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Existence of a weak solution for a nonlinear parabolic problem with fractional derivates

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Abstract

The primary objective of this study was to demonstrate the existence and uniqueness of a weak solution for a nonlinear parabolic problem with fractional derivatives for the spatial and temporal variables on a one-dimensional domain. Using the Nehari manifold method and its relationship with the Fibering maps, the existence of a weak solution for the stationary case was demonstrated. Finally, using the Arzela-Ascoli theorem and Banach's fixed point theorem, the existence and uniqueness of a weak solution for the nonlinear parabolic problem were shown.

Keywords: Fractional calculus, Nehari manifold, Fibering maps, weak Solution.

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1. Introduction

Fractional calculus finds its application in different areas; for example, applications can be cited in viscoelasticity, electronics, chemical reactions, quantum mechanics, semiconductors, propagation of electromagnetic waves and materials, and transport phenomena by convection-diffusion, see [1, 5, 6, 8, 11, 13, 16, 17, 20, 21, 23, 24, 27, 28, 33].

Some studies have made it possible to test different methods in this area of research, such as the work of Hai Pu and Lili Cao [25], who proved the existence and multiplicity of solutions for a fractional differential equation with boundary conditions using the Nehari manifold and Fibering maps; and the work of Goyal and Sreenadh [12], who demonstrated the existence and multiplicity of nonnegative solutions by minimization on the appropriate subset of the Nehari manifold using Fibering maps. In the same way, Meilan et al. [26] proved the existence of a weak solution for a p-Laplace problem and obtained results of the existence of weak solutions using the Nehari variety, the fixed point theorem and the Arzela-Ascoli theorem. Brown et al. [4] studied a differential equation with Dirichlet conditions and showed how the results of the existence and multiplicity of solutions by nature of the Nehari manifold arise. Tsun-Wu [31] studied the number of solutions for a semilinear elliptic system with a weight function that changes sign, and with the Nehari variety method, they showed that the system has at least two nontrivial nonnegative

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solutions. Brown [3] demonstrated the existence of a weak solution for an elliptic problem with the Nehari manifold method, and with bifurcation theory, the nonexistence of solutions was analyzed. Drabek et al. [9] studied the theory of nonlinear boundary value problems for elliptic operators and demonstrated the existence of a weak solution in weighted Sobolev spaces. Similarly, Torres [18] demonstrated the existence of nontrivial solutions for a Dirichlet problem with mixed fractional derivatives using variational methods and the mountain pass theorem. In the same way, Chen et al. [7] used critical point theory to demonstrate the existence of weak solutions for a frontier problem with fractional derivative and p-Laplacian. Similarly, Meilan et al. [26] demonstrated the existence of a weak solution for a nonlinear problem with fractional derivative using the Nehari manifold method. These results are an important antecedent for the objective of this study and are described below:

$$P_1 : \begin{cases} {}_0^cD_t^\beta u(x,t) = -{}_xD_\Lambda^\alpha(|\,{}_0D_x^\alpha u(x,t)|^{p-2}\,{}_0D_x^\alpha u(x,t) + \lambda|u(x,t)|^{p-2}u(x,t), \\ \qquad\qquad +b(x)|u(x,t)|^{q-1}u(x,t), \quad (x,t) \in \Omega_T, \\ u(0,t) = u(\Lambda,t) = 0, \quad t \in [0,T], \\ u(x,0) = \phi(x), x \in [0,\Lambda], \\ u_t(x,0) = \psi(x), x \in [0,\Lambda], \end{cases} \tag{1.1}$$

where $\Omega_T = [0,\Lambda] \times [0,T]$, ${}^cD^\beta$ and D^α are Caputo fractional derivatives of order $1 < \beta < 2$ and $\frac{1}{p} < \alpha < 1$ for the temporary variable, respectively; $1 < q < p-1$ with $2 < p < \infty$, $b : [0,\Lambda] \to \mathbb{R}$ is a continuous function, $b \in L^\infty([0,\Lambda])$, $\phi(x)$, $\psi(x) \in L^\infty[0,\Lambda]$, λ is real positive, and $u \in E_0^{\alpha,p}[0,\Lambda]$ is the fractional space that will be defined in a later section of the article. In addition, the stationary problem associated with the problem P_1 is:

$$P_0 : \begin{cases} {}_xD_\Lambda^\alpha(|\,{}_0D_x^\alpha u(x)|^{p-2}\,{}_0D_x^\alpha u(x)) = \lambda|u(x)|^{p-2}u(x) + b(x)|u(x)|^{q-1}u(x), \; x \in [0,\Lambda], \\ u(0) = u(\Lambda) = 0, \end{cases} \tag{1.2}$$

where $\frac{1}{p} < \alpha < 1$, and $1 < q < p-1$, with $2 < p < \infty$, and $b \in L^\infty[0,\Lambda]$.

To prove the existence of a weak solution for the problem P_1 (1.1), we will prove the existence of a weak solution for the problem P_0 (1.2). To achieve this, we will use the Nehari Manifold because the minimization of energy functional associated with the problem P_0 (1.2) is not possible for all $E_0^{\alpha,p}[0,\Lambda]$. Thus, it is necessary to restrict to the set of critical points on the Nehari manifold, which is a weak solution for the problem P_0 (1.2).

This study is structured in different sections as follows. Preliminaries are described in Section 2; the weak solution of the stationary problem P_0 is described in Section 3; the existence of a weak solution of the fractional parabolic equation is described in Section 4; and conclusions are provided in the final section.

2. Preliminary

Different definitions of fractional derivatives, such as those reported by Riemann Liouville, Grunwald Letnikov, Hadamard, Erdelyi and Caputo, which can be found in the literature by Kilbas [16] and Kenneth [21], are not necessarily equivalent. This study used the Riemann-Liouville and Caputo definitions of fractional derivatives.

Definition 2.1 ([32]). Let $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a real function and $\alpha \in \mathbb{R}^+$. The left and right Riemann-Liouville fractional integral of order α and function u are defined by:

$${}_aI_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}u(s)ds, \quad {}_tI_b^\alpha u(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}u(s)ds,$$

respectively.

Definition 2.2 ([32])**.** Let $u : [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a real function where $\alpha \in \mathbb{R}^+$ and $\lceil \alpha \rceil = n$ is the smallest integer greater than α. The left and right Riemann-Liouville fractional derivates of order α and function u are defined by:

$$ {}_aD_t^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} u(s) ds \tag{2.1}$$

and

$$ {}_tD_b^\alpha u(t) = (-1)^n \frac{d^n}{dt^n} \int_t^b (s-t)^{n-\alpha-1} u(s) ds, \tag{2.2}$$

respectively. Expressions (2.1) and (2.2) can also be written as

$$ {}_aD_t^\alpha u(t) = \frac{d^n}{dt^n} \; [{}_aI_t^{(n-\alpha)} u(s)] \quad \text{and} \quad {}_tD_b^\alpha u(t) = (-1)^n \frac{d^n}{dt^n} [{}_tI_b^{(n-\alpha)} u(s)],$$

where ${}_aI_t^{(n-\alpha)}$, ${}_tI_b^{(n-\alpha)} \in C^n[a, b]$.

Definition 2.3 ([32])**.** Let $\alpha \in \mathbb{R}^+$ and $\lceil \alpha \rceil = n$ be the smallest integer greater than α. The Caputo fractional left and right derivatives of the function $u : [a, b] \subset \mathbb{R} \to \mathbb{R}$ are defined by the Riemann-Liouville fractional derivative as follows:

$$ {}_a^CD_t^\alpha u(t) = {}_aD_t^\alpha \left[u(t) - \sum_{k=0}^{n-1} \frac{u^k(a)}{k!} (t-a)^k \right] \quad \text{and} \quad {}_t^CD_b^\alpha u(t) = {}_tD_b^\alpha \left[u(t) - \sum_{k=0}^{n-1} \frac{u^k(b)}{k!} (b-t)^k \right]$$

for $a \leqslant t \leqslant b$. In particular, when $0 < \alpha < 1$, of Definition 2.3, we have:

$$ {}_a^CD_t^\alpha u(t) = {}_aD_t^\alpha (u(t) - u(a)), \quad {}_t^CD_b^\alpha u(t) = {}_tD_b^\alpha (u(t) - u(a)). \tag{2.3}$$

Proposition 2.4 ([34])**.** *Let* $\alpha > 0$, $n \in \mathbb{N}$ *such that* $\lceil \alpha \rceil = n$ *is the smallest integer greater than* α $(n - 1 < \alpha \leqslant n)$. *If* $u \in AC^n([a, b], \mathbb{R})$ *or* $u \in C^n([a, b], \mathbb{R})$, *then*

$$\begin{aligned} {}_aI_t^\alpha ({}_a^CD_t^\alpha u(t)) &= u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!} (t-a)^k, \quad t \in [a, b], \\ {}_tI_b^\alpha ({}_t^CD_b^\alpha u(t)) &= u(t) - \sum_{k=0}^{n-1} \frac{(-1)^k u^{(k)}(b)}{k!} (b-t)^k, \quad t \in [a, b]. \end{aligned} \tag{2.4}$$

In particular, when $0 < \alpha \leqslant 1$, $u \in AC^n([a, b], \mathbb{R})$ *or* $u \in C^n([a, b], \mathbb{R})$, *then*

$$ {}_aI_t^\alpha ({}_a^CD_t^\alpha u(t)) = u(t) - u(a), \quad {}_tI_b^\alpha ({}_t^CD_b^\alpha u(t)) = u(t) - u(b).$$

Definition 2.5 ([19])**.** Let $u \in L^1(a, b)$, $\alpha \in \langle 0, 1 \rangle$. If $v \in L^1_{Loc}(a, b)$ exists such that:

$$\int_0^\Lambda u(t) \; {}_tD_b^\alpha \varphi(t) dt = \int_0^\Lambda v(t) \varphi(t) dt, \quad \forall \varphi \in C_0^\infty([0, \Lambda], \mathbb{R}),$$

then, v is called the left fractional weak derivative of u and is denoted by: ${}_a\dot{D}_t^\alpha u = v$. Similarly, we have that if there exists $w \in L^1_{Loc}(a, b)$ such that:

$$\int_0^\Lambda u(t) \; {}_aD_t^\alpha \varphi(t) dt = \int_0^\Lambda w(t) \varphi(t) dt, \quad \forall \varphi \in C_0^\infty([0, \Lambda], \mathbb{R}).$$

Then, w is called the right fractional weak derivative of u and is denoted by ${}_t\dot{D}_b^\alpha u = w$.

Definition 2.6 ([15]). For $0 < \alpha \leqslant 1$ and $1 \leqslant p < \infty$, the fractional Sobolev space denoted for $E_0^{\alpha,p}$ is defined for the closure of $C_0^\infty([0,\Lambda],\mathbb{R})$ with respect to the norm of $E^{\alpha,p}[a,b]$,

$$E_0^{\alpha,p}[a,b] = \overline{C_0^\infty[0,\Lambda]}^{\|\cdot\|_{\alpha,p}}.$$

Definition 2.7 ([30]). Let $\frac{1}{p} < \alpha \leqslant 1$ and $1 < p < \infty$, the space of fractional derivatives $E_0^{\alpha,p}[0,\Lambda]$ is defined by:

$$E_0^{\alpha,p}[0,\Lambda] = \{u \in L^p[0,\Lambda] : {}_0D_t^\alpha u \in L^p[0,\Lambda], u(0) = u(\Lambda) = 0\}$$

with

$$\|u\|_{\alpha,p}^p = \int_0^\Lambda |u(t)|^p dt \ + \ \int_0^\Lambda |\, {}_0D_t^\alpha u(t)|^p dt, \ \ \forall u \in E_0^{\alpha,p}[0,\Lambda].$$

In the following, we give some properties of the fractional space $E_0^{\alpha,p}[0,\Lambda]$.

Proposition 2.8 ([30, Poincare-Friedrich inequality]). *Let $0 < \alpha \leqslant 1$ and $1 \leqslant p < \infty$. For all $u \in E_0^{\alpha,p}[0,\Lambda]$, we have*

$$\|u\|_{L^p} \leqslant \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^p}.$$

If $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\|u\|_\infty \leqslant \frac{T^{\alpha-1/p}}{\Gamma(\alpha)((\alpha-1)q+1)^{1/q}} \|{}_0D_t^\alpha u\|_{L^p}.$$

Remark 2.9 ([30]). According to Proposition 2.8, can be considered $E_0^{\alpha,p}[0,\Lambda]$ with respect to the norm

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u(t)\|_{L^p} = \left(\int_0^t |{}_0D_t^\alpha u(t)|^p dt\right)^{1/p}. \tag{2.5}$$

Proposition 2.10 ([14, 18]). *Let $0 < \alpha \leqslant 1$ and $1 \leqslant p < \infty$. Assume that $\alpha > \frac{1}{p}$ and the sequence $\{u_k\}$ converges weakly to u on $E_0^{\alpha,p}[0,\Lambda]$; then, $u_k \to u$ in $C[0,T]$, i.e., $\|u_k f - u\|_\infty \to 0$, $k \to \infty$.*

Theorem 2.11 ([30]). *Let $\alpha \in \langle \frac{1}{p}, 1\rangle$; then, the continuous injection $E_0^{\alpha,p}[0,\Lambda] \hookrightarrow L^p[0,T]$ is compact.*

Theorem 2.12 ([2, Banach fixed point theorem]). *Let X be a Banach space and let $T : X \to X$ be a contraction, that is, there exists $k \in (0,1)$ such that*

$$\|T(u) - T(v)\| \leqslant k\|u - v\|, \ \ \forall u, v \in X.$$

Then, there exists $u_0 \in X$ (unique) such that $T(u_0) = u_0$.

Lemma 2.13 ([16]). *Suppose $y \in C[0,T]$, $0 < T < 1$ and $1 < \alpha \leqslant 2$, then, the problem*

$$D^\alpha u(t) = y(t), \ \ t \in [0,T],$$

has a unique solution

$$u(t) = u_0 + u'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

3. Weak solution of the stationary parabolic problem P_0

Considering the problem P_0, we proceed to make the variation formulation to obtain the energy functional. We have the problem:

$${}_xD_\Lambda^\alpha(|{}_0D_x^\alpha u(x)|^{p-2}\, {}_0D_x^\alpha u(x)) = \lambda|u(x)|^{p-2}u(x) + b(x)|u(x)|^{q-1}u(x), \forall x \in [0,\Lambda], \quad u(0) = u(\Lambda) = 0.$$

The next Theorem 3.1 shows that the function of the problem P_0 is $J_\lambda \in C^1(E_0^{\alpha,p}[0,\Lambda],\mathbb{R})$.

Theorem 3.1. *Let* $J_\lambda : E_0^{\alpha,p}[0,\Lambda] \longrightarrow \mathbb{R}$ *defined by*

$$J_\lambda(u) = \frac{1}{p}\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - \frac{\lambda}{p}\int_{[0,\Lambda]} |u|^p\,dx - \frac{1}{q+1}\int_{[0,\Lambda]} b|u|^{q+1}dx.$$

Then, $J_\lambda \in C^1(E_0^{\alpha,p}[0,\Lambda],\mathbb{R})$ *with*

$$J'_\lambda(u)v = \int_{[0,\Lambda]} |{}_0D_x^\alpha u|^{p-2}\,{}_0D_x^\alpha u\,{}_0D_x^\alpha v dx - \int_{[0,\Lambda]} \lambda|u|^{p-2}uv dx - \int_{[0,\Lambda]} b|u|^{q-1}uv dx;\ \forall v \in E_0^{\alpha,p}[0,\Lambda].$$

Proof. Given $1 < q < p-1,\ 2 < p < \infty,\ \frac{1}{p} < \alpha < 1$ and $v \in C_0^\infty[0,\Lambda]$, the variation formulation of the problem P_0 is:

$$\int_{[0,\Lambda]} {}_xD_\Lambda^\alpha(|{}_0D_x^\alpha u|^{p-2}\,{}_0D_x^\alpha u)\varphi dx = \int_{[0,\Lambda]} \lambda|u|^{p-2}u\varphi dx + \int_{[0,\Lambda]} b|u|^{q-1}u\varphi,\ \ \forall\varphi \in C_0^\infty[0,\Lambda],$$
$$\int_{[0,\Lambda]} |{}_0D_x^\alpha u|^{p-2}\,{}_0D_x^\alpha u\,{}_0D_x^\alpha \varphi dx = \int_{[0,\Lambda]} \lambda|u|^{p-2}u\varphi dx + \int_{[0,\Lambda]} b|u|^{q-1}u\varphi dx,\ \ \forall\varphi \in C_0^\infty[0,\Lambda],$$
$$\int_{[0,\Lambda]} |{}_0D_x^\alpha u|^{p-2}\,{}_0D_x^\alpha u\,{}_0D_x^\alpha v dx = \int_{[0,\Lambda]} \lambda|u|^{p-2}uv dx + \int_{[0,\Lambda]} b|u|^{q-1}uv dx,\ \ \forall v \in \overline{C_0^\infty[0,\Lambda]}, \tag{3.1}$$
$$\int_{[0,\Lambda]} |{}_0D_x^\alpha u|^{p-2}\,{}_0D_x^\alpha u\,{}_0D_x^\alpha v dx = \int_{[0,\Lambda]} \lambda|u|^{p-2}u\, v dx + \int_{[0,\Lambda]} b|u|^{q-1}uv dx,\ \ \forall v \in E_0^{\alpha,p}[0,\Lambda],$$
$$J'_\lambda(u)v = \int_{[0,\Lambda]} |{}_0D_x^\alpha u|^{p-2}\,{}_0D_x^\alpha u\,{}_0D_x^\alpha v dx - \int_{[0,\Lambda]} \lambda|u|^{p-2}uv dx - \int_{[0,\Lambda]} b|u|^{q-1}uv dx;\ \ \forall v \in E_0^{\alpha,p}[0,\Lambda].$$

If this function is the derivative of a functional for some $u \in E_0^{\alpha,p}[0,\Lambda]$ then we have a variational formulation, with $J_\lambda : E_0^{\alpha,p}[0,\Lambda] \to \mathbb{R}$ and

$$J_\lambda(u) = \frac{1}{p}\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - \frac{\lambda}{p}\int_{[0,\Lambda]} |u|^p\,dx - \frac{1}{q+1}\int_{[0,\Lambda]} b|u|^{q+1}dx,\ \ \forall u \in E_0^{\alpha,p}[0,\Lambda].$$

□

According to the Theorem 3.1, J_λ is a functional in $C^1(E_0^{\alpha,p}[0,\Lambda],\mathbb{R})$ with the derivative of Gateaux on $u \in E_0^{\alpha,p}[0,\Lambda]$ given by (3.1) for every address $v \in E_0^{\alpha,p}[0,\Lambda]$. In this case, $u \in E_0^{\alpha,p}[0,\Lambda]$ is a weak solution for the problem P_0 if and only if it is a critical point of the functional J_λ. The following Lemma explains the behavior of the energy functional $J_\lambda(u)$ in the fractional space $E_0^{\alpha,p}[0,\Lambda]$.

Lemma 3.2.

(i) *Suppose* $\lambda < \lambda_1$*, then,* J_λ *is lower bounded on* $E_0^{\alpha,p}[0,\Lambda]$.
(ii) *If* $\lambda > \lambda_1$*, then* J_λ *is not lower bounded.*

Proof.

(i) The first eigenvalue λ_1 of problem P_0 is:

$$\lambda_1 = \min_{u\in E_0^{\alpha,p}} \frac{\int_0^\Lambda |\,{}_0D_x^\alpha u(x)|^p\,dx}{\int_0^\Lambda |u(x)|^p\,dx},\quad u \neq 0,$$

Also

$$\lambda_1\int_{[0,\Lambda]} |u|^p\,dx \leqslant \int_{[0,\Lambda]} |{}_0D_x^\alpha u|^p\,dx,$$
$$\lambda_1\int_{[0,\Lambda]} |u|^p\,dx - \lambda\int_{[0,\Lambda]} |u|^p\,dx \leqslant \int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - \lambda\int_{[0,\Lambda]} |u|^p\,dx, \tag{3.2}$$
$$\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - \lambda\int_{[0,\Lambda]} |u|^p\,dx \geqslant (\lambda_1-\lambda)\int_{[0,\Lambda]} |u|^p\,dx,\ \ \forall u \in E_0^{\alpha,p}.$$

Then, we have

$$J_\lambda(u) \geqslant \frac{(\lambda_1-\lambda)}{p}\int_{[0,\Lambda]}|u|^p\,dx - \frac{1}{q+1}\int_{[0,\Lambda]} b|u|^{q+1}dx,$$

and

$$\begin{aligned} J_\lambda(u) &\geqslant \frac{1}{p}(\lambda_1-\lambda)\int_{[0,\Lambda]}|u|^p\,dx - \frac{\bar{b}}{q+1}\int|u|^{q+1}dx,\\ J_\lambda(u) &\geqslant \frac{1}{p}(\lambda_1-\lambda)\int_{[0,\Lambda]}|u|^p\,dx - \frac{\bar{b}}{q+1}|\Lambda|^{1-(q+1)/p}\left(\int|u|^p\,dx\right)^{(q+1)/p}. \end{aligned} \tag{3.3}$$

Therefore, J_λ is lower bounded on $E_0^{\alpha,p}[0,\Lambda]$ when $\lambda<\lambda_1$.

(ii) If $\lambda>\lambda_1$, we look at the address of the primary eigenfunction $\phi_1\in E_0^{\alpha,p}$, and we see that, when $t\to\infty$, the functional J_λ goes to $-\infty$, that is

$$\begin{aligned} \lim_{t\to\infty} J_\lambda(t\phi_1) &= \lim_{t\to\infty}\left[\frac{\lambda_1}{p}\int_{[0,\Lambda]}|t\phi_1|^p\,dx - \frac{\lambda}{p}\int_{[0,\Lambda]}|t\phi_1|^p\,dx - \frac{1}{q+1}\int_{[0,\Lambda]} b|t\phi_1|^{q+1}dx\right],\\ \lim_{t\to\infty} J_\lambda(t\phi_1) &= \lim_{t\to\infty}|t|^p\left[\frac{(\lambda_1-\lambda)}{p}\int_{[0,\Lambda]}|\phi_1|^p\,dx - \frac{1}{(q+1)t^{p-(q+1)}}\int_{[0,\Lambda]} b|\phi_1|^{q+1}dx\right], \end{aligned}$$

we have that $\lim_{t\to\infty} J_\lambda(t\phi_1)=-\infty$, therefore, J_λ is not lower bounded on $E_0^{\alpha,p}[0,\Lambda]$ when $\lambda>\lambda_1$. □

Minimization is not possible in the entire space $E_0^{\alpha,p}[0,\Lambda]$. In this case, we can consider the Nehari Manifold [22] for the problem P_0 defined by:

$$N_\lambda = \{u\in E_0^{\alpha,p}[0,\Lambda] : \langle J'_\lambda(u),u\rangle = 0, u\neq 0\}. \tag{3.4}$$

The set N_λ (3.4) to be Negari manifold should satisfy some condition, such as $N_\lambda\neq\emptyset$, closed and C^1 class.

The next proposition shows that N_λ is closed in $E_0^{\alpha,p}[0,\Lambda]$.

Proposition 3.3. *There exists $c_0>0$ such that $\|u\|_{E_0^{\alpha,p}[0,\Lambda]}\geqslant c_0$ for all $u\in N_\lambda$. Consequently, N_λ is a closed subset of $E_0^{\alpha,p}[0,\Lambda]$.*

Proof. From the Poincaré inequality [18, Proposition 2.6], we have

$$\|u\|_{L^p}\leqslant\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}\|{}_0D_t^\alpha u\|_{L^p}=\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}\|u\|_{E_0^{\alpha,p}[0,\Lambda]}.$$

Additionally, as $b\in L^\infty[0,\Lambda]$ by continuous injection $L^p[0,\Lambda]\hookrightarrow L^{q+1}[0,\Lambda]$, there exists a constant c, such that $\|u\|_{L^{q+1}[0,\Lambda]}\leqslant c\|u\|_{L^p[0,\Lambda]}$; now considering $b<\|b\|_{L^\infty[0,\Lambda]}$, it follows that:

$$\int_{[0,\Lambda]} b|u|^{q+1}dx < \|b\|_{L^\infty[0,\Lambda]}\|u\|_{L^{q+1}}^{q+1} < \|b\|_{L^\infty[0,\Lambda]}c^{q+1}\|u\|_{L^p}^{q+1} < \|b\|_{L^\infty[0,\Lambda]}c^{q+1}\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}\|u\|_{E_0^{\alpha,p}[0,\Lambda]}.$$

Because $u\in N_\lambda$, we have that

$$\begin{aligned} &\int_{[0,\Lambda]}|{}_0D_x^\alpha u|^p\,dx - \lambda\int_{[0,\Lambda]}|u|^p\,dx - \int_{[0,\Lambda]} b|u|^{q+1}dx = 0,\\ &\int_{[0,\Lambda]}|{}_0D_x^\alpha u|^p\,dx - \lambda\int_{[0,\Lambda]}|u|^p\,dx = \int_{[0,\Lambda]} b|u|^{q+1}dx,\\ &\int_{[0,\Lambda]}|{}_0D_x^\alpha u|^p\,dx + \int_{[0,\Lambda]}|u|^p\,dx - \int_{[0,\Lambda]}|u|^p\,dx - \lambda\int_{[0,\Lambda]}|u|^p\,dx = \int_{[0,\Lambda]} b|u|^{q+1}dx,\\ &\|u\|_{E_0^{\alpha,p}[0,\Lambda]}^p - (1+\lambda)\|u\|_{L^p}^p = \int_{[0,\Lambda]} b|u|^{q+1}dx < \|b\|_{L^\infty[0,\Lambda]}c^{q+1}\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}\|u\|_{E_0^{\alpha,p}[0,\Lambda]}^{q+1},\\ &\|u\|_{E_0^{\alpha,p}[0,\Lambda]}^p \leqslant (1+\lambda)\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}\|u\|_{E_0^{\alpha,p}[0,\Lambda]}^p + \|b\|_{L^\infty[0,\Lambda]}c^{q+1}\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}\|u\|_{E_0^{\alpha,p}[0,\Lambda]}^{q+1}. \end{aligned}$$

Considering $c_1 = (1+\lambda)\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}$ and $c_2 = \|b\|_{L^\infty[0,\Lambda]} c^{q+1}\frac{\Lambda^\alpha}{\Gamma(\alpha+1)}$, we have the next inequality,

$$\begin{aligned}
\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} &\leqslant c_1\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} + c_2\|u\|^{q+1}_{E_0^{\alpha,p}[0,\Lambda]},\\
\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} &\geqslant -c_1\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} + c_2\|u\|^{q+1}_{E_0^{\alpha,p}[0,\Lambda]},\\
\|u\|_{E_0^{\alpha,p}[0,\Lambda]} &\geqslant [\frac{c_2}{1+c_1}]^{\frac{1}{p-(q+1)}} = c_0 > 0.
\end{aligned}$$

That is, $\|u\|_{E_0^{\alpha,p}[0,\Lambda]} \geqslant c_0 > 0$, $\forall u \in N_\lambda$, therefore, N_λ is a closed subset of $E_0^{\alpha,p}[0,\Lambda]$. □

For the functional J_λ, minimizing points should be coercive and lower bounded on N_λ. The following Theorem 3.4 shows this.

Theorem 3.4. *The functional J_λ is coercive and lower bounded on N_λ.*

Proof. From the definition of J_λ, as well as $b \in L^\infty[0,\Lambda]$, using equivalence (2.5) and continuous injection of $L^p[0,\Lambda] \hookrightarrow L^{q+1}[0,\Lambda]$, there exists C_1 such that $\|u\|_{L^{q+1}[0,\Lambda]} \leqslant C_1\|u\|_{L^p[0,\Lambda]}$:

$$J_\lambda(u) \geqslant \frac{1}{p}\|u\|_{E_0^{\alpha,p}[0,\Lambda]} - \frac{\lambda}{p}\|u\|^p_{L^p[0,\Lambda]} - \frac{1}{q+1}\|b\|_\infty C_1\|u\|^{q+1}_{L^p[0,\Lambda]}.$$

Also, from the continuous injection of $E_0^{\alpha,p}[0,\Lambda] \hookrightarrow L^p[0,\Lambda]$, there exists C_2 such that $\|u\|_{L^p[0,\Lambda]} \leqslant C_2\|u\|_{E_0^{\alpha,p}[0,\Lambda]}$. Having

$$J_\lambda(u) \geqslant \frac{1}{p}\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} - \frac{\lambda}{p}C_2\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} - \frac{1}{q+1}\|b\|_\infty C_1C_2\|u\|^{q+1}_{E_0^{\alpha,p}[0,\Lambda]},$$

implies that:

$$J_\lambda(u) \geqslant \left(\frac{1}{p} - \frac{\lambda}{p}C_2\right)\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} - \frac{1}{q+1}\|b\|_\infty C_3\|u\|^{q+1}_{E_0^{\alpha,p}[0,\Lambda]}.$$

Because $1 < q < p-1$, then $2 < q+1 < p$, and it follows that

$$J_\lambda(u) \to +\infty, \quad \text{when} \quad \|u\|_{E_0^{\alpha,p}([0,\Lambda])} \to \infty.$$

The functional J_λ is bounded below. Indeed, if J_λ is coercive and given $M = 1$, there exists $R > 0$ such that:

$$J_\lambda(u) \geqslant 1 \quad \text{for} \quad \|u\|_{E_0^{\alpha,p}([0,\Lambda])} \geqslant R. \tag{3.5}$$

If $u \in E_0^{\alpha,p}[0,\Lambda]$ and $\|u\|_{E_0^{\alpha,p}[0,\Lambda]} \leqslant R$, then

$$\begin{aligned}
|J_\lambda(u)| &\leqslant \frac{1}{p}\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} + \frac{\lambda}{p}\|u\|^p_{L^p[0,\Lambda]} + \frac{1}{q+1}\|b\|_\infty C_1\|u\|^{q+1}_{L^p[0,\Lambda]}\\
&\leqslant \frac{1}{p}\|u\|^p_{E_0^{\alpha,p}([0,\Lambda])} + \frac{\lambda}{p}C_2\|u\|^p_{E_0^{\alpha,p}[0,\Lambda]} + \frac{1}{q+1}\|b\|_\infty C_1C_2\|u\|^{q+1}_{E_0^{\alpha,p}[0,\Lambda]}\\
&\leqslant \frac{1}{p}R^p + \frac{\lambda}{p}C_2R^p + \frac{1}{q+1}\|b\|_\infty C_3R^{q+1} = K.
\end{aligned}$$

Therefore, we have

$$J_\lambda(u) \geqslant -K. \tag{3.6}$$

From (3.5) and (3.6):

$$J_\lambda(u) \geqslant -K, \quad \forall u \in E_0^{\alpha,p}[0,\Lambda],$$

proving that J_λ is lower bounded in $E_0^{\alpha,p}[0,\Lambda]$. □

The Nehari Manifold is associated with the behavior of Fibering maps of the form $\phi_u(t) : t \mapsto J_\lambda(tu)$ $(t > 0)$. The **Fibering maps** were introduced by Drabek and Pohozaev [9] and by Brown and Zhang [4]. The fiber map for problem P_0 is defined by following.

Definition 3.5. Let $t \in \mathbb{R}^+$, the Fibering maps $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ are defined by

$$\phi_u(t) = J_\lambda(tu) = \frac{t^p}{p}\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - \frac{t^{q+1}}{q+1}\int_{[0,\Lambda]} b|u|^{q+1}dx.$$

Then, the derivative of $\phi_u(t)$, is:

$$\phi_u'(t) = J_\lambda'(tu)u = t^{p-1}\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - t^q\int_{[0,\Lambda]} b|u|^{q+1}dx. \tag{3.7}$$

Because $t > 0$ it follows from (3.7) that

$$\phi_u'(t) = \frac{1}{t}J_\lambda'(tu)tu. \tag{3.8}$$

This result implies that $t > 0$ is the critical point of ϕ_u if and only if $tu \in N_\lambda$; for example, $u \in N_\lambda$ if and only if $t = 1$ is a critical point of ϕ_u. Thus, the task of proving that $N_\lambda \neq \emptyset$ can be replaced by finding critical points for the Fibering maps. Explicitly finding the critical points of ϕ_u is infeasible; thus, the following helper function is defined:

$$m_u(t) = \int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - t^{q-(p-1)}\int_{[0,\Lambda]} b|u|^{q+1}dx. \tag{3.9}$$

The derivative of (3.9) is:

$$m_u'(t) = [(p-1)-q]\,t^{q-p}\int_{[0,\Lambda]} b|u|^{q+1}dx, \quad m_u''(t) = [(p-1)-q]\,(q-p)t^{q-p-1}\int_{[0,\Lambda]} b|u|^{q+1}dx. \tag{3.10}$$

Factoring t^{p-1} in the equation (3.7), we have:

$$\begin{aligned}\phi_u'(t) &= t^{p-1}\left(\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - t^{q-(p-1)}\int_{[0,\Lambda]} b|u|^{q+1}dx\right)\\ &= t^{p-1}\left(\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - t^{q-(p-1)}\int_{[0,\Lambda]} b|u|^{q+1}dx - \int_{[0,\Lambda]} \lambda|u|^p\,dx\right)\\ &= t^{p-1}\left(m_u(t) - \int_{[0,\Lambda]} \lambda|u|^p\,dx\right).\end{aligned} \tag{3.11}$$

Thus, from (3.11) and (3.8), we obtain

$$tu \in N_\lambda \Leftrightarrow \phi_u'(t) = 0 \Leftrightarrow m_u(t) = \lambda\int_{[0,\Lambda]} |u|^p\,dx, \quad t > 0. \tag{3.12}$$

Remark 3.6. Given $t > 0$ will be a critical point of ϕ_u if and only if, it is a solution of the equation

$$m_u(t) = \lambda\int_{[0,\Lambda]} |u|^p\,dx. \tag{3.13}$$

If $u \in N_\lambda$, then $t = 1$ is a critical point of the function ϕ_u. Thus, the critical point can be characterized according to the sign of the second derivative of ϕ_u, that is, verifying if $\phi_u''(1) > 0$, $\phi_u''(1) < 0$, o $\phi_u''(1) =$

0. With problem P_0, this characterization is equivalent to verifying if the critical point is a local minimum, local maximum or inflection point. Thus, the second derivative of ϕ_u is:

$$\begin{aligned}\phi_u''(t) &= (p-1)t^{p-2}\int_{[0,\Lambda]}(|\,{}_0D_x^{\alpha}u|^p - \lambda|u|^p)\,dx - qt^{q-1}\int_{[0,\Lambda]} b|u|^{q+1}dx \\ &= \frac{1}{t^2}\left((p-1)\int_{[0,\Lambda]}(|\,{}_0D_x^{\alpha}tu|^p - \lambda|tu|^p)\,dx - q\int_{[0,\Lambda]} b|tu|^{q+1}dx\right) = \frac{1}{t^2}\phi_{tu}''(1), \quad t>0.\end{aligned} \tag{3.14}$$

Thus, similar to the method used by Tarantello [29], N_λ is subdivided into three subsets:

$$N_\lambda^+ = \left\{u \in N_\lambda : \phi_u''(1) > 0\right\}, \quad N_\lambda^- = \left\{u \in N_\lambda : \phi_u''(1) < 0\right\}, \quad N_\lambda^0 = \left\{u \in N_\lambda : \phi_u''(1) = 0\right\}.$$

(3.14) and (3.10) imply that:

$$\phi_u''(t) = \frac{1}{t^2}\phi_{tu}''(1) = t^{p-1}m_u'(t). \tag{3.15}$$

Equation (3.15) tells us that to characterize a critical point of ϕ_u, it is sufficient to observe the sign of the first derivative of m_u' relative to that point. Once the subsets of N_λ have been defined, we are in a position to state the following theorem, which gives us a sufficient condition for the set N_λ to be a differentiable variety.

Theorem 3.7. *If $N_\lambda^0 = \emptyset$, then the set N_λ is a manifold of class $C^1[0,\Lambda]$.*

Proof. We have $N_\lambda = G_\lambda^{-1}(\{0\})$, where $G_\lambda : E_0^{\alpha,p}[0,\Lambda]\setminus\{0\} \longrightarrow \mathbb{R}$, is a function defined by

$$G_\lambda(u) = \langle J_\lambda'(u), u\rangle = \int_{[0,\Lambda]} |\,{}_0D_x^{\alpha}u|^p\,dx - \lambda\int_{[0,\Lambda]}|u|^p\,dx - \int_{[0,\Lambda]} b|u|^{q+1}dx, \tag{3.16}$$

where G_λ is a function of class $C^1[0,\Lambda]$, whose gateaux derivative is $u \in E_0^{\alpha,p}[0,\Lambda]\setminus\{0\}$, in the direction of vector v, which is given by

$$\langle G_\lambda'(u), v\rangle = p\int_{[0,\Lambda]} |\,{}_0D_x^{\alpha}u|^{p-1}\,{}_0D_x^{\alpha}v dx - \lambda p\int_{[0,\Lambda]}|u|^{p-1}v dx - (q+1)\int_{[0,\Lambda]} b|u|^{q-1}uv dx.$$

We want to prove that $N_\lambda = G_\lambda^{-1}(\{0\})$ is a variety. 0 will be proved to be a regular value of $G_\lambda(u)$, which is equivalent to proving that, for all $u \in N_\lambda$, the function $G_\lambda : E_0^{\alpha,p}[0,\Lambda]\setminus\{0\} \longrightarrow \mathbb{R}$ is surjective. Thus, there exists $v \in E_0^{\alpha,p}[0,\Lambda]$ such that $\langle G_\lambda'(u), v\rangle \neq 0$. However, because $u \in N_\lambda$, just taking $v = u$, we have

$$\begin{aligned}\langle G_\lambda'(u), u\rangle &= p\int_{[0,\Lambda]} |\,{}_0D_x^{\alpha}u|^p\,dx - \lambda p\int_{[0,\Lambda]}|u|^p\,dx - (q+1)\int_{[0,\Lambda]} b|u|^{q+1}dx \\ &= (p-1)\int_{[0,\Lambda]} |\,{}_0D_x^{\alpha}u|^p\,dx - \lambda(p-1)\int_{[0,\Lambda]}|u|^p\,dx - \int_{[0,\Lambda]} b|u|^{q+1}dx \\ &\quad + \int_{[0,\Lambda]} |\,{}_0D_x^{\alpha}u|^p\,dx - \lambda\int_{[0,\Lambda]}|u|^p\,dx - q\int_{[0,\Lambda]} b|u|^{q+1}dx \\ &= \phi_u''(1) + \langle J_\lambda'(u), u\rangle = \phi_u''(1).\end{aligned} \tag{3.17}$$

Because $N_\lambda^0 = \emptyset$, we have $\phi_u''(1) \neq 0$, and therefore, $G_\lambda : E_0^{\alpha,p}[0,\Lambda]\setminus\{0\} \longrightarrow \mathbb{R}$ is surjective for every $u \in N_\lambda$. The proof is thus concluded. □

The following proposition relates the Nehari manifold and fibering maps.

Remark 3.8. Given $u \in N_\lambda$, we can write the functional $J_\lambda : E_0^{\alpha,p}[0,\Lambda] \to \mathbb{R}$ as

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_{[0,\Lambda]} b|u|^{q+1} dx$$

as a consequence of

$$\int_{[0,\Lambda]} (|\,_0D_x^\alpha u|^p dx - \lambda|u|^p)\, dx = \int_{[0,\Lambda]} b|u|^{q+1} dx. \tag{3.18}$$

Remark 3.9. If $u \in N_\lambda$, that is, if $\phi'_u(1) = 0$, from (3.18) and (3.14), we have

$$\phi''_u(1) = [(p-1) - q] \int_{[0,\Lambda]} b|u|^{q+1} dx. \tag{3.19}$$

Lemma 3.10. *If* $tu \in N_\lambda$, *it follows from* (3.19) *and* (3.10) *that*

$$\phi''_{tu}(1) = t^{p+1} m'_u(t).$$

Proof. If $\phi'_u(1) = 0$ of (3.7), we have:

$$\phi''_u(1) = [(p-1) - q] \int_{[0,\Lambda]} b|u|^{q+1} dx.$$

Then, letting $tu \in N_\lambda$, we have

$$\phi''_{tu}(1) = [(p-1) - q]\, t^{q+1} \int_{[0,\Lambda]} b|u|^{q+1} dx = t^{p+1}.t^{q-p}\, [(p-1) - q] \int_{[0,\Lambda]} b|u|^{q+1} dx.$$

From (3.10), we have

$$\phi''_{tu}(1) = t^{p+1} m'_u(t).$$

□

Lemma 3.11. *Given* $u \in N_\lambda$ *and* $m_u(t)$ *defined in* (3.9) *implies that:*

$$u \in N_\lambda^+ \iff m'_u(t) > 0 \;\; \textit{and} \;\; u \in N_\lambda^- \iff m'_u(t) < 0.$$

Proof. This result follows from Lemma (3.10):

$$tu \in N_\lambda^+ \iff \phi''(1) \iff m'_u(t) > 0, \quad tu \in N_\lambda^- \iff \phi''(1) \iff m'_u(t) < 0.$$

□

The following lemma exhibits a sufficient condition for minimization over Nehari to generate critical points for the functional J_λ.

Lemma 3.12. *Suppose that* $u_0 \in N_\lambda$ *is a local maximum or minimum point for* J_λ *in* N_λ. *Therefore, if* $u_0 \notin N_\lambda^0$, *then* u_0 *is a critical point of* J_λ *at* $E_0^{\alpha,p}$.

Proof. If u_0 is a local maximum or minimum point of J_λ in N_λ, then u_0 is a solution of the following optimization problem:

$$\text{Maximize(Minimize) } J_\lambda \text{ subject to } N_\lambda, \text{where } N_\lambda = G_\lambda^{-1}\{0\} \text{ and } G_\lambda \text{ defined in (3.16).}$$

Then, by Lagrange's Multipliers Theorem, there exists $\delta \in \mathbf{R}$ such that:

$$\langle J'_\lambda(u_0), v\rangle = \delta\, \langle G'(u_0), v\rangle, \text{ for all } v \in E_0^{\alpha,p}[0,\lambda]. \tag{3.20}$$

Taking $v = u_0$ and considering that $u_0 \in N_\lambda$, it follows from (3.17) that $\langle G'(u_0), u_0\rangle = \phi''_{u_0}(1)$, which is different from zero, by hypothesis. Therefore, from (3.20) it follows that $\delta = 0$. Therefore, u_0 is the critical point of J_λ. □

3.1. Behavior of the function m_u

The behavior of the functions m_u and ϕ_u depend on the sign of the integrals $\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)dx$ and $\int_{[0,\Lambda]} b|u|^{q+1}dx$, (in the graphs, $A(u)$ is $\int_{[0,\Lambda]}|\,_0D_x^\alpha u|^p dx$ and $H(u)$ is $\int_{[0,\Lambda]}|u|^p dx$). We present all possible cases for the behavior of the function m_u:

Case A. If $\int_{[0,\Lambda]} b|u|^{q+1}dx > 0$, the function m_u satisfies the following properties.

(a) We deduce from (3.10) that m_u is a strictly increasing function on $\langle 0, +\infty\rangle$.

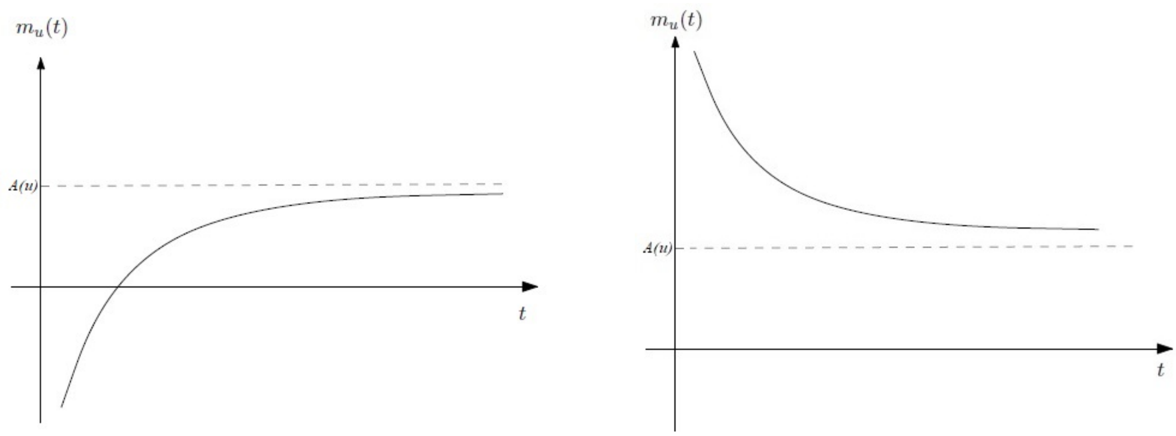
(b) If $t = 0$, the derivative of the function m_u is not defined.

(c) $\lim_{t\to\infty} m_u(t) = \int_{[0,\Lambda]}|\,_0D_x^\alpha u|^p dx$ (see graph (a) in Figure 1)

(d) $\lim_{t\to 0^+} m_u(t) = -\infty$.

(e) If $\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx < 0$, then there is no value t that it is a critical point and therefore that it satisfies the equivalence (3.12).

(f) If $\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx > 0$, there is only one value $\bar{t} = \left[\frac{\int_{[0,\Lambda]} b|u|^{q+1}dx}{\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)dx}\right]^{\frac{1}{(p-1-q)}}$, which is a critical point and thus satisfies the equivalence (3.12).



(a) $\int_{[0,\Lambda]} b|u|^{q+1}dx > 0$ (b) $\int_{[0,\Lambda]} b|u|^{q+1}dx < 0$

Figure 1: Possible graph of the function m_u.

Case B. If $\int_{[0,\Lambda]} b|u|^{q+1}dx < 0$, then the function m_u satisfies the following properties.

(a) We deduce from (3.10) that m_u is a strictly decreasing function $\langle 0, +\infty\rangle$.

(b) If $t = 0$, the derivative of the function m_u is not defined.

(c) $\lim_{t\to\infty} m_u(t) = \int_{[0,\Lambda]}|\,_0D_x^\alpha u|^p dx$ (see graph (b) in Figure 1).

(d) $\lim_{t\to 0^+} m_u(t) = +\infty$.

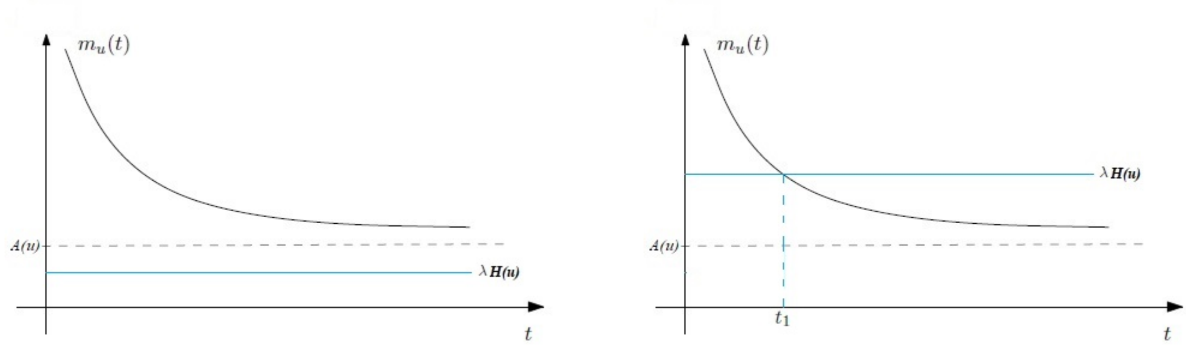
(e) If $\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx > 0$, then there is no value t that it is a critical point and therefore that it satisfies the equivalence (3.12)

(f) If $\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx < 0$, then there is only one value of

$$\bar{t} = \left[\frac{\int_{[0,\Lambda]} b|u|^{q+1}dx}{\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)dx}\right]^{\frac{1}{(p-1-q)}},$$

which is a critical point and thus satisfies the equivalence (3.12).

Thus, we can conclude that if $\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx$, $\int_{[0,\Lambda]} b|u|^{q+1}dx$ have the same sign, then for $u \in E_0^{\alpha,p}[0,\Lambda]$, the function ϕ_u has a unique critical point at $\bar{t}$, therefore, there exists $t \in \mathbb{R}$ such that $tu \in N_\lambda$. If $\int_{[0,\Lambda]}(|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx$ and $\int_{[0,\Lambda]} b|u|^{q+1}dx$ have different signs, then ϕ_u does not have critical points; therefore, there are no multiples of u in N_λ.



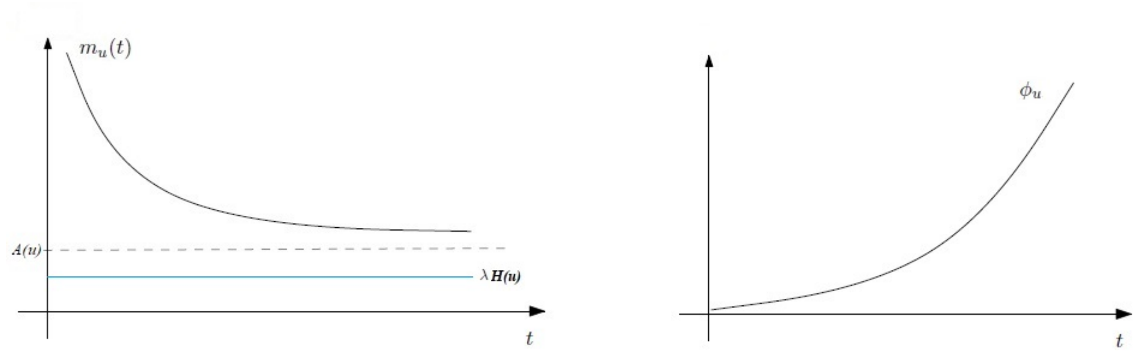
(a) $\int_{[0,\Lambda]} |\,_0D_x^\alpha u|^p\,dx > \lambda \int_{[0,\Lambda]} |u|^p\,dx$ (b) $\int_{[0,\Lambda]} |\,_0D_x^\alpha u|^p\,dx < \lambda \int_{[0,\Lambda]} |u|^p\,dx$

Figure 2: Possible graph of the function m_u in **Case B**.

3.2. Analysis of Fibering maps

Using the behavior of auxiliary function m_u, we analyze the fiber maps considering four cases.

Case1. If $\int_{[0,\Lambda]} b|u|^{q+1}dx < 0$ and $\int_{[0,\Lambda]} (|\,_0D_x^\alpha u|^p\,dx - \lambda|u|^p)\,dx > 0$, then we have the scenario described in property (e) of f Case B, which is shown in graph (a) of Figure 2. Then, $\phi_u(t)$ is increasing (see graph (b) of Figure 3) because (3.7) we have that $\phi_u'(t) > 0$. Thus, the equivalence (3.12) is not fulfilled; therefore, it is concluded that no multiple of u is in N_λ.



(a) Graph of m_u (b) Graph of ϕ_u

Figure 3: Possible graph of the function ϕ_u in **Case 1.**

Case2. If $\int_{[0,\Lambda]} b|u|^{q+1}dx < 0$ and $\int_{[0,\Lambda]} (|\,_0D_x^\alpha u|^p\,dx - \lambda|u|^p)\,dx < 0$, then we have the scenario described in the property (f) of **Case B**, this is observed in (b) of Figure 2. Also, we have that $m_u(t)$ is continuous and $\lim_{t\to 0} m_u(t) = \infty$; thus, for a sufficiently small t_1, we have:

$$m_u(t_1) > \lambda \int_{[0,\Lambda]} |u|^p\,dx.$$

Additionally, $\lambda \int_{[0,\Lambda]} |u|^p\,dx > \int_{[0,\Lambda]} |_0D_x^\alpha u|^p\,dx$ and $\lim_{t\to 0} m_u(t) = \int_{[0,\Lambda]} |_0D_x^\alpha u|^p\,dx$, then there exists a t_2 that is sufficiently large such that:

$$m_u(t_2) < \lambda \int_{[0,\Lambda]} |u|^p\,dx.$$

Defining $m_u : [t_1, t_2] \to \mathbf{R}$, $m_u(t)$ is a continuous function with:

$$m_u(t_1) < \lambda \int_{[0,\Lambda]} |u|^p\,dx < m_u(t_2),$$

Then, by the intermediate value theorem, there exists $t_u \in \langle t_1, t_2 \rangle$ such that

$$m_u(t_u) = \lambda \int_{[0,\Lambda]} |u|^p \, dx.$$

Additionally

$$m_u'(t) = [(p-1) - q] \, t^{q-p} \int_{[0,\Lambda]} b|u|^{q+1} dx.$$

Also

$$m_u'(t) < 0, \text{ since } t > 0, \ 1 < q < p-1, \ 2 < p < \infty.$$

Therefore, $m_u(t)$ is a strictly decreasing function. Then, we can conclude that t_u is unique, and the equation (3.13) has a unique solution t_u. We now proceed to prove that $t_u u \in N_\lambda$. Because $m_u(t)$ has a unique solution, substituting (3.13) into (3.9), we have

$$\lambda \int_{[0,\Lambda]} |u|^p \, dx = \int_{[0,\Lambda]} | \, {}_0D_x^\alpha u|^p \, dx - t_u^{q-(p-1)} \int_{[0,\Lambda]} b|u|^{q+1} dx.$$

Thus

$$\int_{[0,\Lambda]} | \, {}_0D_x^\alpha u|^p \, dx - \lambda \int_{[0,\Lambda]} |u|^p \, dx - t_u^{q-(p-1)} \int_{[0,\Lambda]} b|u|^{q+1} dx = 0. \tag{3.21}$$

Multiplying equation (3.21) by t_u^{p-1}, we obtain:

$$t_u^{p-1} \int_{[0,\Lambda]} (| \, {}_0D_x^\alpha u|^p - \lambda |u|^p) \, dx - t_u^q \int_{[0,\Lambda]} b|u|^{q+1} dx = 0, \tag{3.22}$$

which is the same as $J_\lambda'(t_u u) t_u u = 0.$. As a consequence, $t_u u \in N_\lambda$. Given $t_u u \in N_\lambda$, $m_u'(t_u) < 0$ and $t > 0$, by remark 3.10:

$$\phi_{t_u u}''(1) = t^{p+1} m_u'(t_u) < 0.$$

Thus, $t_u u \in N_\lambda^-$. Also, $\phi_u'(t_u) = 0$, which means that ϕ_u has a single critical point at $t = t_u$, which is a local maximum point. From (3.22), we know that

$$t_u^{p-1} \int_{[0,\Lambda]} (| \, {}_0D_x^\alpha u|^p - \lambda |u|^p) \, dx - t_u^q \int_{[0,\Lambda]} b|u|^{q+1} dx = 0. \tag{3.23}$$

Dividing the equation (3.23) by $t_u \neq 0$, we have

$$t_u^{p-2} \int_{[0,\Lambda]} (| \, {}_0D_x^\alpha u|^p - \lambda |u|^p) \, dx - t_u^{q-1} \int_{[0,\Lambda]} b|u|^{q+1} dx = 0.$$

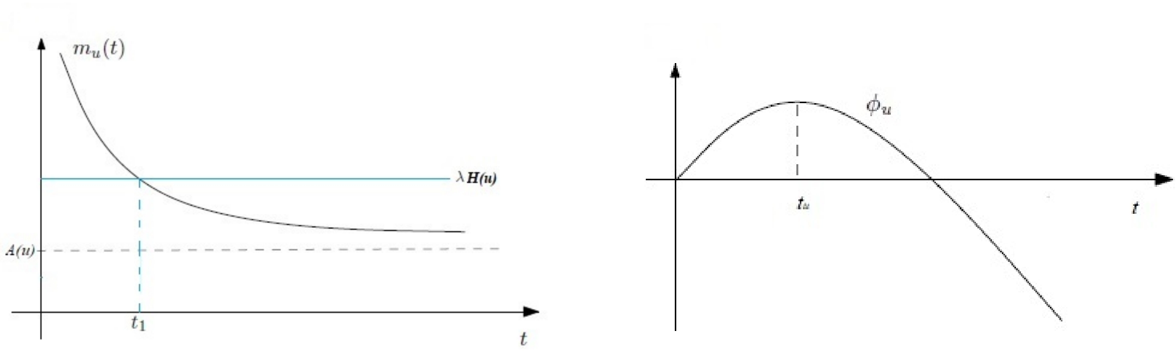
Also

$$\lim_{t\to\infty} \phi_u(t) = \lim_{t\to\infty} \left[\frac{t^p}{p} \int_{[0,\Lambda]} (| \, {}_0D_x^\alpha u|^p - \lambda |u|^p) \, dx - \frac{t^{q+1}}{q+1} \int_{[0,\Lambda]} b|u|^{q+1} dx \right] = -\infty,$$

and

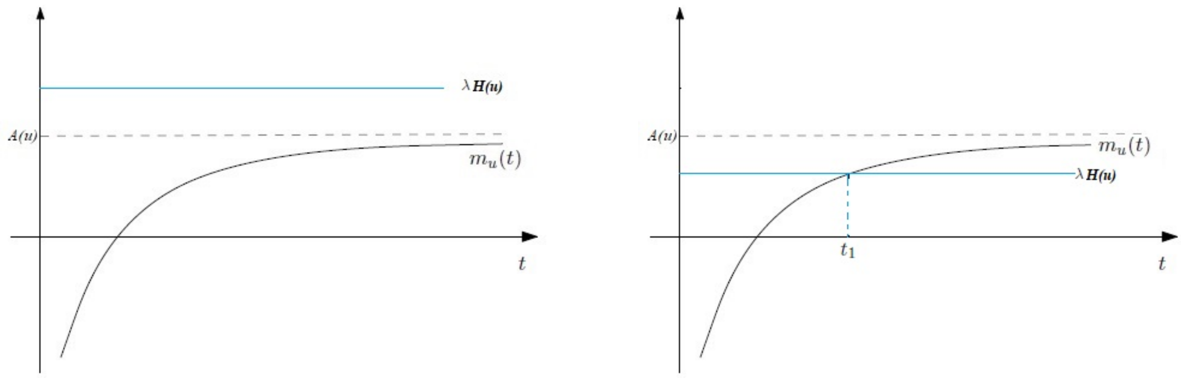
$$\lim_{t\to 0^+} \phi_u(t) = \lim_{t\to 0^+} \left[\frac{t^p}{p} \int_{[0,\Lambda]} (| \, {}_0D_x^\alpha u|^p - \lambda |u|^p) \, dx - \frac{t^{q+1}}{q+1} \int_{[0,\Lambda]} b|u|^{q+1} dx \right] = 0.$$

Based on this analysis, the graph of ϕ_u is similar to (b) of Figure 4.



(a) Graph of m_u (b) Graph of ϕ_u

Figure 4: Possible graph of ϕ_u in **Case 2.**



(a) $\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx < \lambda \int_{[0,\Lambda]} |u|^p\,dx$ (b) $\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx > \lambda \int_{[0,\Lambda]} |u|^p\,dx$

Figure 5: Possible graph of the function m_u in **Case A**.

Case3. If $\int_{[0,\Lambda]} b|u|^{q+1} dx > 0$ and $\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p\,dx - \lambda|u|^p)\,dx > 0$, then we have the scenario described in property (f) of **Case A**, which is observed in graph (a) of Figure 5. Also

$$\begin{aligned}\lim_{t\to\infty} m_u(t) &= \lim_{t\to\infty}\left[\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - t^{q-(p-1)}\int_{[0,\Lambda]} b|u|^{q+1}dx\right]\\ &= \int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx > \lambda\int_{[0,\Lambda]} |u|^p\,dx\end{aligned}$$

and

$$\lim_{t\to 0^+} m_u(t) = \lim_{t\to 0^+}\left[\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - t^{q-(p-1)}\int_{[0,\Lambda]} b|u|^{q+1}dx\right] = -\infty.$$

Because $m_u(t)$ is a continuous function with

$$\lim_{t\to 0^+} m_u(t) < \lambda\int_{[0,\Lambda]} |u|^p\,dx < \lim_{t\to\infty} m_u(t),$$

by the intermediate value theorem, there exists $t_u \in \langle 0, +\infty\rangle$ such that

$$m_u(t_u) = \lambda\int_{[0,\Lambda]} |u|^p\,dx.$$

Also

$$m_u'(t) = [(p-1)-q]\,t^{q-p}\int_{[0,\Lambda]} b|u|^{q+1}dx$$

and

$$m'_u(t) > 0, \text{ since } t > 0,\ 1 < q < p-1,\ 2 < p < \infty.$$

Thus, m_u is a strictly increasing function, and we conclude that the equation (3.13) has t_u as a unique solution. Similarly, $t_u u \in N_\lambda$. Because $m_u(t)$ has a unique solution, substituting (3.13) into (3.9), we have

$$\lambda \int_{[0,\Lambda]} |u|^p\,dx = \int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - t_u^{q-(p-1)} \int_{[0,\Lambda]} b|u|^{q+1}dx.$$

Thus

$$\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p\,dx - \lambda \int_{[0,\Lambda]} |u|^p\,dx - t_u^{q-(p-1)} \int_{[0,\Lambda]} b|u|^{q+1}dx = 0. \tag{3.24}$$

Multiplying equation (3.24) by t_u^{p-1}, we have

$$t_u^{p-1} \int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - t_u^q \int_{[0,\Lambda]} b|u|^{q+1}dx = 0,$$

which is the same as $J'_\lambda(t_u u)t_u u = 0$. Thus, $t_u u \in N_\lambda$. Because $t_u u \in N_\lambda$, $m'_u(t_u) > 0$ and $t > 0$

$$\phi''_{t_u u}(1) = t^{p+1} m'_u(t_u) > 0,$$

that is, $t_u u \in N_\lambda^+$. Also, $\phi'_u(t_u) = 0$, meaning that ϕ_u has a critical point that is a local minimum point at $t = t_u$. Indeed

$$t_u^{p-1} \int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - t_u^q \int_{[0,\Lambda]} b|u|^{q+1}dx = 0. \tag{3.25}$$

Dividing equation (3.25) by t_u yields

$$t_u^{p-2} \int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - t_u^{q-1} \int_{[0,\Lambda]} b|u|^{q+1}dx = 0.$$

Also

$$\lim_{t\to\infty} \phi_u(t) = \lim_{t\to\infty} \left[\frac{t^p}{p} \int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - \frac{t^{q+1}}{q+1} \int_{[0,\Lambda]} b|u|^{q+1}dx \right] = \infty$$

and

$$\lim_{t\to 0^+} \phi_u(t) = \lim_{t\to 0^+} \left[\frac{t^p}{p} \int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx - \frac{t^{q+1}}{q+1} \int_{[0,\Lambda]} b|u|^{q+1}dx \right] = 0.$$

From this analysis, we conclude that the graph of ϕ_u is like (b) of Figure 6.

Case4. If $\int_{[0,\Lambda]} b|u|^{q+1}dx > 0$ and $\int_{[0,\Lambda]} |\,{}_0D_x^\alpha u|^p - \lambda \int_{[0,\Lambda]} |u|^p\,dx < 0$, thus, we have the scenario described in property (e) of **Case A**, which is observed in graph (b) of Figure 5. Then, $\phi_u(t)$ is decreasing (see graph (b) of Figure 7). Also, because (3.14), $\phi'_u(t) < 0$. Thus, the equivalence (3.12) is not satisfied; therefore, it is concluded that no multiple of u is in N_λ.

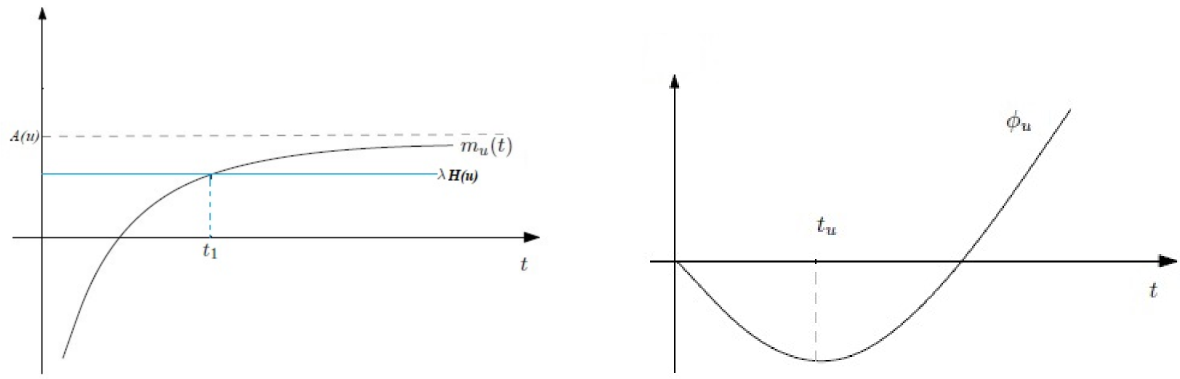
After this analysis, we can now define:

$$L_+(\lambda) = \left\{ u \in E_0^{\alpha,p} : \|u\| = 1, \int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\,dx > 0 \right\},$$

$$B_+ = \left\{ u \in E_0^{\alpha,p} : \|u\| = 1, \int_{[0,\Lambda]} b|u|^{q+1}dx > 0 \right\}.$$

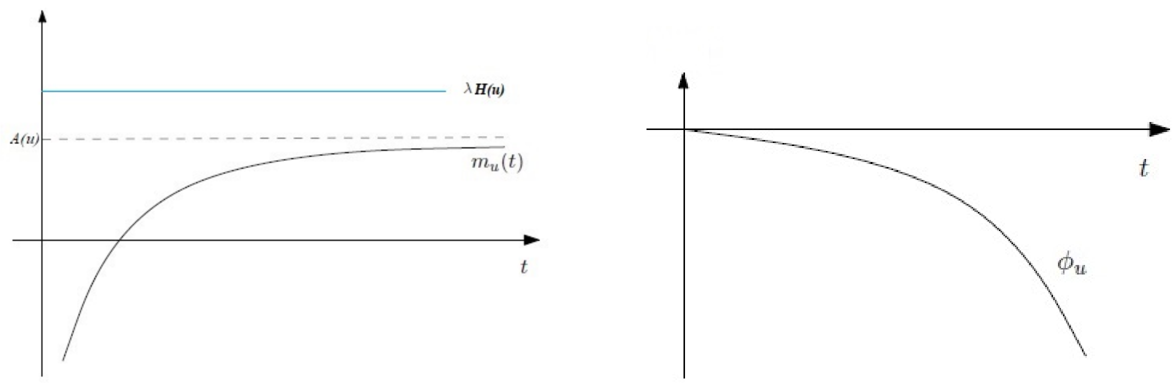
Analogously, we can define $L_-(\lambda), B_-, L_0(\lambda), B_0$. In an appropriate way, we can state the following.

(i) If $u \in L_+(\lambda) \cap B_+$, then $t \to \phi_u(t)$ has a local minimum $t = t(u)$ y $t(u)u \in N_\lambda^+$.
(ii) If $u \in L_-(\lambda) \cap B_-$, then $t \to \phi_u(t)$ has a local maximum $t = t(u)$ and $t(u)u \in N_\lambda^-$.
(iii) If $u \in L_+(\lambda) \cap B_-$, then $t \to \phi_u(t)$ is strictly increasing and no multiple of u is in N_λ.
(iv) If $u \in L_-(\lambda) \cap B_+$, then $t \to \phi_u(t)$ is strictly decreasing and no multiple of u is in N_λ;



(a) Graph of m_u (b) Graph of ϕ_u

Figure 6: Possible graph of ϕ_u in **Case 3.**



(a) Graph of m_u (b) Graph of ϕ_u

Figure 7: Possible graph of ϕ_u in **Case 4.**

3.3. *Properties of the Nehari manifold* N_λ

In this section, we discuss the fundamental role that the condition $L_-(\lambda) \subseteq B_-$ plays in determining the nature of the Nehari manifold.

- When $\lambda < \lambda_1$, by (3.2), we have $\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u|^p - \lambda|u|^p)\, dx > 0$, for all $u \in E_0^{\alpha,p}[0,\Lambda]$. Therefore,

$$L_+(\lambda) = \{u \in E_0^{\alpha,p}[0,\Lambda] : \|u\| = 1\}$$

  and $L_-(\lambda) = \emptyset$, $L_0(\lambda) = \emptyset$.
- When $\lambda = \lambda_1$, we have $L_-(\lambda) = \emptyset$, $L_0(\lambda) = \{\phi_1\}$.
- When $\lambda > \lambda_1$, $L_-(\lambda)$ is nonempty.

According to the previous considerations, the condition $L_-(\lambda) \subseteq B_-$ is always fulfilled when $\lambda < \lambda_1$, because the set $L_-(\lambda) = \emptyset$ in this case.

Theorem 3.13. *Assuming that there exists* $\hat{\lambda}$ *such that, for all* $\lambda < \hat{\lambda}$, $L_-(\lambda) \subseteq B_-$, *then,* $\forall \lambda < \hat{\lambda}$, *it holds that:*

(i) $L_0(\lambda) \subseteq B_-$ *and so* $L_0(\lambda) \cap B_0 = \emptyset$;
(ii) N_λ^+, *is bounded;*
(iii) $0 \notin \overline{N_\lambda^-}$, *and* N_λ^- *is closed;*

(iv) $\overline{N_\lambda^+} \cap N_\lambda^- = \emptyset$.

Proof.

(i). Suppose by contradiction that $L_0(\lambda) \nsubseteq B_-$. Then, there exists $u \in L_0(\lambda)$ such that $u \notin B_-$. Then

$$u \in L_0(\lambda) \Rightarrow u \in E_0^{\alpha,p}[0,\Lambda], \|u\| = 1, \quad \int_{[0,\Lambda]} (|{}_0D_x^\alpha u|^p dx - \lambda |u|^p) dx = 0,$$

and

$$u \notin B_- \Rightarrow \int_{[0,\Lambda]} b \left(\frac{|u|}{\|u\|}\right)^{q+1} dx \geqslant 0.$$

If $\lambda < \mu < \hat{\lambda}$, then

$$0 = \int_{[0,\Lambda]} (|{}_0D_x^\alpha u|^p - \lambda |u|^p) dx > \int_{[0,\Lambda]} (|{}_0D_x^\alpha u|^p - \mu |u|^p) dx \Rightarrow u \in L_-(\mu),$$

so that $L_-(\mu) \nsubseteq B_-$, and what we obtained contradicts the hypothesis of the theorem. Then, $L_0(\lambda) \subseteq B_-$ and being $B_- \cap B_0 = \emptyset$, we have $L_0(\lambda) \cap B_0 = \emptyset$.

(ii). Suppose that N_λ^+, is not bounded. Then, there exists $\{u_n\} \subseteq N_\lambda^+$, such that $\|u_n\| \to \infty$ when $n \to \infty$.

Let $v_n = \frac{u_n}{\|u_n\|}$. Thus, we have that $\{v_n\}$ is bounded, and without loss of generality, it can be assumed that $v_n \rightharpoonup v_0$ in $E_0^{\alpha,p}[0,\Lambda]$. Thus, $v_n \to v_0$ in $L^p([0,\Lambda])$ and in $L^{q+1}([0,\Lambda])$, because $1 < q < p-1$. Like $u_n \in N_\lambda^+$:

$$\int_{[0,\Lambda]} b|v_n|^{q+1} dx = \frac{1}{\|u_n\|^{q+1}} \int_{[0,\Lambda]} b|u_n|^{q+1} dx > 0,$$

Also

$$\int_{[0,\Lambda]} b|v_0|^{q+1} dx \geqslant 0. \tag{3.26}$$

Because $u_n \in N_\lambda^+ \subseteq N_\lambda$, we have

$$\int_{[0,\Lambda]} (|{}_0D_x^\alpha u_n|^p - \lambda |u_n|^p)\, dx = \int_{[0,\Lambda]} b|u_n|^{q+1} dx.$$

Then, dividing by $\|u_n\|^p$ yields

$$\int_{[0,\Lambda]} \left(\frac{|{}_0D_x^\alpha u_n|^p}{\|u_n\|^p} - \lambda \frac{|u_n|^p}{\|u_n\|^p}\right) dx = \int_{[0,\Lambda]} b \frac{|u_n|^{q+1}}{\|u_n\|^{q+1}} \frac{\|u_n\|^{q+1}}{\|u_n\|^p} dx,$$

$$\int_{[0,\Lambda]} (|{}_0D_x^\alpha v_n|^p - \lambda |v_n|^p)\, dx = \int_{[0,\Lambda]} b|v_n|^{q+1} \frac{1}{\|u_n\|^{p-(q+1)}} dx \to 0,$$

on $L^p([0,\Lambda])$ because $b|v_n|^{q+1}$ is bounded on $L^{q+1}([0,\Lambda])$ and $\|u_n\|^{p-(q+1)} \to \infty$. Suppose now that $v_n \nrightarrow v_0$ in $E_0^{\alpha,p}[0,\Lambda]$. By the convergence theorem, we have

$$\int_{[0,\Lambda]} |{}_0D_x^\alpha v_0|^p dx < \liminf_{n\to\infty} \int_{[0,\Lambda]} |{}_0D_x^\alpha v_n|^p.$$

Also

$$\int_{[0,\Lambda]} (|{}_0D_x^\alpha v_0|^p - \lambda |v_0|^p)\, dx < \lim_{n\to\infty} \int_{[0,\Lambda]} (|{}_0D_x^\alpha v_n|^p - \lambda |v_n|^p)\, dx = 0$$

and thus, we have $\frac{v_0}{\|v_0\|} \in L_-(\lambda)$. By hypothesis of the theorem, we have $L_-(\lambda) \subseteq B_-$ and this allows $\frac{v_0}{\|v_0\|} \in B_-$, which is a contradiction by (3.26). Now, suppose $v_n \to v_0$ in $E_0^{\alpha,p}[0,\Lambda]$. Thus, $\|v_0\| = 1$ and

$$\int_{[0,\Lambda]} (|{}_0D_x^\alpha v_0|^p - \lambda |v_0|^p)\, dx = \lim_{n\to\infty} \int_{[0,\Lambda]} (|{}_0D_x^\alpha v_n|^p - \lambda |v_n|^p)\, dx = 0.$$

Thus, $v_0 \in L_0(\lambda)$ and for part (i) $L_0(\lambda) \subseteq B_-$, this allows us to obtain $v_0 \in B_-$, which is again a contradiction, because

$$\int_{[0,\Lambda]} b|v_0|^{q+1}dx \geqslant 0.$$

Therefore, N_λ^+ is bounded.

(iii). Suppose $0 \in \overline{N_\lambda^-}$; then, there exists $\{u_n\} \subseteq N_\lambda^-$ such that $\lim_{n\to\infty} u_n = 0$.

Taking $v_n = \frac{u_n}{\|u_n\|}$, we have that $\{v_n\}$ is bounded, and it can be assumed, without a loss of generality, that $v_n \rightharpoonup v_0$ in $E_0^{\alpha,p}$. Thus, $v_n \to v_0$ in $L^p([0,\Lambda])$ and $L^{q+1}([0,\Lambda])$. Because $u_n \in N_\lambda^- \subseteq N_\lambda$, we have

$$\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u_n|^p - \lambda|u_n|^p)\,dx = \int_{[0,\Lambda]} b|u_n|^{q+1}dx < 0$$

and multiplying by $\|u_n\|^{-p}$, we obtain

$$\int_{[0,\Lambda]} \left(\frac{|\,{}_0D_x^\alpha u_n|^p}{\|u_n\|^p} - \lambda\frac{|u_n|^p}{\|u_n\|^p}\right)dx = \int_{[0,\Lambda]} b\frac{|u_n|^{q+1}}{\|u_n\|^{q+1}}\frac{\|u_n\|^{q+1}}{\|u_n\|^p}dx.$$

Then, we have

$$\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha v_n|^p - \lambda|v_n|^p)\,dx = \frac{1}{\|u_n\|^{p-(q+1)}}\int_{[0,\Lambda]} b|v_n|^{q+1}dx,$$

$$\|u_n\|^{p-(q+1)}\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha v_n|^p - \lambda|v_n|^p)\,dx = \int_{[0,\Lambda]} b|v_n|^{q+1}dx \leqslant 0.$$

We know that $\{v_n\}$ is bounded on $E_0^{\alpha,p}[0,\Lambda]$, b is regular on $\overline{[0,\Lambda]}$ and $\lim\limits_{n\to\infty}\|u_n\| = 0$, and we obtain

$$\lim_{n\to\infty}\int_{[0,\Lambda]} b|v_n|^{q+1}dx = 0.$$

Thus

$$\int_{[0,\Lambda]} b|v_0|^{q+1}dx = 0. \tag{3.27}$$

Because $b|v_0|^{q+1}$ is bounded on $[0,\Lambda]$ and the term $\|u_n\|^{p-(q+1)} \to \infty$. Suppose that $v_n \to v_0$ in $E_0^{\alpha,p}[0,\Lambda]$; thus, $\|v_0\| = 1$ and

$$\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha v_0|^p - \lambda|v_0|^p)\,dx = \lim_{n\to\infty}\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha v_n|^p - \lambda|v_n|^p)\,dx \leqslant 0,$$

which allows $v_0 \in L_0(\lambda)$ or $v_0 \in L_-(\lambda)$. $L_\lambda \in B_0(\lambda)$ by hypothesis of the theorem and $L_0(\lambda) \subseteq B_-$ this for (i). In both cases, we would have $v_0 \in B_-$, which contradicts (3.27). Thus, $v_n \nrightarrow v_0$ in $E_0^{\alpha,p}[0,\Lambda]$; thus, by the convergence theorem, we have that:

$$\int_{[0,\Lambda]} |\,{}_0D_x^\alpha v_0|^p\,dx < \lim_{n\to\infty}\int_{[0,\Lambda]} |\,{}_0D_x^\alpha v_n|^p\,dx.$$

Also, $\{v_n\}$ is bounded on $E_0^{\alpha,p}[0,\Lambda]$ and by the dominated convergence theorem

$$\lim_{n\to\infty}\int_{[0,\Lambda]} |v_n|^p\,dx = \int_{[0,\Lambda]} \lim_{n\to\infty}|v_n|^p\,dx.$$

Thus

$$\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha v_0|^p - \lambda|v_0|^p)\,dx < \lim_{n\to\infty}\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha v_n|^p - \lambda|v_n|^p)\,dx \leqslant 0.$$

Then, $\frac{v_0}{\|v_0\|} \in L_-(\lambda) \cap B_0$, which is again a contradiction, because $L_-(\lambda) \subseteq B_-$ and in $B_- \cap B_0 = \emptyset$. Therefore, $0 \notin \overline{N_\lambda^-}$. Now, we continue with the proof that $\overline{N_\lambda^-}$ is closed. To achieve this, we must show that $\overline{N_\lambda^-} \subset N_\lambda^-$. Let $\{u_n\} \subseteq N_\lambda^-$; thus, there exists $\{u_n\} \in \overline{N_\lambda^-}$ such that $u_n \to u$ into $E_0^{\alpha,p}[0,\Lambda]$. Therefore, $u \in \overline{N_\lambda^-}$, and as we saw before, u cannot be identically null, that is, $u \neq 0$. Also, we have the following result:

$$\int_{[0,\Lambda]} (|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx = \int_{[0,\Lambda]} b|u|^{\alpha+1}dx \leqslant 0. \tag{3.28}$$

If both integrals are equal to 0, then $\frac{u}{\|u\|} \in L_0(\lambda) \cap B_0$, which contradicts (i). Hence, by (3.28), both integrals must be negative, which allows us to $u \in N_\lambda^-$. Thus, N_λ^- is closed.

(iv). Suppose that there exists $u \in \overline{N_\lambda^+} \cap N_\lambda^-$, because $u \in N_\lambda^-$, by (iii), we have that u is not identically null, that is, $u \neq 0$, and it is evident that

$$\int_{[0,\Lambda]} b|u|^{q+1}dx < 0.$$

Additionally, because $u \in \overline{N_\lambda^+}$,

$$\int_{[0,\Lambda]} b|u|^{q+1}dx \geqslant 0,$$

which is a contradiction because we would have to

$$\int_{[0,\Lambda]} (|\,_0D_x^\alpha u|^p - \lambda|u|^p)\,dx = \int_{[0,\Lambda]} b|u|^{q+1}dx = 0,$$

which is impossible. Therefore, we conclude that $\overline{N_\lambda^+} \cap N_\lambda^- = \emptyset$. □

When analyzing the Fibering maps, we observed that $J_\lambda(u) > 0$ in N_λ^- and $J_\lambda(u) < 0$ in N_λ^+. That is, the following theorem states that $J_\lambda(u) > 0$ in N_λ^- and the behavior of $J_\lambda(u)$ in N_λ^+.

Theorem 3.14. *Suppose that there exists $\hat{\lambda}$ such that, for all $\lambda < \hat{\lambda}$, $L_-(\lambda) \subseteq B_-$. Then, $\forall \lambda < \hat{\lambda}$, it holds that (hypothesis of Theorem 3.13):*

(i) *J_λ is lower bounded on N_λ^+;*
(ii) *$\inf_{u \in N_\lambda^-} J_\lambda(u) > 0$, showing that N_λ^- is nonempty.*

Proof.

(i). The proof of (i) is an immediate consequence of the bounding of N_λ^+.

(ii). Note that $J_\lambda(u) \geqslant 0$ for $u \in N_\lambda^-$. Indeed, if $u \in N_\lambda^-$ then $u \in N_\lambda$ and

$$J_\lambda(u) = \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_{[0,\Lambda]} (|\,_0D_x^\alpha u|^p - \lambda|u|^p)dx = \left(\frac{1}{p} - \frac{1}{q+1}\right) \int_{[0,\Lambda]} b|u|^{q+1}dx \geqslant 0.$$

Now, suppose that $\inf_{u \in N_\lambda^-} J_\lambda(u) = 0$. Then, there exists $\{u_n\} \subseteq N_\lambda^-$ such that $\lim_{n\to\infty} J_\lambda(u_n) = 0$. By Theorem 2.11:

$$\int_{[0,\Lambda]} (|\,_0D_x^\alpha u_n|^p - \lambda|\,_0D_x^\alpha u_n|^p)dx \to 0, \quad \int_{[0,\Lambda]} b|u_0|^{q+1}dx = \lim_{n\to\infty} \int_{[0,\Lambda]} b|u_n|^{q+1}dx \to 0,$$

when $n \to \infty$.

Now, let $v_n = \frac{u_n}{\|u_n\|}$, as $0 \notin \overline{N_\lambda^-}$; then, $\|u_n\|$ is bounded, that is, there exists $C > 0$ such that $\{\|u_n\|\} > C$. Then

$$\lim_{n\to\infty}\int_{[0,\Lambda]}(|{}_0D_x^\alpha v_n|^p - \lambda|{}_0D_x^\alpha v_n|^p)dx = \lim_{n\to\infty}\frac{1}{\|u_n\|^p}\int_{[0,\Lambda]}(|{}_0D_x^\alpha u_n|^p - \lambda|{}_0D_x^\alpha u_n|^p)dx = 0$$

and

$$\lim_{n\to\infty}\int_{[0,\Lambda]} b|v_n|^{q+1}dx = \lim_{n\to\infty}\frac{1}{\|u_n\|^p}\int_{[0,\Lambda]} b|u_n|^{q+1}dx = 0.$$

Being v_n bounded, we can assume, without a loss of generality, that $v_n \rightharpoonup v_0$ in $E_0^{\alpha,p}[0,\Lambda]$. Therefore, $v_n \to v_0$ on $E_0^{\alpha,p}[0,\Lambda]$, and we have $\|v_0\| = 1$ and $v_n \to v_0$ on $L^p([0,\Lambda])$ and $L^{q+1}([0,\Lambda])$. Because b is a regular function on $[0,\Lambda]$, using the dominated convergence theorem, we conclude that

$$\lim_{n\to\infty}\int_{[0,\Lambda]} b|v_n|^{q+1}dx = \int_{[0,\Lambda]} b\lim_{n\to\infty}|v_n|^{q+1}dx = \int_{[0,\Lambda]} b|v_0|^{q+1}dx = 0,$$

$$\int_{[0,\Lambda]}(|{}_0D_x^\alpha v_0|^p - \lambda|{}_0D_x^\alpha v_0|^p)dx = 0.$$

Thus, $v_0 \in L_0(\lambda)$. Conversely, if $v_n \nrightarrow v_0$ into $E_0^{\alpha,p}[0,\Lambda]$, we have

$$\int_{[0,\Lambda]}(|{}_0D_x^\alpha v_0|^p - \lambda|{}_0D_x^\alpha v_0|^p)dx < 0,$$

that is, $\frac{v_0}{\|v_0\|} \in L_-(\lambda)$. However, in both cases, $\frac{v_0}{\|v_0\|} \in B_0$, which is a contradiction, since we know $L_-(\lambda) \subseteq B_-$ and $L_0(\lambda) \cap B_0 = \emptyset$. Therefore

$$\inf_{u\in N_\lambda^-} J_\lambda(u) > 0.$$

□

3.4. Existence of weak solution of problem P_0

In this section, we show that there exists a minimizer at $N_\lambda^+(N_\lambda^-)$, which is a critical point of $J_\lambda(u)$ and thus a nontrivial solution of the boundary problem P_0:

Theorem 3.15. *Suppose* $L_-(\lambda) \subseteq B_-(\lambda)$, *then for all* $\lambda < \hat{\lambda}$

(i) *there is a minimizing point for* J_λ *at* N_λ^+;
(ii) *there is a minimizing point for* J_λ *in* N_λ^-, *whenever* $L_-(\lambda)$ *is nonempty.*

Proof.
(i). By Theorem 3.14, J_λ is lower bounded on N_λ^+. By the definition of infimum, there exists $\{u_n\} \subseteq N_\lambda^+$ a minimizing sequence such that:

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^+} J_\lambda(u) < 0.$$

and

$$J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{(q+1)}\right)\int_{[0,\Lambda]} b|u_n|^{q+1},$$

as $\left(\frac{1}{p} - \frac{1}{(q+1)}\right) < 0$ and $\int_{[0,\Lambda]} b|v_0|^{q+1}dx > 0$ for all n, we have that $J_\lambda(u_n) < 0$. Also, by part (ii) of Theorem 3.14, N_λ^+ is bounded; thus, we can assume that $u_n \rightharpoonup u_0$ in $E_0^{\alpha,p}[0,\Lambda]$ and $u_n \to u_0$ in $L^{q+1}([0,\Lambda])$. Therefore, we follow that

$$\int_{[0,\Lambda]} b|u_0|^{q+1}dx = \lim_{n\to\infty}\int_{[0,\Lambda]} b|u_n|^{q+1}dx > 0$$

and so $\frac{u_0}{\|u_0\|} \in B_+$. Therefore, $J_\lambda(u_n) = \left(\frac{1}{p} - \frac{1}{q+1}\right)\int_{[0,\Lambda]} b|u_n|^{q+1}$. By Theorem 3.13, $L_0(\lambda) \subseteq B_-$, $L_-(\lambda) \subseteq B_-$ and we also have $B_- \cap B_+ = \emptyset$. Thus, $\frac{u_0}{\|u_0\|} \in L_+(\lambda) \cap B_+$, and by the previous results, we obtain that ϕ_{u_0} has a unique minimum in t_{u_0} such that $t_{u_0}u_0 \in N_\lambda^+$. We must prove that u_0 is in the Nehari manifold. To do this, suppose that $u_n \nrightarrow u_0$ in $E_0^{\alpha,p}$, then

$$\int_{[0,\Lambda]} (|\,_0D_x^\alpha u_0|^p - \lambda|\,_0D_x^\alpha u_0|^p)dx < \lim_{n\to\infty}\int_{[0,\Lambda]} (|\,_0D_x^\alpha u_n|^p - \lambda|\,_0D_x^\alpha u_n|^p)dx,$$

$$\lim_{n\to\infty}\int_{[0,\Lambda]} b|u_n|^{q+1}dx = \int b|u_0|^{q+1}dx.$$

Therefore

$$t_{u_0} = \left[\frac{\int_{[0,\Lambda]} b|u_0|^{q+1}dx}{\int_{[0,\Lambda]}(|\,_0D_x^\alpha u_0|^p - \lambda|u_0|^p)dx}\right]^{\frac{1}{p-(q+1)}} > 1.$$

Also

$$\begin{aligned} J_\lambda(u_0) &= \int_{[0,\Lambda]} (|\,_0D_x^\alpha u_0|^p - \lambda|\,_0D_x^\alpha u_0|^p)dx - \ int_{[0,\Lambda]} b|u_0|^{q+1}dx \\ &< \lim_{n\to\infty}\int_{[0,\Lambda]} (|\,_0D_x^\alpha u_n|^p - \lambda|\,_0D_x^\alpha u_n|^p)dx - \int_{[0,\Lambda]} b|u_n|^{q+1}dx = \lim_{n\to\infty} J_\lambda(u_n). \end{aligned} \tag{3.29}$$

Because ϕ_{u_0} has a unique minimum at t_{u_0} such that $t_{u_0}u_0 \in N_\lambda^+$, it follows that:

$$\phi_{u_0}(t_{u_0}) = J_\lambda(t_{u_0}u_0) < \phi_{u_0}(t), \quad \forall\, t \in \mathbb{R}^+$$

In particular, the inequality holds for $t = 1$,

$$J_\lambda(t_{u_0}u_0) < J_\lambda(u_0). \tag{3.30}$$

Then, by (3.29) and (3.30), we have that

$$J_\lambda(t_{u_0}u_0) < J_\lambda(u_0) < \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^+} J_\lambda(u),$$

which is impossible because $t_{u_0}u_0 \in N_\lambda^+$. Therefore, $u_n \to u_0$ in $E_0^{\alpha,p}[0,\Lambda]$ and $u_0 \in N_\lambda^+$. Then, we follow that u_0 is a minimizer for J_λ in N_λ^+.

Conversely, $J_\lambda(u) = J_\lambda(|u|)$, and we can assume that u_0 is nonnegative in $[0,\Lambda]$. Therefore, $J_\lambda(u_0) < 0$, u_0 is a local minimum for J_λ in N_λ^+. We follow from Lemma 3.12 that u_0 is a critical point of J_λ and thus is a weak solution of the boundary problem P_0.

(ii). Let $\{u_n\} \subseteq N_\lambda^-$ be a minimizing sequence for J_λ in N_λ^-. After Theorem 3.14, we have that

$$\lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^-} J_\lambda(u) > 0.$$

Suppose that $\{u_n\}$ is unbounded; thus, we can assume that $\|u_n\| \to \infty$ when $n \to \infty$. Consider $v_n = \frac{u_n}{\|u_n\|}$. Being $\{J_\lambda(u_n)\}$ bounded, it follows that

$$\left\{\int_{[0,\Lambda]} (|\,_0D_x^\alpha u_n|^p - \lambda|u_n|^p)dx\right\} \text{ and } \left\{\int_{[0,\Lambda]} (b|u_n|^{q+1})dx\right\}$$

are bounded and therefore

$$\lim_{n\to\infty}\int_{[0,\Lambda]} (|\,_0D_x^\alpha v_n|^p - \lambda|v_n|^p)dx = \lim_{n\to\ infty}\int_{[0,\Lambda]} b|v_n|^{q+1}dx = \lim_{n\to\infty}\frac{1}{\|u_n\|^p}\int_{[0,\Lambda]} b|u_n|^{q+1}dx = 0.$$

Because $\{v_n\}$ is bounded, we can assume that $v_n \rightharpoonup v_0$ on $E_0^{\alpha,p}[0,\Lambda]$ and $v_n \to v_0$ on $L^p([0,\Lambda])$ and $L^{q+1}([0,\Lambda])$; thus

$$\int_{[0,\Lambda]} b|v_0|^{q+1}dx = 0.$$

If $v_n \to v_0$ in $E_0^{\alpha,p}[0,\Lambda]$, we see that $v_0 \in L_0(\lambda) \cap B_0$, which is not possible by the (i) part of Theorem 3.13. From there $v_n \nrightarrow v_0$ in $E_0^{\alpha,p}[0,\Lambda]$ and

$$\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha v_0|^p - \lambda|v_0|^p)dx < \lim_{n\to\infty} \int_{[0,\Lambda]} |\,{}_0D_x^\alpha v_n|^p - \lambda|v_n|^p dx = 0.$$

Therefore, $v_0 \neq 0$ and $\frac{v_0}{\|v_0\|} \in L_-(\lambda) \cap B_0$, which is also impossible. Therefore, $\{u_n\}$ is bounded, and we can assume that $u_n \rightharpoonup u_0$ in $E_0^{\alpha,p}[0,\Lambda]$ and $u_n \to u_0$ in $L^p([0,\Lambda])$ and $L^{q+1}([0,\Lambda])$. Suppose that $u_n \nrightarrow u_0$ in $E_0^{\alpha,p}0,\Lambda$; then, we have

$$\int_{[0,\Lambda]} b|u_0|^{q+1}dx = \lim_{n\to\infty} \int_{[0,\Lambda]} b|u_n|^{q+1}dx = \left(\frac{1}{p} - \frac{1}{q+1}\right)^{-1} \lim_{n\to\infty} J_\lambda(u_n) < 0$$

and

$$\begin{aligned}\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u_0|^p - \lambda|u_0|^p)dx &< \lim_{n\to\infty} \int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u_n|^p - \lambda|u_n|^p)dx \\ &= \lim_{n\to\infty} \int_{[0,\Lambda]} b|u_n|^{q+1}dx = \int b|u_0|^{q+1}dx < 0.\end{aligned}$$

Therefore, $\frac{v_0}{\|v_0\|} \in L_-(\lambda) \cap B_-(\lambda)$ and $t_{u_0}u_0 \in N_\lambda^-$, where

$$t_{u_0} = \left[\frac{\int_{[0,\Lambda]} b|u_0|^{q+1}dx}{\int_{[0,\Lambda]} (|\,{}_0D_x^\alpha u_0|^p - \lambda|u_0|^p)dx}\right]^{\frac{1}{p-(q+1)}} < 1.$$

Additionally, $t_{u_0}u_n \rightharpoonup t_{u_0}u_0$, but $t_{u_0}u_n \nrightarrow t_{u_0}u_0$ in $E_0^{\alpha,p}[0,\Lambda]$, then

$$J_\lambda(t_{u_0}u_0) < \lim_{n\to\infty} J_\lambda(t_{u_0}u_n).$$

Because the operator $t \to J_\lambda(t(u_n)$, it reaches its maximum at $t = 1$,

$$\lim_{n\to\infty} J_\lambda(t_{u_0}u_0) \leqslant \lim_{n\to\infty} J_\lambda(u_n) = \inf_{u\in N_\lambda^-} J_\lambda(u).$$

Therefore, $J_\lambda(t_{u_0}u_0) < \inf_{u\in N_\lambda^-} J_\lambda(u)$, which is a contradiction. In that sense, $u_n \to u_0$ in $E_0^{\alpha,p}[0,\Lambda]$, and it follows that u_0 is a minimizing point for $J_\lambda(u)$ in N_λ^-. Because $J_\lambda(u) = J_\lambda(|u|)$ [19], we can assume that u_0 is nonnegative in $[0,\Lambda]$, and as N_λ^- is closed, u_0 is a local minimum point for J_λ in N_λ. We follow from Lemma 3.12 that u_0 is a critical point of J_λ and thus is a weak solution of the problem P_0. □

4. Existence of a weak solution of the fractional order equation P_1

In this section, we investigate the existence of a weak solution of the fractional order equation (P_1) in the fractional space $E_0^{\alpha,p}[0,\Lambda]$. By Definition 2.1, [30, Theorem 2.2], [18, Theorem 2.1], and Lemma 2.13, we can write the problem P_1 as an integral equation shown in Theorem 4.2.

Definition 4.1. Consider $F : [0,\Lambda] \times [0,T] \to \mathbb{R}$ such that:

$$F(x,u(x)) = -\,{}_xD_\Lambda^\alpha(|\,{}_0D_x^\alpha u(x,s)|^{p-2}\,{}_0D_x^\alpha u(x,s)) + \lambda|u(x,s)|^{p-2}u(x,s) + b(x)|u(x,s)|^{q-1}u(x,s)$$

a continuous function on a flat enclosure $G \subset [0,\Lambda] \times [0,T]$ that contains $u(x,0) = \phi(x)$ and satisfies the

Lipschitz condition with respect to t:

$$|F(x,t_1)-F(x,t_2)| \leqslant M|t_1-t_2|.$$

Theorem 4.2. *Let* $1<\beta\leqslant 2$, $\lceil\beta\rceil=n$ *and* $F(x,u(x))$ *defined in Definition 4.1. A function* $u\in C^2[0,T]$ *is a solution of the problem* P_1 *if and only if it is a solution of the integral equation*

$$E_1\begin{cases} u(x,t)=\phi(x)+\psi(x)t+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}F(x,u(x))ds,\\ u(0,t)=u(\Lambda,t)=0\ \ ,\ \textit{for all } t\in\Omega=[0,T].\end{cases}$$

Proof.

$\Longrightarrow$ Let the continuous function $F(u):[0,\Lambda]\times[0,T]\to\mathbb{R}$ with:

$$F(u(x,t))=-\,{}_xD_T^{\alpha}(|\,{}_0D_x^{\alpha}u(x,s)|^{p-2}\,{}_0D_x^{\alpha}u(x,s))+\lambda|u(x,s)|^{p-2}u(x,s)+b(x)|u(x,s)|^{q-1}u(x,s).$$

From the problem P_1 we have the equation

$$ {}_0^C D_t^{\beta}u=F(u) \tag{4.1}$$

subject to initial conditions

$$u(x,0)=\phi(x) \text{ and } u_t(x,0)=\psi(x),\quad \text{with }\ x\in[0,\Lambda]. \tag{4.2}$$

Applying the fractional Riemann-Liouville integral of order β from left to (4.1):

$$ {}_0I_t^{\beta}(\,{}_0^C D_t^{\beta}u)\ =\ {}_0I_t^{\beta}(F(u)).$$

Then, given that $1<\beta<2$ the value of $n=2$ and of the property (2.4), we can determine that:

$$u(t)-\sum_{k=0}^{1}\frac{u^{(k)}(0)}{k!}(t-0)^k={}_0\,I_t^{\beta}(F(u)),\ \ t\in[0,T],\ \ u(t)-u(0)-u'(0)t=\frac{1}{\Gamma(\beta)}\int_0^t (ts)^{\beta-1}F(u)ds,$$

then substituting the conditions (4.2),

$$u(t)-\phi(x)-\psi(x)t=\frac{1}{\Gamma(\beta)}\int_0^t (ts)^{\beta-1}F(u)ds.$$

We thus obtain the integral equation E_1.

$\Longleftarrow$ In the integral equation E_1, we apply the Caputo fractional derivative of order β:

$$u(x,t)=\phi(x)+\psi(x)t+\frac{1}{\Gamma(\beta)}\int_0^t (ts)^{\beta-1}F(u)ds,$$
$$ {}_0^C D_t^{\beta}u(x,t)=\ {}_0^C D_t^{\beta}\phi(x)+\ {}_0^C D_t^{\beta}\psi(x)t+\ {}_0^C D_t^{\beta}(\,{}_0I_t^{\beta}F(u)),$$

then with the property that connect the fractional derivative of Riemann-Liouville and Caputo [34], we follow what

$$ {}_0^C D_t^{\beta}u(x,t)=0+\frac{1}{\Gamma(2-\alpha)}\int_0^t (ts)^{2-\alpha-1}t^{(2)}ds+\ {}_0^C D_t^{\beta}(\,{}_0I_t^{\beta}F(u)),\quad {}_0^C D_t^{\beta}u(x,t)=F(u(x,t)).$$

To obtain the initial conditions, we consider $u(x,0)$ of the equation E_1,

$$u(x,0)=\phi(x)+\psi(x)0+\frac{1}{\Gamma(\beta)}\int_0^0 (0-s)^{\beta-1}F(u)ds,\quad u(x,0)=\phi(x),$$

further differentiating $u(x,t)$ and replacing $t=0$,

$$u_t(x,0)=\psi(x)+\frac{1}{\Gamma(\beta)}\int_0^0 (0-s)^{\beta-1}F(u)ds,\quad u_t(x,0)=\psi(x),$$

we obtain the problem P_1. □

Definition 4.3. We say that $u \in C([0,T]; E_0^{\alpha,p}([0,\Lambda]))$ for $0 < T < 1$ is a weak solution of the differential equation of fractional order P_1, if it satisfies

$$\int_{[0,\Lambda]} (u - \Phi(u))v dx = 0,\ \forall t \in [0,T],\ \text{for each } v \in E_0^{\alpha,p}([0,\Lambda]),$$

where

$$\begin{cases} \Phi(u) = \phi(x) + \psi(x)t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (-\,{}_xD_T^\alpha(|\,{}_0D_x^\alpha u(x,s)|^{p-2}\,{}_0D_x^\alpha u(x,s)) \\ \qquad + \lambda|u(x,s)|^{p-2}u(x,s) + b(x)|u(x,s)|^{q-1}u(x,s))ds, \quad \forall (x,t) \in \Omega_T, \\ u(0,t) = u(\Lambda,t) = 0 \ \ , \ \ \text{for all } t \text{ in} \Omega = [0,T]. \end{cases}$$

Lemma 4.4. *Let* $b \in L^\infty[0,\Lambda]$*, then, the operator*

$$\Phi(u) : E_0^{\alpha,p}[0,\Lambda] \to E^{\alpha,p}[0,\Lambda]$$

is completely continuous.

Proof. Given

$$F(u) = -\,{}_xD_\Lambda^\alpha(|\,{}_0D_x^\alpha u(x,s)|^{p-2}\,{}_0D_x^\alpha u(x,s)) + \lambda|u(x,s)|^{p-2}u(x,s) + b(x)|u(x,s)|^{q-1}u(x,s),$$

then, we can write

$$\Phi(u) = \phi(x) + \psi(x)t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} F(u) ds.$$

For each $v \in E_0^{\alpha,p}([0,\Lambda])$ and $\|v\|_{E_0^{\alpha,p}} = 1$, we have that,

$$\begin{aligned} \langle F(u), v\rangle &= \int_{[0,\Lambda]} (-|\,{}_0D_x^\alpha u(x,s)|^{p-2}\,{}_0D_x^\alpha u(x,s)\,{}_0D_x^\alpha v(x,s) + \lambda|u(x,s)|^{p-2}u(x,s)v(x,s) \\ &\quad + b(x)|u(x,s)|^{q-1}u(x,s)v(x,s))dx, \text{ for each } v \in E_0^{\alpha,p}, \\ |\langle F(u), v\rangle| &= \left| \int_{[0,\Lambda]} (-|\,{}_0D_x u|^{p-2}\,{}_0D_x u\,{}_0D_x^\alpha v + \lambda|u|^{p-2}uv + b|u|^{q-1}uv) dx \right| . \end{aligned} \tag{4.3}$$

By Lemma 3.12, we know that $r(u) = \langle J_\lambda'(u), u\rangle = 0$:

$$\int_{[0,\Lambda]} (|\,{}_0D_x u|^p) dx = \int_{[0,\Lambda]} \lambda|u|^p dx + \int_{[0,\Lambda]} b|u|^{q+1} dx.$$

Additionally, $E_0^{\alpha,p}[0,\Lambda] \hookrightarrow L^p[0,\Lambda]$, we know by Poincaré's inequality, $\|u\|_{L^p[0,\Lambda]} \leqslant \frac{\Lambda^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_{L^p[0,\Lambda]}$, let us remember that $\|{}_0D_t^\alpha u\|_{L^p[0,\Lambda]} = \|u\|_{E_0^{\alpha,p}[0,\Lambda]}$, then $\|u\|_{L^p[0,\Lambda]} \leqslant \frac{\Lambda^\alpha}{\Gamma(\alpha+1)} \|u\|_{E_0^{\alpha,p}[0,\Lambda]}$. Thus

$$\int_{[0,\Lambda]} \lambda|u|^p dx \leqslant |\lambda| \int_{[0,\Lambda]} |u|^p dx = |\lambda| \|u\|_{L^p[0,\Lambda]}^p \leqslant |\lambda| \frac{\Lambda^\alpha}{\Gamma(\alpha+1)} \|u\|_{E_0^{\alpha,p}[0,\Lambda]}^p.$$

Additionally, by (3.3) and [18, Proposition 2.6], we have

$$\int_{[0,\Lambda]} b|u|^{q+1} dx \leqslant \|b\|_{L^\infty([0,\Lambda])} \frac{\Lambda^{1-(q+1)/p+\alpha(q+1)}}{\Gamma(\alpha+1)^{q+1}} \|u\|_{\alpha,p}^{q+1}.$$

Let $S=\frac{\Lambda^{\alpha}}{\Gamma(\alpha+1)}$ and $C=\frac{\Lambda^{1-(q+1)/p+\alpha(q+1)}}{\Gamma(\alpha+1)^{q+1}}$, thus

$$\begin{aligned}
\|u\|^{p}_{E_0^{\alpha,p}} &\leqslant |\lambda|S^{p}\|u\|^{p}_{E_0^{\alpha,p}}+\|b\|_{L^{\infty}[0,\Lambda]}C^{q+1}\|u\|^{q+1}_{E_0^{\alpha,p}},\\
\|u\|^{p}_{E_0^{\alpha,p}}-|\lambda|S^{p}\|u\|^{p}_{E_0^{\alpha,p}} &\leqslant \|b\|_{L^{\infty}[0,\Lambda]}C^{q+1}\|u\|^{q+1}_{E_0^{\alpha,p}},\\
\|u\|^{p}_{E_0^{\alpha,p}}(1-|\lambda|S^{p}) &\leqslant \|b\|_{L^{\infty}[0,\Lambda]}C^{q+1}\|u\|^{q+1}_{E_0^{\alpha,p}},\\
\frac{\|u\|^{p}_{E_0^{\alpha,p}}}{\|u\|^{q+1}_{E_0^{\alpha,p}}} &\leqslant \frac{\|b\|_{L^{\infty}[0,\Lambda]}C^{q+1}}{(1-|\lambda|S^{p})},\\
\|u\|^{p-(q+1)}_{E_0^{\alpha,p}} &\leqslant \frac{\|b\|_{L^{\infty}[0,\Lambda]}C^{q+1}}{(1-|\lambda|S^{p})},\\
\|u\|_{E_0^{\alpha,p}} &\leqslant \left(\frac{\|b\|_{L^{\infty}([0,\Lambda])}C^{q+1}}{(1-|\lambda|S^{p})}\right)^{\frac{1}{(p-(q+1)}}.
\end{aligned} \tag{4.4}$$

Now, we continue with (4.3) and have

$$\begin{aligned}
|\langle F(u),v\rangle| &= \left|\int_{[0,\Lambda]}(-|\,{}_0D_x^{\alpha}u|^{p-2}\,{}_0D_x^{\alpha}u\,{}_0D_x^{\alpha}v+\lambda|u|^{p-2}uv+b|u|^{q-1}uv)dx\right|\\
&\leqslant \left|\int_{[0,\Lambda]}|\,{}_0D_x^{\alpha}u|^{p-1}\,{}_0D_x^{\alpha}vdx\right|+\left|\int_{[0,\Lambda]}\lambda|u|^{p-1}vdx\right|+\left|\int_{[0,\Lambda]}b|u|^{q}vdx\right|.
\end{aligned} \tag{4.5}$$

Then, by (2.8), (2.5), and (4.4) and the inequality of Hölder, we have that

$$\begin{aligned}
\int_{[0,\Lambda]}|\,{}_0D_x^{\alpha}u|^{p-1}\,{}_0D_x^{\alpha}vdx &\leqslant \left(\int_{[0,\Lambda]}|\,{}_0D_x^{\alpha}u|^{(p-1)\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{[0,\Lambda]}|\,{}_0D_x^{\alpha}v|^{p}dx\right)^{\frac{1}{p}}\\
&= \|\,{}_0D_x^{\alpha}u\|^{p-1}_{L^p}\|\,{}_0D_x^{\alpha}v\|_{L^p}=\|u\|^{p-1}_{\alpha,p}\|v\|_{\alpha,p}.
\end{aligned}$$

Also

$$\begin{aligned}
\int_{[0,\Lambda]}|u|^{p-1}vdx &\leqslant \left(\int_{[0,\Lambda]}|u|^{(p-1)\frac{p}{p-1}}dx\right)^{\frac{p-1}{p}}\left(\int_{[0,\Lambda]}|v|^{p}dx\right)^{1/p}\\
&= \|u\|^{p-1}_{L^p}\|v\|_{L^p}\\
&\leqslant \left(\frac{\Lambda^{\alpha}}{\Gamma(\alpha+1)}\right)^{p-1}\|u\|^{p-1}_{E_0^{\alpha,p}}\left(\frac{\Lambda^{\alpha}}{\Gamma(\alpha+1)}\right)\|v\|_{E_0^{\alpha,p}}\\
&= \left(\frac{\Lambda^{\alpha}}{\Gamma(\alpha+1)}\right)^{p}\|u\|^{p-1}_{\alpha,p}\|v\|_{\alpha,p}=S^{p}\|u\|^{p-1}_{E_0^{\alpha,p}}\|v\|_{E_0^{\alpha,p}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{[0,\Lambda]}b|u|^{q}vdx &\leqslant \|b\|_{L^{\infty}[0,\Lambda]}\left(\int_{[0,\Lambda]}|u|^{q\frac{p}{q}}dx\right)^{\frac{q}{p}}\left(\int_{[0,\Lambda]}|v|^{\frac{p}{p-q}}dx\right)^{\frac{p-q}{p}}\\
&\leqslant \|b\|_{L^{\infty}[0,\Lambda]}\|u\|^{q}_{L^p}\left(\int_{[0,\Lambda]}|1|^{\frac{p-q}{p-q-1}}dx\right)^{\frac{p-q-1}{p-q}}\left(\int_{[0,\Lambda]}|v|^{\frac{p}{p-q}(p-q)}dx\right)^{\frac{1}{p-q}\frac{p-q}{p}\frac{p}{p-q}}\\
&= \|b\|_{L^{\infty}[0,\Lambda]}\|u\|^{q}_{L^p}|\Lambda|^{\frac{p-q-1}{p-q}}\|v\|^{\frac{p}{p-q}}_{L^p}\\
&\leqslant \|b\|_{L^{\infty}[0,\Lambda]}\left(\frac{\Lambda^{\alpha}}{\Gamma(\alpha+1)}\right)^{q}\|u\|^{q}_{E_0^{\alpha,p}}|\Lambda|^{\frac{p-q-1}{p-q}}\left(\frac{\Lambda^{\alpha}}{\Gamma(\alpha+1)}\right)^{\frac{p}{p-q}}\|v\|^{\frac{p}{p-q}}_{E_0^{\alpha,p}}\\
&\leqslant \|b\|_{L^{\infty}[0,\Lambda]}\frac{\Lambda^{q\alpha+\frac{p-q-1}{p-q}+\frac{p\alpha}{p-q}}}{\Gamma(\alpha+1)^{q\alpha+\frac{p}{p-q}}}\|u\|^{q}_{E_0^{\alpha,p}}\|v\|^{\frac{p}{p-q}}_{E_0^{\alpha,p}}.
\end{aligned}$$

Substituting in (4.5) the previous estimations and considering $M_1 = \frac{\Lambda^{q\alpha+\frac{p-q-1}{p-q}+\frac{p\alpha}{p-q}}}{\Gamma(\alpha+1)^{q\alpha+\frac{p}{p-q}}}$, we also know that $\|v\|_{E_0^{\alpha,p}} = 1$, and $1 < q < p-1$ and $2 < p < \infty$, then, we have

$$
\begin{aligned}
|\langle F(u), v\rangle| &\leqslant \|u\|^{p-1}_{E_0^{\alpha,p}} + |\lambda| S^p \|u\|^{p-1}_{E_0^{\alpha,p}} + \|b\|_{L^\infty([0,\Lambda])} M_1 \|u\|^{q}_{E_0^{\alpha,p}} \\
|\langle F(u), v\rangle| &\leqslant (1 + |\lambda| S^p)\, \|u\|^{p-1}_{E_0^{\alpha,p}} + \|b\|_{L^\infty([0,\Lambda])} M_1 \|u\|^{q}_{E_0^{\alpha,p}} \\
|\langle F(u), v\rangle| &\leqslant (1 + |\lambda| S^p) \left(\frac{\|b\|_{L^\infty([0,\Lambda])} C^{q+1}}{1 - |\lambda| S^p} \right)^{\frac{p-1}{p-(q+1)}} \\
&\qquad + \|b\|_{L^\infty([0,\Lambda])} M_1 \left(\frac{\|b\|_{L^\infty([0,\Lambda])} C^{q+1}}{1 - |\lambda| S^p} \right)^{\frac{q}{p-(q+1)}} = M, \\
|\langle F(u), v\rangle| &\leqslant M,
\end{aligned}
$$

S, C, M_1 are constants that we obtain using the Poincaré-Friederich inequality (2.8). Then,

$$
\begin{aligned}
\|\Phi(u)\|_{(E_0^{\alpha,p})^*} &= \sup_{\|v\|_{E_0^{\alpha,p}} \leqslant 1} |\langle \Phi(u), v\rangle| \\
&= \sup_{\|v\|_{E_0^{\alpha,p}} \leqslant 1} \left| \langle \phi(x), v\rangle + \langle \psi(x), v\rangle t + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \langle F(u), v\rangle ds \right| \\
&\leqslant |\langle \phi(x), v\rangle| + |\langle \psi(x), v\rangle t| + \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \langle F(u), v\rangle ds \right| \\
&\leqslant \|\phi(x)\|_{L^\infty([0,\Lambda])} \|v\|_{\alpha,p} + \|\psi(x)\|_{L^\infty([0,\Lambda])} \|v\|_{\alpha,p} T + |\langle F(u), v\rangle| \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} ds \right| \\
&\leqslant \|\phi(x)\|_{L^\infty([0,\Lambda])} + \|\psi(x)\|_{L^\infty([0,\Lambda])} T + \frac{M}{\Gamma(\beta)} \left| \int_0^t (t-s)^{\beta-1} ds \right| \\
&\leqslant \|\phi(x)\|_{L^\infty([0,\Lambda])} + \|\psi(x)\|_{L^\infty([0,\Lambda])} T + \frac{M}{\beta\Gamma(\beta)} t^{\beta} \\
&\leqslant \|\phi(x)\|_{L^\infty([0,\Lambda])} + \|\psi(x)\|_{L^\infty([0,\Lambda])} T + \frac{M}{\beta\Gamma(\beta)} T^{\beta}.
\end{aligned}
$$

Therefore, $\Phi(u)$ is bounded. Therefore, for each $v \in E_0^{\alpha,p}[0,\Lambda]$, $t_1 < t_2$; $t_1, t_2 \in [0,T]$, $T > 0$ and $t_2 - t_1 < \delta$, see the following:

$$
\begin{aligned}
\|\Phi u(t_2) - \Phi u(t_1)\| &= \sup_{\|v\|_{E_0^{\alpha,p}} \leqslant 1} |\langle \Phi u(t_2) - \Phi u(t_1), v\rangle| \\
&= \sup_{\|v\|_{E_0^{\alpha,p}} \leqslant 1} \Bigg| \langle \psi(x), v\rangle (t_2 - t_1) + \frac{1}{\Gamma(\beta)} \int_0^{t_2} (t_2 - s)^{\beta-1} \langle F(u), v\rangle ds \\
&\qquad - \frac{1}{\Gamma(\beta)} \int_0^{t_1} (t_1 - s)^{\beta-1} \langle F(u), v\rangle ds \Bigg|, \\
\|\Phi u(t_2) - \Phi u(t_1)\| &\leqslant \|\psi(x)\|_{L^\infty([0,\Lambda])} \|v\|_{E_0^{\alpha,p}} |t_2 - t_1| + \frac{1}{\Gamma(\beta)} |\langle F(u), v\rangle| \int_{t_1}^{t_2} |t_2 - s|^{\beta-1} ds \\
&\qquad + \frac{1}{\Gamma(\beta)} |\langle F(u), v\rangle| \int_0^{t_1} |(t_2 - s)^{\beta-1} - (t_1 - s)^{\beta-1}| ds \\
&= \|\psi(x)\|_{L^\infty([0,\Lambda])} \|v\|_{E_0^{\alpha,p}} |t_2 - t_1| + \frac{1}{\Gamma(\beta)} |\langle F(u), v\rangle| (t_2 - t_1)^{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\beta)}|\langle F(u),\nu\rangle|{t_2}^{\beta}-\frac{1}{\Gamma(\beta)}|\langle F(u),\nu\rangle|(t_2-t_1)^{\beta}-\frac{1}{\Gamma(\beta)}|\langle F(u),\nu\rangle|t_1^{\beta}\\
&\leqslant\|\psi(x)\|_{L^\infty([0,\Lambda])}|t_2-t_1|+\frac{M}{\beta\Gamma(\beta)}t_2^{\beta}-\frac{M}{\beta\Gamma(\beta)}t_1^{\beta}\\
&=\|\psi(x)\|_{L^\infty([0,\Lambda])}|t_2-t_1|+\frac{M}{\beta\Gamma(\beta)}(t_2^{\beta}-t_1^{\beta})
\end{aligned}
$$

In the following, we divide the proof into two cases. Additionally, for case 1, consider $f:(\delta;1)\to\mathbb{R}$, defined by $f(t)=t^{\beta}$.

Case 1: $\delta\leqslant t_1<t_2<T$, because $1<\beta\leqslant 2$, it follows that:

$$
\|\Phi u(t_2)-\Phi u(t_1)\|_{(E_0^{\alpha,p})^*}=\sup_{\|\nu\|_{E_0^{\alpha,p}}\leqslant 1}|\langle\Phi u(t_2)-\Phi u(t_1),\nu\rangle|\leqslant\|\psi(x)\|_{L^\infty([0,\Lambda])}|t_2-t_1|+\frac{M}{\beta\Gamma(\beta)}(t_2^{\beta}-t_1^{\beta})
$$

with $t_1<t<t_2$ and applying the mean value theorem,

$$
\begin{aligned}
t_2^{\beta}-t_1^{\beta}=\beta t^{\beta-1}(t_2-t_1)&=\|\psi(x)\|_{L^\infty([0,\Lambda])}|t_2-t_1|+\frac{M}{\beta\Gamma(\beta)}\beta t^{\beta-1}(t_2-t_1)\\
&\leqslant\|\psi(x)\|_{L^\infty([0,\Lambda])}|t_2-t_1|+\frac{M}{\Gamma(\beta)\delta^{1-\beta}}|t_2-t_1|\\
&=\|\psi(x)\|_{L^\infty([0,\Lambda]}\delta+\frac{M}{\Gamma(\beta)}\delta^{\beta}\\
&=\|\psi(x)\|_{L^\infty([0,\Lambda])}\delta^{\beta}+\frac{M}{\Gamma(\beta)}\delta^{\beta}\\
&=\left(\|\psi(x)\|_{L^\infty([0,\Lambda])}+\frac{M}{\Gamma(\beta)}\right)\delta^{\beta}\leqslant\varepsilon,
\end{aligned}
$$

if

$$
|t_2-t_1|<\delta=\left\{\left(\|\psi(x)\|_{L^\infty([0,\Lambda])}+\frac{M}{\Gamma(\beta)}\right)^{-1}\varepsilon\right\}^{1/\beta}.
$$

Case 2: $0\leqslant t_1<\delta,\ t_2<\beta^{\frac{1}{\beta}}\delta$.

$$
\begin{aligned}
\|\Phi u(t_2)-\Phi u(t_1)\|_{(E_0^{\alpha,p})^*}&=\sup_{\|\nu\|_{H_0^1}\leqslant 1}|\langle\Phi u(t_2)-\Phi u(t_1),\nu\rangle|\\
&\leqslant\|\psi(x)\|_{L^\infty([0,\Lambda])}|t_2-t_1|+\frac{M}{\beta\Gamma(\beta)}(t_2^{\beta}-t_1^{\beta})\\
&\leqslant\|\psi(x)\|_{L^\infty([0,\Lambda])}\delta+\frac{M}{\beta\Gamma(\beta)}(\beta^{\frac{1}{\beta}}\delta)^{\beta}\\
&\leqslant\|\psi(x)\|_{L^\infty([0,\Lambda])}+\frac{M}{\Gamma(\beta)}\delta^{\beta}\\
&=\left(\|\psi(x)\|_{L^\infty([0,\Lambda])}+\frac{M}{\Gamma(\beta)}\right)\delta^{\beta}\leqslant\varepsilon.
\end{aligned}
$$

Therefore, given $\epsilon>0$ and setting

$$
\delta=\left\{\left(\|\psi(x)\|_{L^\infty([0,\Lambda])}+\frac{M}{\Gamma(\beta)}\right)^{-1}\varepsilon\right\}^{1/\beta},
$$

for each $v \in E_0^{\alpha,p}([0,\Lambda])$, $t_1 < t_2$; $t_1, t_2 \in [0,T]$, $T > 0$ and $t_2 - t_1 < \delta$, we have

$$\|\Phi u(t_2) - \Phi u(t_1)\| = \sup_{\|v\|_{E_0^{\alpha,p}} \leqslant 1} |\langle \Phi u(t_2) - \Phi u(t_1), v\rangle| \leqslant \varepsilon.$$

Therefore, $\Phi(u)$ is equicontinuous. Using the Arzela-Ascoli Theorem, we have that there exists a subsequence $\{\Phi(u_{k_j})\}_{j=1}^{\infty} \subseteq \{\Phi(u_k)\}_{k=1}^{\infty}$ such that

$$\Phi(u_{k_j}) \to \Phi(u)$$

uniformly on $E_0^{\alpha,p}[0,\Lambda]$. Therefore, $\Phi(u) : E_0^{\alpha,p}[0,\Lambda] \to E_0^{\alpha,p}[0,\Lambda]$ is completely continuous. □

Then, from Definition 2.13, Lemma 4.4, Banach Fixed Point Theorem 2.12, Theorem 4.2, and Definition 4.3, it is proven that the problem of nonlinear parabolic with fractional derivatives P_1 has a unique weak solution $u \in C([0,T]; E_0^{\alpha,p}[0,\Lambda])$.

5. Conclusion

Problem P_1 has a unique weak solution in fractional Sobolev space $E_0^{\alpha,p}[0,\Lambda]$, with the hypothesis of $\lambda < \hat{\lambda} < \lambda_1$, where λ_1 is the first eigenvalue associated with problem P_0, region $\Omega_T = [0,\Lambda] \times [0,T]$, and Caputo fractional derivatives ${}^cD^{\beta}$ and D^{α} with order $1 < \beta < 2$ and $\frac{1}{p} < \alpha < 1$ for temporal and spatial variables, those were defined by Riemann-Liouville fractional derivative with conditions $u(0) = u(\Lambda) = 0$, where $1 < q < p-1$ with $2 < p < \infty$. Also, continuous functions were established b, ϕ, and ψ such that $b : [0,\Lambda] \to \mathbb{R}$, $b \in L^{\infty}[0,\Lambda]$, $\phi(x)$, $\psi(x) \in L^{\infty}[0,\Lambda]$ and $u \in E_0^{\alpha,p}[0,\Lambda]$. The same conditions were established for problem P_0.

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