the entropy problem.

Following that, additional topological space generalizations appeared. Minimal structure [30], weak structure [21], generalized weak structure [18], ordered structure [8] and so on are examples of these types of structures. Many researchers have also examined these concepts in depth; see, [19, 22, 33]. In fact, over the last two decades, the entire research field has developed.

Topology has become an important tool to model some real-life problems; this matter was investigated using somewhere dense [6], somewhat open sets [7], ideal structures [14, 25, 26], neighbourhood systems [29, 35], and near open sets [36]. Recently, extensions of topology have been applied to cope with uncertainty issues, for example, infra-topologies [1], minimal structures [24], infra-fuzzy topologies [17] and supra topologies [28].

Recently, Al-Shami and Alshammari [15] have introduced novel rough sets models inspired by supra topological spaces. Then, Al-Shami and Mhemdi [15] have successfully applied infra-topologies to initiate new kinds of approximation operators and address some medical issues. In fact, these two contributions showed the important role of two generalizations of topology to describe some practical problems and open the door for more investigations in this area of research.

Al-Odhari [3] formally introduced the concept of an infra-topological space in 2015, although it has received little attention from researchers. In addition, as Witczak [39] pointed out, infra-topological spaces are interesting to investigate. As a result, we attempt to conduct a comprehensive investigation of infra-topological spaces. Many aspects of topological spaces are still applicable to infra-topological spaces, as we can see. We demonstrate, using examples, that specific relationships between certain topological notions are simpler in infra-topological spaces.

Definition 1.1 ([3]). Let $X \neq \emptyset$. The family $\tau \subseteq 2^X$ is called an infra-topology on X if $\emptyset \in \tau$ and for each finite $\mathcal{H} \subseteq \tau$ implies $\bigcap \mathcal{H} \in \tau$. We call (X, τ) an infra-topological space. Each $H \in \tau$ is called infra-open and H^c (or $R \in \tau^c$) is called infra-closed.

Definition 1.2 ([3]). For a subset H of (X, τ), the infra-interior, referred to $\mathtt{iInt}(H)$ and infra-closure, referred to $\mathtt{iCl}(H)$, of H are respectively defined as follows:

(i) $\mathtt{iInt}(H) = \cup\{G : G \in \tau, G \subseteq H\}$;
(ii) $\mathtt{iCl}(H) = \cap\{F : F \in \tau^c, H \subseteq F\}$.

Proposition 1.3 ([39]). *For subsets H, R of (X, τ), the following are true:*

(i) *if $H \in \tau$, then $\mathtt{iInt}(H) = H$;*
(ii) *if $R \in \tau^c$, then $\mathtt{iCl}(R) = R$.*

Proposition 1.4 ([39]). *The following properties hold for a subset H of (X, τ):*

(i) *$x \in \mathtt{iInt}(H)$ iff there exists $G \in \tau$ with $x \in G \subseteq H$;*
(ii) *$x \in \mathtt{iCl}(H)$ iff $H \bigcap G \neq \emptyset$ for each $G \in \tau$ with $x \in G$.*

Proposition 1.5 ([39]). *For a subset H of (X, τ), we have:*

(i) *$(\mathtt{iInt}(H))^c = \mathtt{iCl}(H^c)$;*
(ii) *$(\mathtt{iCl}(H))^c = \mathtt{iInt}(H^c)$.*

Theorem 1.6 ([39]). *Let G, H be subsets of (X, τ). The infra-interior operator has the next properties:*

(i) $\mathrm{iInt}(X) = X$;
(ii) $\mathrm{iInt}(G) \subseteq G$;
(iii) *if* $G \subseteq H$, *then* $\mathrm{iInt}(G) \subseteq \mathrm{iInt}(H)$;
(iv) $\mathrm{iInt}(\mathrm{iInt}(G)) = \mathrm{iInt}(G)$;
(v) $\mathrm{iInt}(G \bigcap H) = \mathrm{iInt}(G) \bigcap \mathrm{iInt}(H)$.

Theorem 1.7 ([39]). *Let* E, F *be subsets of* (X, τ). *The infra-closure operator has the next properties:*

(i) $\mathrm{iCl}(\emptyset) = \emptyset$;
(ii) $E \subseteq \mathrm{iCl}(E)$;
(iii) *if* $E \subseteq F$, *then* $\mathrm{iCl}(E) \subseteq \mathrm{iCl}(F)$;
(iv) $\mathrm{iCl}(\mathrm{iCl}(E)) = \mathrm{iCl}(E)$;
(v) $\mathrm{iCl}(E \bigcup F) = \mathrm{iCl}(E) \bigcup \mathrm{iCl}(F)$.

The remainder of this work is structured as follows. Section 2 investigates certain mappings between infra-topological structures. We characterize these mappings and show that some of their topological properties are evaporated. Then, we investigate the concept of infra-quotient topology in Section 3. In Section 4, we explore the celebrated separation axioms via infra-topological structures with the help of interesting examples. Finally, in Section 5, we summarize the important contributions and suggest some further work.

2. Homeomorphism between infra-topological spaces

The concepts of continuity, openness, closedness, and homeomorphism from an infra-topological space into another infra-topological space are introduced in this section. We scrutinize their counterpart properties via topological spaces that are still valid and construct some counterexamples to show which one of them is invalid.

Definition 2.1. Let $g : (X, \tau) \to (Y, \sigma)$ be a mapping. It is said that g is

(i) infra-continuous if $g^{-1}(H) \in \tau$ for every $H \in \sigma$;
(ii) infra-open (resp. infra-closed) if $g(H) \in \sigma$ (resp. $g(F) \in \sigma^c$) for each $H \in \tau$ (resp. $F \in \tau^c$).

Proposition 2.2. *Let* $g : (X, \tau) \to (Y, \sigma)$ *be a mapping. Then* g *is infra-continuous iff* $g^{-1}(F) \in \tau^c$ *for every* $F \in \sigma^c$.

Proof. It follows from Definition 2.1 (i) and the fact that $g^{-1}(A^c) = (g^{-1}(A))^c$ for every subset $A \subseteq Y$. □

Theorem 2.3. *Let* $g : (X, \tau) \to (Y, \sigma)$ *be a mapping. If* g *is infra-continuous, then the following hold:*

(i) $\mathrm{iCl}(g^{-1}(H) \subseteq g^{-1}(\mathrm{iCl}(H))$ *for each set* H *in* Y;
(ii) $g(\mathrm{iCl}(F)) \subseteq \mathrm{iCl}(g(F))$ *for each set* F *in* X;
(iii) $g^{-1}(\mathrm{iInt}(H)) \subseteq \mathrm{iInt}(g^{-1}(H))$ *for each set* H *in* Y.

Proof. To prove (i), let $x \notin g^{-1}(\mathrm{iCl}(H))$. Then, $g(x) \notin \mathrm{iCl}(H)$, so there exists $V \in \sigma$ such that $g(x) \in V$ and $V \bigcap H = \emptyset$. This automatically means that $g^{-1}(V) \bigcap g^{-1}(H) = \emptyset$. By hypothesis, $g^{-1}(V) \in \tau$, which means that $x \notin \mathrm{iCl}(g^{-1}(H))$. Thus, $\mathrm{iCl}(g^{-1}(H) \subseteq g^{-1}(\mathrm{iCl}(H))$. Similarly, one can prove (ii). To prove (iii), let $x \in g^{-1}(\mathrm{iInt}(H))$. Then, $g(x) \in \mathrm{iInt}(H)$, so there exists $V \in \sigma$ such that $g(x) \in V \subseteq H$. Therefore, $x \in g^{-1}(V) \subseteq g^{-1}(H)$. By hypothesis, $g^{-1}(V) \in \tau$, which means that $x \in \mathrm{iInt}(g^{-1}(H))$. Hence, $g^{-1}(\mathrm{iInt}(H)) \subseteq \mathrm{iInt}(g^{-1}(H))$, as required. □

Proposition 2.4. *Let* $g : (X, \tau) \to (Y, \sigma)$ *be a mapping. If* g *is infra-open, then* $g(\mathrm{iInt}(H)) \subseteq \mathrm{iInt}(g(H))$ *for each set* H *in* Y.

Proof. Let $y \in g(\mathtt{iInt}(H))$. Then there is $x \in \mathtt{iInt}(H)$ with $g(x) = y$. Therefore, there is $G \in \tau$ such that $x \in G \subseteq H$. Evidently, $y = g(x) \in g(G) \subseteq g(H)$. By assumption, $g(G)$ is infra-open, which means that $y \in \mathtt{iInt}(g(H))$. Hence, the result is proved. □

Corollary 2.5. *Let $g : (X, \tau) \to (Y, \sigma)$ be a mapping. If g is infra-open, then $f(N)$ is an infra-neighborhood of $g(x)$ for each infra-neighborhood N of $x \in X$.*

Proposition 2.6. *Let $g : (X, \tau) \to (Y, \sigma)$ be a surjection. If g is infra-open, then $g^{-1}(iCl(F)) \subseteq iCl(g^{-1}(F))$ for each set F in Y.*

Proof. Let $x \notin iCl(g^{-1}(F))$. Then, there exists $V \in \tau$ along with $x \in V$ and $V \bigcap g^{-1}(F) = \emptyset$. Since g is surjective, $g(V) \bigcap F = \emptyset$. By hypothesis, $g(V) \in \sigma$, which means that $g(x) \notin iCl(F)$. Thus, $x \notin g^{-1}(iCl(F))$. This finishes the proof that $g^{-1}(iCl(F)) \subseteq iCl(g^{-1}(F))$. □

Proposition 2.7. *Let $g : (X, \tau) \to (Y, \sigma)$ be a mapping. If g is infra-closed, then $iCl(g(H)) \subseteq g(iCl(H))$ for each set H in X.*

Proof. If $y \notin g(iCl(H))$, then $g^{-1}(y) \bigcap iCl(H) = \emptyset$. This implies that for each $x \in g^{-1}(y)$, there is $U \in \tau$ with $x \in U$ such that $U \cap H = \emptyset$. Directly, $g(H) \subseteq g(U^c)$. Now, $g(x) = y \notin g(U^c)$. As g is infra-closed, we obtain $iCl(g(H)) \subseteq g(U^c)$, which means that $y \notin iCl(g(H))$. Hence, the proof. □

Proposition 2.8. *A bijection $g : (X, \tau) \to (Y, \sigma)$ is infra-open iff it is infra-closed.*

Proof. Since $g(H^c) = (g(H))^c$ for each $H \subseteq X$, the proof follows. □

In point of fact, the structure of infra-topology prevents several characterizations of continuous, open, and closed mappings, which are the counterparts of infra-continuous, infra-open, and infra-closed mappings. For instance, the three properties given in Theorem 2.3 are equivalent to the continuity in topological structures. Also, the sufficient conditions given in Propositions 2.4, 2.6, and 2.7 are satisfied via topological structures. But this matter is not satisfied via infra-topological structures as shown in the next example.

Example 2.9. Consider $g : (\mathcal{R}, \tau) \to (\mathcal{R}, \sigma)$ and $f : (\mathcal{R}, \sigma) \to (\mathcal{R}, \tau)$ as identity mappings, where $\tau = \{\mathbb{R}, G \subseteq \mathbb{R} : G \text{ is finite}\}$ is an infra-topology on $\mathcal{R}$, and σ is the discrete topology on $\mathcal{R}$. It can be noted that the three properties given in Theorem 2.3 hold for a mapping g, whereas g is not infra-continuous. Also, note that the sufficient conditions given in Propositions 2.4, 2.6, and 2.7 hold for a mapping f, whereas f is neither infra-open nor infra-closed.

Proposition 2.10. *Let a mapping $g : (X, \tau) \to (Y, \sigma)$ be infra-open. The restriction mapping $g_U : (U, \tau_U) \to (Y, \sigma)$ is infra-open whenever $U \in \tau$.*

Proof. Let $G \in \tau_U$. Then there is $V \in \tau$ such that $G = V \cap U$. Therefore, $G \in \tau$. By hypothesis, $g_U(G) = g(G) \in \sigma$ and hence, g_U is infra-open. □

Proposition 2.11. *The following statements hold for mappings $g : (X, \tau) \to (Y, \sigma)$ and $f : (Y, \sigma) \to (Z, \mu)$:*

(i) *if g, f are infra-continuous (resp. infra-open), then $f \circ g$ is infra-continuous (resp. infra-open);*
(ii) *if $f \circ g$ is infra-open, g is infra-continuous surjective, then f is infra-open;*
(iii) *if $f \circ g$ is infra-open, f is infra-continuous injective, then g is infra-open.*

Proof.

(i). Evident.

(ii). If H is infra-open in Y, by continuity of g, $g^{-1}(H)$ is infra-open in X. Thus, $(f \circ g)(g^{-1}(H))$ is infra-open in Z. By surjectivity of g, we have that $(f \circ g)(g^{-1}(H)) = f(g(g^{-1}(H))) = f(H)$. Therefore, the mapping f is infra-open.

If G is an infra-open set in X, by assumption, $(f \circ g)(G)$ is infra-open in Z. Then, $f^{-1}(f \circ g(G))$ is infra-open in Y. Since f is an injection, so $f^{-1}(f \circ g(G)) = (f^{-1}f)(g(G)) = g(G)$. Hence, g is infra-open. □

The next result can be proven using similar arguments as those used in the previous proof.

Proposition 2.12. *For mappings* $g : (X, \tau) \to (Y, \sigma)$, $f : (Y, \sigma) \to (Z, \mu)$, *the following statements are true:*

(i) *if* g, f *are infra-closed, then* $f \circ g$ *is infra-closed;*
(ii) *if* $f \circ g$ *is infra-closed,* g *is infra-continuous surjective, then* f *is infra-closed;*
(iii) *if* $f \circ g$ *is infra-closed,* f *is infra-continuous injective, then* g *is infra-closed.*

Proposition 2.13. *Suppose* $\{(X_i, \tau_i) : i \in I\}$ *is any indexed set of infra-topological spaces. Then* $\tau = \{\prod_{i \in I} U_i : U_i \in \tau_i\}$ *forms an infra-topology on* $X = \prod_{i \in I} X_i$.

Proof. One can easily check that $X, \emptyset \in \tau$. Now, give any members $\prod_{i \in I} U_i$ and $\prod_{i \in I} V_i$ of τ. Since for each coordinate i, $U_i \cap V_i \in \tau_i$, this implies that $\left(\prod_{i \in I} U_i\right) \cap \left(\prod_{i \in I} V_i\right) = \prod_{i \in I} (U_i \cap V_i))$ and so τ is includes all of its finite intersections. Hence, the proof is completed. □

The τ, as defined in the preceding assertion, is a product of infra-topologies, and (X, τ) is a product of infra spaces.

Definition 2.14. For a collection of mappings $\{g_i : (X_i, \tau_i) \to (Y_i, \sigma_i) : i \in I\}$, the product of mappings is a mapping $g : (X, \tau) \to (Y, \sigma)$ which is defined by $g((x_i)_{i \in I}) = (g(x_i))_{i \in I}$.

Whenever I is countable, then g can be written as:

$$g((x_0, x_1, \ldots)) = (g(x_0), g(x_1), \ldots).$$

Theorem 2.15. *Let* $\{g_i : (X_i, \tau_i) \to (Y_i, \sigma_i) : i \in I\}$ *be a collection of infra-continuous mappings. Then the product of* g_i *is infra-continuous.*

Proof. Suppose $g : (X, \tau) \to (Y, \sigma)$ is the product of infra-continuous mappings $\{g_i : (X_i, \tau_i) \to (Y_i, \sigma_i) : i \in I\}$. If U is an infra-open set in Y, since $U = \prod_{i \in I} U_i$ where $U_i \in \sigma_i$ for each i, then $g^{-1}(U) = g^{-1}\left(\prod_{i \in I} U_i\right) = \prod_{i \in I} g_i^{-1}(U_i)$. By assumption, $g_i^{-1}(U_i)$ is an infra-open set in X_i for each i. Therefore, $g^{-1}(U)$ is an infra-open set in X, according to the concept of the product of infra spaces. As a result, we achieve the intended result. □

It can be proved the next two results following similar technique displayed in the above proof.

Theorem 2.16. *Let* $\{g_i : (X_i, \tau_i) \to (Y_i, \sigma_i) : i \in I\}$ *be a collection of infra-open mappings. Then the product of* g_i *is infra-open.*

Corollary 2.17. *Let* $\{g_i : (X_i, \tau_i) \to (Y_i, \sigma_i) : i \in I\}$ *be a collection of infra-closed bijections. Then the product of* g_i *is infra-closed.*

Definition 2.18. Let $g : (X, \tau) \to (Y, \sigma)$ be a bijective mapping. Then g is called an infra homeomorphism if it is infra-continuous and infra-open.

We eliminate the proofs of the next two consequences since they are simple:

Proposition 2.19. *If* $g : (X, \tau) \to (Y, \sigma)$, $f : (Y, \sigma) \to (Z, \mu)$ *are infra-homeomorphisms, then* $f \circ g$ *is an infra-homeomorphism.*

Proposition 2.20. *For a bijection* $g : (X, \tau) \to (Y, \sigma)$*, the statements that follow are equivalent:*

(i) g *is an infra-homeomorphism;*
(ii) g, g^{-1} *are infra-continuous;*
(iii) g *is infra-closed and infra-continuous.*

Proposition 2.21. *For a subset* $H \subseteq X$ *under an infra-homeomorphism* $g : (X, \tau) \to (Y, \sigma)$*, the following statements hold:*

(i) $g(\mathrm{iInt}(H)) = \mathrm{iInt}(g(H))$;
(ii) $g(\mathrm{iCl}(H)) = \mathrm{iCl}(g(H))$.

Proof.
(i). The direction, $g(\mathrm{iInt}(H)) \subseteq \mathrm{iInt}(g(H))$, follows from Proposition 2.4. Conversely, let $y \in \mathrm{iInt}(g(H))$. Then there exists $G \in \sigma$ for which $y \in G \subseteq g(H)$. By assumption, $x = g^{-1}(y) \in g^{-1}(G) \subseteq H$, where $g^{-1}(G) \in \tau$. Therefore, $x \in \mathrm{iInt}(H)$, which means that $y \in g(\mathrm{iInt}(H))$. Thus, $\mathrm{iInt}(g(H)) \subseteq g(\mathrm{iInt}(H))$. Hence, we obtain the desired result.

The proof of (ii) comes from Theorem 2.3 and Proposition 2.7. □

Definition 2.22. A property is called an infra-topological invariant if it is the property that is preserved by infra-homeomorphisms.

Theorem 2.23. *Being an infra-dense set is an infra-topological invariant.*

Proof. Let D be an infra-dense set in X (i.e., $\mathrm{iCl}(D) = X$) and let $g : (X, \tau) \to (Y, \sigma)$ be an infra-homeomorphism. We need to show that the image of D under g is infra-dense in Y. By the use of Proposition 2.21 (ii), we have that $\mathrm{iCl}(g(D)) = g(\mathrm{iCl}(D)) = g(X) = Y$. But this means that $g(D)$ is infra-dense in Y, and we are done. □

Theorem 2.24. *The product of two infra-homeomorphisms is again an infra-homeomorphism.*

Proof. Since the product of bijective mappings is bijective, so Theorems 2.15 and 2.16 finish the proof. □

Proposition 2.25. *Let* $\{(X_i, \tau_i) : i \in I\}$ *be a collection of nonempty pairwise disjoint infra-topological spaces. The set*

$$\tau = \{G : G \cap X_i \text{ is infra-open in } (X_i, \tau_i) \text{ for all } i \in I\}$$

is an infra-topology on $X = \bigcup_{i \in I} X_i$.

Proof. Evidently, both X and $\emptyset$ belong to τ. It lefts to prove that τ is closed under finite intersections. Let $G_1, G_2 \in \tau$. Then, $G_1 \cap X_i \in \tau_i$ and $G_2 \cap X_i \in \tau_i$ for all $i \in I$. This implies that $(G_1 \cap G_2) \cap X_i \in \tau_i$ for each $i \in I$. Therefore, $G_1 \cap G_2 \in \tau$ and hence, τ is an infra-topology on X. □

An infra-topological space presented in the above assertion is referred to as a sum (coproduct) of infra-topological spaces and is symbolized by $(\oplus X_i, \tau)$.

Proposition 2.26. *A subset* H *of* $(\oplus X_i, \tau)$ *is infra-closed iff* $H \cap X_i$ *is infra-closed in* X_i *for each* $i \in I$.

Proof. For each $i \in I$, a subset H of $\oplus X_i$ is infra-closed iff $H^c \cap X_i$ is infra-open in X_i iff $H \cap X_i$ is infra-closed in X_i. □

Definition 2.27. For a collection $\{g_i : (X_i, \tau_i) \to (Y_i, \sigma_i) : i \in I\}$ of mappings, a mapping $g : (\oplus X_i, \tau) \to (\oplus Y_i, \sigma)$ is defined by $g(x) = g_i(x)$ for $x \in X_i$. The following is the image and inverse image of a set under g:

(i) the image of $U \subseteq \oplus X_i$ is given by $g(U) = \bigcup_{i \in I} g_i(U \cap X_i)$;

(ii) the inverse image of $V \subseteq \oplus Y_i$ is given by $g^{-1}(V) = \bigcup_{i \in I} g_i^{-1}(V \cap X_i)$.

Theorem 2.28. *A mapping $g : (\oplus X_i, \tau) \to (\oplus Y_i, \sigma)$ is infra-continuous (resp. infra-open, infra closed) iff all mappings $g_i : (X_i, \tau_i) \to (Y_i, \sigma_i)$ are infra-continuous (resp. infra-open, infra-closed).*

Proof. In case of infra-open mapping.

Assume $g : (\oplus X_i, \tau) \to (\oplus Y_i, \sigma)$ is an infra-open mapping. For $j \in I$, consider a mapping $g_j : (X_j, \tau_j) \to (Y_j, \sigma_j)$. If G is an infra-open set in X_j, then G is infra-open in $\oplus X_i$. Therefore, $g(G)$ is infra-open in $\oplus Y_i$. Therefore, $g(G) = g_j(G)$, since $G \cap X_i = \emptyset$ for each $i \neq j$. Thus, $g_j(G)$ is an infra-open set in Y_j.

Conversely, for all $i \in I$, we assume $g_i : (X_i, \tau_i) \to (Y_i, \sigma_i)$ be an infra-open mapping. If G is an infra-open set in $\oplus X_i$, then $g_i(G \cap X_i)$ is an infra-open set in Y_i for each $i \in I$ as $G \cap X_i$ is an infra-open set in X_i. From the definition of σ on $\oplus Y_i$, $\bigcup_{i \in I} g_i(G \cap X_i)$ is infra-open in $\oplus Y_i$. But, $g(G) = \bigcup_{i \in I} g_i(G \cap X_i)$, and so $g(G)$ is infra-open in $\oplus Y_i$. Thus, the proof is completed.

Similar arguments can be used to demonstrate the case of infra-closedness and infra-continuity of mappings. □

Corollary 2.29. *A mapping g from $(\oplus X_i, \tau)$ onto $(\oplus Y_i, \sigma)$ is infra-homeomorphism iff all g_i (X_i, τ_i) onto (Y_i, σ_i) are infra-homeomorphisms.*

3. Infra-quotient topology

The main goal of the present section is to define the concept of infra-quotient topology. We give some descriptions for this concept and discuss some of the fundamental properties.

Definition 3.1. Let $q : (X, \tau) \to (Y, \sigma)$ be a mapping. A family $\sigma \subseteq 2^Y$ is called an infra-quotient topology on Y related to q if σ is the greatest infra-topology which makes q infra-continuous. Notice that $\sigma = \{U \subseteq Y : q^{-1}(U) \in \tau\}$.

The way of constructing an infra-quotient topology is illustrated in the next example.

Example 3.2. Let $\tau = \{\emptyset, X, \{1\}, \{2\}, \{1, 2, 3\}\}$ be an infra-topology on $X = \{0, 1, 2, 3\}$. Let $Y = \{a, b, c\}$ be another set, and let $q : X \to Y$ be a mapping defined as follows:

$$q(0) = c,\ q(1) = a, \text{ and } q(2) = q(3) = b.$$

Then $\sigma = \{\emptyset, Y, \{a\}, \{a, b\}\}$ is the infra-quotient topology on Y relate to q. Note that any infra-topology on Y is strictly finer than σ leads to that q is not infra-continuous.

Theorem 3.3. *Consider the infra-continuous mapping $g : (X, \tau) \to (Y, \sigma)$. Then the arguments that follow are equivalent:*

(i) *σ is an infra-quotient topology;*

(ii) *G is infra-open in Y iff $g^{-1}(G)$ is infra-open in X;*

(iii) *H is infra-closed in Y iff $g^{-1}(H)$ is infra-closed in X.*

Proof.

(i) $\Longrightarrow$ (ii): The first direction follows from the fact that g is infra-continuous. To prove the other part, let $g^{-1}(G)$ be an infra-open set in X. Since σ is an infra-quotient topology, then $G \in \sigma$, as required.

(ii) $\Longrightarrow$ (i): Suppose otherwise that there is a larger infra-topology ν than σ makes g infra-continuous. Now, let $G \in \nu$. Then $g^{-1}(G) \in \tau$. By assumption, $G \in \sigma$. Therefore, $\sigma = \nu$, we are done.

(ii) $\Longleftrightarrow$ (iii): It is an immediate consequence of the statement that $g^{-1}(G^c) = (g^{-1}(G))^c$ for any subset G. □

Corollary 3.4. *Let $g : (X, \tau) \to (Y, \sigma)$ be an infra-continuous mapping. If g in addition is either infra-open or infra-closed, then σ is an infra-quotient topology.*

Proof. The infra-continuity of g guarantees that $g^{-1}(H)$ is infra-open in X for each infra-open H in Y. Now, since g is infra-open surjective, $H = g(g^{-1}(H))$ is infra-open. The part (ii) in the above theorem assures that σ is an infra-quotient topology.

When g is an infra-closed mapping can be proved by a similar technique. □

Proposition 3.5. *Let a mapping f from (X, τ) onto an infra-quotient topological space (Y, σ) be infra-continuous and let (Z, μ) be an infra topological space. A mapping $g : (Y, \sigma) \to (Z, \mu)$ is infra-continuous iff $g \circ f$ is infra-continuous.*

Proof. This first part follows from Proposition 2.11 (i).

Conversely, if V is infra-open in Z, then $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is infra-open in X as $g \circ f$ is infra-continuous. But σ is an infra-quotient topology, so by Theorem 3.3, $g^{-1}(V)$ is infra-open in Y. Hence, the proof. □

Definition 3.6. Let $g : X \to Y$ be any mapping. Then g is called an infra-quotient mapping if g is surjective and Y is endowed with the infra-quotient topology related to g.

In other words, $g : X \to Y$ is an infra-quotient mapping whenever g is surjective and possesses the property that a set V in Y is infra-open iff $g^{-1}(V)$ is infra-open in X.

4. Infra T_i-spaces ($i = 0, 1, \ldots, 4$)

This part of our work studies the T_i-spaces $(i = 0, 1, \ldots, 4)$ in the frame of infra-topological spaces. With the aid of counterexamples, we elucidate the relationships between them as well as determine some topological features that are missing. It can be noted that one of the advantages which gives infra-topological spaces their own beauty is that many topological features of these spaces are still valid.

Definition 4.1. An infra-topological space (X, τ) is said to be:

(i) infra T_0 if for each $r, s \in X$ with $r \neq s$, there is an infra-open set G such that $r \in G$, $s \notin G$, or $s \in G$, $r \notin G$;
(ii) infra T_1 if for each $r, s \in X$ with $r \neq s$, there are infra-open sets G and H such that $r \in G \setminus H$ and $s \in H \setminus G$;
(iii) infra T_2 (or infra-Hausdorff) if for each $r, s \in X$ with $r \neq s$, there are infra-open sets G and H such that $r \in G$, $s \in H$ and $G \bigcap H = \emptyset$;
(iv) infra-regular if for each infra-closed set F and every $r \in X$ such that $r \notin F$, there are infra-open sets G and H such that $r \in G$, $F \subseteq V$ and $G \bigcap H = \emptyset$;
(v) infra-normal if each disjoint infra-closed sets can be separated by disjoint infra-open sets;
(vi) infra T_3 if X is infra-regular and infra T_1;
(vii) infra T_4 if X is infra normal and infra T_1.

Proposition 4.2. *If (X, τ) is infra T_i-space, then (X, τ) is infra T_{i-1} whenever $i \in \{1, 2, 3\}$.*

Proof. We only prove the case of $i = 3$, the other cases are easier. Let (X, τ) be an infra T_3-space and let r, s such that $r \neq s$. Since X is infra-T_1, one can find an infra-open set H such that $r \in H$ and $s \notin H$. Now, H^c is infra-closed with $r \notin H^c$. By infra regularity of X, there are infra-open sets U and V such that $H^c \subseteq U$, $r \in V$, and $U \bigcap V = \emptyset$. Thus, X is infra T_2. □

The next examples demonstrate that we cannot, in general, reverse the above result.

Example 4.3. Let $\tau = \{\emptyset, X, \{t\}, \{u\}\}$ be an infra-topology on $X = \{t, u, v\}$. Then (X, τ) is infra T_0 but not infra T_1.

Example 4.4. The infra-topology $\tau = \{\emptyset, G \subseteq \mathbb{N} : G \text{ is finite}\}$ on the set of natural numbers $\mathbb{N}$ is infra T_1 but not infra T_2.

Example 4.5. Let $\tau = \{\emptyset, X, \{t\}, \{u\}, \{v\}\}$ be an infra-topology on $X = \{t, u, v\}$. It can be seen that (X, τ) is infra T_2. On the other hand, a point v does not belong to an infra-closed set $\{t, u\}$, but no infra-open sets can separate them. Thus (X, τ) is not infra-regular and so it is not infra T_3.

Note that (X, τ) given in Example 4.5 is infra T_4. As a result, the preceding example and the next example indicate that infra T_3 and infra T_4-spaces are unrelated of one another.

Example 4.6. It is well known from general topology that Niemytzki plane (it is also called Moore plane) is a topological space with a T_3 axiom but it does not satisfy a T_4 axiom. So we suffice with this infra-topological space as an example of T_3 which is not T_4.

Now, we investigate and analyze some more aspects of infra T_i-spaces.

Proposition 4.7. *For finite* X, *a space* (X, τ) *is infra* T_1 *iff each singleton subset is infra-open.*

Proof. Let $r \in X$. Then for each $s \in X$ distinct with r, one can find infra-open sets U_r and V_s along with $r \in U_r \setminus V_s$ and $s \in V_s \setminus U_r$. So, the family $\{U_r^c : s \in U_r^c\}$ includes infra-closed sets such that $X \setminus \{r\} = \bigcup U_r^c$. As X is finite, so $X \setminus \{r\}$ is infra-closed, and hence $\{r\}$ is infra-open. The reverse part is obvious. □

Corollary 4.8. *If* (X, τ) *is a finite infra* T_1*-space, then* $|\tau| \geqslant |X| + 2$.

Proposition 4.9. *If each singleton subset of* (X, τ) *is infra-closed, then* (X, τ) *is infra* T_1.

Proof. Given $r, s \in X$ such that $r \neq s$, since $\{r\}, \{s\}$ are infra-closed sets, then $\{r\}^c$, $\{s\}^c$ are infra-open. Clearly, $s \in \{r\}^c \setminus \{s\}^c$ and $r \in \{s\}^c \setminus \{r\}^c$, thus (X, τ) is infra T_1. □

Note that every singleton subset of (X, τ) given in Example 4.4 is not an infra-closed set in spit of (X, τ) is infra T_1, as a result, the converse of the preceding argument is false in general.

Proposition 4.10. *A space* (X, τ) *is infra* T_1 *iff* $iCl(\{r\}) = \{r\}$ *for each* $r \in X$.

Proof. Take an arbitrary point $r \in X$, then for any $s \neq r \in X$ there is an infra-open set G such that $s \in G$ and $r \notin G$. This means that $s \notin iCl(\{r\})$ for each $s \neq r$. Thus, $iCl(\{r\}) = \{r\}$. Conversely, let $r, s \in X$ with $r \neq s$. By assumption, $iCl(\{r\}) = \{r\}$ and $iCl(\{s\}) = \{s\}$, then there exist infra-open sets G, H respectively including r, s such that $G \bigcap \{s\} = \emptyset$ and $H \bigcap \{r\} = \emptyset$. Hence, (X, τ) is infra T_1. □

Proposition 4.11. *A space* (X, τ) *is infra* T_1 *iff* $\{r\} = \bigcap \{G : r \in G \in \tau\}$ *for each* $r \in X$.

Proof. Take a point $r \in X$, then for any $s \neq r \in X$ we can find infra-open G such that $r \in G_s$ and $s \notin G_s$. This means that $r \in \bigcap_{r \in G_s} G_s$ and $s \notin \bigcap_{r \in G_s} G_s$ for each $s \in X \setminus \{r\}$. Thus, $\{r\} = \bigcap \{G : r \in G \in \tau\}$. Conversely, let $r, s \in X$ with $r \neq s$. By assumption, $\{r\} = \bigcap \{G_r : r \in G_r \in \tau\}$ and $\{s\} = \bigcap \{G_s : s \in G_s \in \tau\}$, which imply there exist infra-open sets G_r, G_s containing r, s, respectively, such that $s \notin G_r$ and $r \notin G_s$. Thus, (X, τ) is infra T_1. □

Proposition 4.12. *A space* (X, τ) *is infra* T_2 *iff for each* $r \neq s$, *there is* $U \in \tau$ *with the property that* $r \in U$ *and* $s \notin iCl(U)$.

Proof. Let r, s with $r \neq s$ be points of infra T_2-space X. Then there exist infra-open sets U, V that are disjoint and $r \in U$, $s \in V$. Now, $U \subseteq V^c$ which means $iCl(U) \subseteq V^c$. Since $s \notin V^c$, $s \notin iCl(U)$. Conversely, let $r, s \in X$ with $r \neq s$. If U is any infra-open set having the property $r \in U$ and $s \notin iCl(U)$, then we can find infra-open G such that $s \in G$, $U \cap G = \emptyset$. This shows that (X, τ) is infra T_2. □

Proposition 4.13. *If a space* (X,τ) *is infra-regular, then for any* $r \in X$ *and any infra-open set* H *with* $r \in H$, *there is infra-open* G *such that* $r \in G \subseteq iCl(G) \subseteq H$.

Proof. Let $H \in \tau$ with $r \in H$. Then H^c is infra-closed such that $r \notin H^c$. Since (X,τ) is infra-regular, then we have disjoint infra-open sets U and G such that $H^c \subseteq U$ and $r \in G$. Evidently, $G \subseteq U^c \subseteq H$, and so $iCl(G) \subseteq U^c \subseteq H$. □

Note that the sufficient condition of the above proposition is satisfied for (X,τ) given in Example 4.4, but (X,τ) is not infra-regular. As a result, the converse of the preceding proposition is false in general.

Theorem 4.14. *The following properties are equivalent whenever a space* (X,τ) *is infra-regular:*

(i) (X,τ) *is infra* T_2;
(ii) (X,τ) *is infra* T_1;
(iii) (X,τ) *is infra* T_0.

Proof. Proposition 4.2 proves the forward implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii).

We now show that (iii) $\Longrightarrow$ (i). Given $r,s \in X$ with $r \neq s$, since (X,τ) is infra T_0, then we can find an infra-open set U along with the property that $r \in U$, $s \notin U$, or $s \in U$, $r \notin U$. If $r \in U$ and $s \notin U$, then surely $r \notin U^c$ and $s \in U^c$. Since (X,τ) is infra-regular, then one have two disjoint infra-open sets G and H such that $r \in G$ and $s \in G^c \subseteq H$. Since G, H are disjoint, and $s \notin G$, $r \notin H$, this proves that (X,τ) is infra T_2. □

Proposition 4.15. *Let* (X,τ) *be a finite infra* T_2*-space. Then* $|\tau| \geqslant |X|+2$, *whenever* $|X| > 1$.

Proof. We start the proof by using induction. If $|X| = 2$, we shall have two different infra-open sets other than $\emptyset$ and X. Then the claim is true for base step $i = 2$. Assume the statement hold for $|X| = n$, i.e., $|\tau| = n+2$. W.l.o.g, we choose (X,τ) to be the least infra T_2-space. Now, let $|X| = n+1$. That is, $X' = X\bigcup\{r\}$, where $r \notin X$. One can easily check that $\tau' = \tau\bigcup\{r\}$ is the least infra T_2 topology on X'. Thus, the conclusion is proved. □

Theorem 4.16. *Every subspace* (A,τ_A) *of an infra* T_i*-space* (X,τ) *is infra* T_i, *where* $i \in \{0,1,2,3\}$.

Proof. We only give proof to $i = 1,3$, the others are comparable.

Let us start proving that (A,τ_A) is an infra T_1-space. Consider $r,s \in A \subseteq X$ with $r \neq s$. Since (X,τ) is infra T_1, τ contains U and V such that $r \in U\setminus V$ and $s \in V \setminus U$. Set $G = A\cap U$ and $H = A \cap V$. Then $G,H \in \tau_A$ with the property that $r \in G\setminus H$ and $s \in H \setminus G$. Thus, (A,τ_A) is infra T_1.

We now show that (A,τ_A) is infra-regular. Take $s \in A$. Let F be infra-closed A with $s \notin F$. Then $F = X\cap H$ for some infra-closed set H in X. Since (X,τ) is infra-regular, then there are disjoint infra-open sets U and V such that $H \subseteq U$ and $s \in V$. Therefore, $A\cap U, A\cap V \in \tau_A$ having the property that $F \subseteq A\cap U$, $s \in A\cap V$, and $(A\cap U)\bigcap(A\cap V) = \emptyset$. Hence, (A,τ_A) is infra-regular, and consequently (A,τ_A) is infra T_3. □

Using the same technique as described before, one can show the following result.

Theorem 4.17. *Let* (X,τ) *be an infra* T_4*-space and let* $A \subseteq X$ *be an infra-closed set. Then* (A,τ_A) *is infra* T_4.

Theorem 4.18. *The finite product of infra* T_i*-spaces is infra* T_i, *where* $i = 0,1,2$.

Proof. When $i = 2$, we provide proof for the result. The remaining cases proceed in a similar manner. W.l.o.g, we consider infra T_2-spaces (X_1,τ_1) and (X_2,τ_2). If $(r_1,s_1),(r_2,s_2)$ are in $X_1\times X_2$ with $(r_1,s_1) \neq (r_2,s_2)$, then either $r_1 \neq r_2$ or $s_1 \neq s_2$. Say, $r_1 \neq r_2$. By assumption, τ_1 contains two disjoint infra-open sets U,V such that $r_1 \in U\setminus V$ and $r_2 \in V\setminus U$. Now, $U\times X_2$ and $V\times X_2$ are disjoint infra-open sets such that $(r_1,s_1) \in (U\times X_2)\setminus(V\times X_2)$ and $(r_2,s_2) \in (V\times X_2)\setminus(U\times X_2)$. Hence, $X_1\times X_2$ is infra T_2. □

Proposition 4.19. *Let* $f : (X, \tau) \to (Z, \sigma)$ *be an infra-continuous injection and let* (Z, σ) *be an infra* T_i*-space. Then* (X, τ) *is infra* T_i *for* $i \in \{0, 1, 2\}$*.*

Proof. For $i = 2$, let $r, s \in X$ with $r \neq s$. Since f is injective, there are $x, y \in Z$ such that $x \neq y$, and $f(r) = x$ and $f(s) = y$. Since Z is infra-T_2, there are two infra-open sets U, V that are disjoint, and $x \in U \setminus V$ and $y \in V \setminus U$. By infra-continuity of f, $f^{-1}(U)$ and $f^{-1}(V)$ are infra-open sets in X such that $r \in f^{-1}(U) \setminus f^{-1}(V)$ and $s \in f^{-1}(V) \setminus f^{-1}(U)$, and $f^{-1}(U) \bigcap f^{-1}(V) = \emptyset$. Hence, the proof is completed. The proof is clear when $i = 0, 1$. □

The following results can be established in a similar fashion:

Proposition 4.20. *Let* $f : (X, \tau) \to (Z, \sigma)$ *be an infra-continuous bijection and let* (Z, σ) *be an infra* T_i*-space. Then* (X, τ) *is infra* T_i *for* $i \in \{0, 1, 2, 3, 4\}$*.*

Proposition 4.21. *Let* $f : (X, \tau) \to (Z, \sigma)$ *be an infra-open bijection and let* (X, τ) *be an infra* T_i*-space. Then* (Z, σ) *is infra* T_i *for* $i \in \{0, 1, 2, 3, 4\}$*.*

Merging the last two results yields following.

Corollary 4.22. *Let* $f : (X, \tau) \to (Z, \sigma)$ *be an infra-homeomorphism. Then* (X, τ) *is infra* T_i *iff* (Z, σ) *is infra* T_i*, for* $i \in \{0, 1, 2, 3, 4\}$*.*

Definition 4.23. Let $\{(X_i, \tau_i) : i \in I\}$ be a collection of infra-topological spaces with property $\mathcal{P}$. Then $\mathcal{P}$ is called an *additive* property if topological sum $(\oplus_{i \in I} X_i, \tau)$ owns the property $\mathcal{P}$.

Theorem 4.24. *The infra separation axioms* T_i *are additive, where* $i \in \{0, 1, 2, 3, 4\}$*.*

Proof. When $i = 2$. Let $r \neq s \in \oplus_{i \in I} X_i$. We consider two cases, the first one: If $r, s \in X_{i_0}$ for some $i_0 \in I$, then there are infra-open sets G and H in infra T_2-space (X_{i_0}, τ_{i_0}) such that $r \in G$ and $s \in H$, and $G \cap H = \emptyset$. By the definition of infra-topological sum, G and H are disjoint infra-open subsets of $(\oplus_{i \in I} X_i, \tau)$. The second case is there exist $i_0 \neq j_0 \in I$ such that $r \in X_{i_0}$ and $s \in X_{j_0}$. Obviously, X_{i_0} and X_{j_0} are disjoint infra-open subsets of (X_{i_0}, τ_{i_0}) and (X_{j_0}, τ_{j_0}), respectively. Therefore, they are disjoint infra-open subsets of $(\oplus_{i \in I} X_i, \tau)$. Summing up the above two cases, the infra T_2-spaceness of $(\oplus_{i \in I} X_i, \tau)$ is proved. One can complete the proof of the result when $i = 0, 1$ by a similar technique.

In order to check the result for $i = 3, 4$, it is enough to show the additivity of infra-regularity and infra-normality. To do this, take an infra-closed set F in $\oplus_{i \in I} X_i$ with $r \notin F$. Then, for all $i \in I$, $F \bigcap X_i$ is infra-closed in X_i. But r is point in $\oplus_{i \in I} X_i$, so $r \in X_{i_0}$ for only $i_0 \in I$. Since (X_{i_0}, τ_{i_0}) is infra regular, then one can find infra-open subsets G and H in X_{i_0} such that $F \bigcap X_{i_0} \subseteq G$ and $r \in H$, and $G \bigcap H = \emptyset$. Therefore, $G \bigcup_{i \neq i_0} X_i$ is infra-open in $(\oplus_{i \in I} X_i, \tau)$ that contains F. One can easily conclude that both $U \bigcup_{i \neq i_0} X_i$ and V are disjoint. Hence $(\oplus_{i \in I} X_i, \tau)$ is infra-regular.

To prove the property of infra-normality, assume F and E are disjoint infra-closed sets in $(\oplus_{i \in I} X_i, \tau)$. Then, $F \bigcap X_i$ and $E \bigcap X_i$ are infra-closed in (X_i, τ_i) for all $i \in I$. By infra-normality of each (X_i, τ_i), we have infra-open sets U_i and V_i in (X_i, τ_i) with the property that $F \bigcap X_i \subseteq U_i$ and $H \bigcap X_i \subseteq V_i$, and $U_i \bigcap V_i = \emptyset$. This yields that $F \subseteq \bigcup_{i \in I} U_i$, $H \subseteq \bigcup_{i \in I} V_i$ and $[\bigcup_{i \in I} U_i] \bigcap [\bigcup_{i \in I} V_i] = \emptyset$. Hence, $(\oplus_{i \in I} X_i, \tau)$ is infra-normal. □

5. Conclusion and future work

For different purposes, topology was generalized to various relaxed structures; infra-topology is one of the more recent generalizations. This article is originated to contribute to this structure. We briefly outline the main fulfillments of this article in the following.

(1) Study the concepts of continuity, openness, and homeomorphism of mappings between infra-topological spaces, and explore their main characterizations.

(2) Apply infra-continuity to define infra-quotient topologies.
(3) Introduce infra T_i-spaces $(i = 0, 1, \ldots, 4)$ and clarify the relationships between them.
(4) Demonstrate some topological properties that are invalid via infra-topologies with the help of some interesting counterexamples.
(5) Investigate the behavior of the concepts given herein with respect to topological and hereditary properties, sum and finite product of infra-topological spaces.

We intend to perform the following in future articles.

(1) Present the basic topological concepts and notions such as different types of connectedness and compactness.
(2) Investigate the maximal compact and maximal connected topologies, and minimal separation axioms in the lattice of all infra-topologies on a domain set.
(3) Discuss the concepts of functions and separation axioms given in [2, 32] via infra-topological structures.
(4) Study the interrelations between infra structures and rough approximations spaces.
(5) It will be interesting to examine the different types of separation axioms discussed via supra topologies in [9, 10, 12, 16, 23] in the frame of infra topologies by using their counterparts of generalizations of infra open sets.

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