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Covering Fuzzy Uniform Spaces

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Abstract

Topological, proximal and uniform structures on the fuzzy spaces are defined using different set of axioms and basic terms. There are various equivalent definitions in literature even for each of these structures. Since the late seventies and early eighties uniformity in fuzzy topology was studied by three authors. B. Hutton[5] , U. Höhle[3,4], and R. Lowen[8]. The approaches of U. Höhle and R. Lowen underpinned by power sets of the form $I^{X \times X}$ or $T^{X \times X}$ and B. Hutton approach is based on exponential power sets of the form $\left(T^X\right)^{T^X}$.

Recently, a different approach to fuzzy uniformities, in terms of T -covers was introduced which seems the most natural one.

In this paper we investigate some properties of T - valued uniformities and we will show that a functor between Hutton uniformities and T - valued uniformities.

Keywords: fuzzy spaces, covering fuzzy uniform spaces, Hutton uniform spaces.

1. Introduction

Note: In this paper T will always denote a frame.

Topological, proximal and uniform structures on the T -valued spaces are defined using different set of axioms and basic terms. There are various equivalent definitions in literature even for each of these structures. Fuzzy uniformity was studied by three authors : B. Hutton [5], R. Lowen[8], U. Höhle[3,4]. While all these authors, as a starting point, took certain counterpart of the filter approach to uniformities in fuzzy context ([14], as different from the approach based on uniform covers[13]in set theory). The

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ideas on which these authors based their works, the applied technique and the obtained results are essentially different.

In [9] the author present a different approach to fuzzy uniformities in terms of T - covers that is called T -valued uniformity. In this paper we investigate some properties of T - valued uniformities and we will show a functor between Hutton uniformities and T - valued uniformities.

2.Preliminaries

Definition 2.1 [6] : A frame L is a complete lattice satisfying the distribution law $x \wedge \vee(A) = \vee\{x \wedge a \mid a \in A\}$, the bottom resp top of L will be denoted by 0 resp 1. The pseudo complement of x $\in$ L by $\bar{x}$. An element x is said to be complemented if $\bar{x} \vee x = 1$.

Definitions 2.2 [6,12] : A cover of a frame L is a subset U $\subset$ L such that $\vee_{x \in U} x = 1$
the set of covers of L will be denoted by φ (L), φ (L) can be preordered, a cover U refines a cover V, $U \leq V$ if for each x$\in$U there is y$\in$V with x < y. This is a preordered set with meets and joins. Take for U $\wedge$ V the cover {x $\wedge$ y|x $\in$U, y$\in$V} and for U$\vee$V just the union U$\bigcup$V . For x$\in$L, the element st(x, U) = $\vee$ {y$\in$U |y$\wedge$x $\neq$ 0} is called the star of x in U . For U$\in$ φ (L), Put U $^{(2)}$ = {x $\vee$ y|x, y$\in$U, x$\wedge$y$\neq$0}, U * = {st(x, U)|x$\in$U }.

Definitions 2.3 [6]A family U$\subset$ φ (L) is said to be a uniformity on L if

(i) U is a filter of φ (L)

(ii) every V$\in$ U has a star-refinement i.e. for every U$\in$ U there is V$\in$ U with V * $\leq$U .

Let (L, U) and (M, V) be uniform frames. A frame homomorphism h : L$\to$M is a uniform homomorphism if, for every U$\in$U , h[U] = {h(x)|x$\in$U }$\in$V.We denote by UFrm the category of uniform frames and uniform homomorphisms.

T -valued Spaces 2.4 [1] : Let T be a complete lattice and let T $_1$ = T \ {1}. For any set X, T ^{X} is a frame.
A T -valued topological space is a pair (X, L) consisting of a set X and a family L of mappin g: X$\to$T .
(1) Containing the constant zero and constant one maps.
(2) Closed under finite meets and arbitrary joints.
If T is a frame then T -valued topologies, being subframes of the frame T ^{X}.
A T -valued continuous map (X, L)$\to$ (Y , M) is a map f : X$\to$Y such that the Correspondence v$\to$v.f maps M into L. The resulting category will be denoted by T -Top.

Definition 2.5 [5] : Hutton generalized the concept of uniformity to fuzzy case as follows. Let T be a complete lattice and let H (T , X) be the set of maps e : T ^{X}$\to$T ^{X} which satisfies
(1) e(ϕ) = ϕ
(2) e(μ) $\geq$ μ
(3) e($\vee_{i\in\Gamma} \mu_i$) =$\vee_{i\in\Gamma} e(\mu_i)$ for $\{\mu_i\}_{i\in\Gamma} \subset$ T ^{X}.
For every e, h $\in$ H (T , X), we have the following properties.
(1) For all $\mu \in$T ^{X}, e^{-1}(μ) = $\wedge$ {p $\in$ T ^{X} |e(p') $\leq$ μ '}. Then e^{-1} $\in$H (T , X).
(2) For all $\mu \in$T ^{X}, e$\wedge$h(μ) =$\wedge$ {e(μ_1)$\vee$ h(μ_2)| $\mu_1 \vee \mu_2 = \mu$} then e$\wedge$h$\in$H(T , X).
(3) For all $\mu \in$T ^{X}, eoh(μ) = e(h(μ)). Then eoh$\in$H (T , X).
(4) α is e-small if $\alpha \leq$e(β) whenever $\alpha \wedge \beta \neq 0$.
A Hutton uniformity on X is a subset D of H (T , X) such that it satisfies the following axioms:

H U1) $D \neq \emptyset$
H U2) $e \in D$ and $e \leq h$ implies $h \in D$
H U3) $e \in D$ and $h \in D$ implies $e \wedge h \in D$
H U4) $e \in D$ implies there exist $h \in D$ such that $hoh \leq e$
H U5) $e \in D$ implies $e\text{-}1 \in D$.

Definition 2.6 : Let (X, D), (X' , D') be two Hutton uniform spaces .A mapping $f : X \to X'$ is said to be uniformly homomorphism such that for every $e \in D$, there exists $g \in D'$with $g.\vec{f} \leq \vec{f}.e$ The resulting category will be denoted by Hutt-Unif.

3. T - valued Uniform Spaces

Definition 3.1 : Let $\underline{1}$ denote the top element of T^X, that is $\underline{1} : X \to 1$. We say that

$$U = (U_i : X \to T)_{i \in I}$$

is a T -cover of X if $\vee_I U_i = \underline{1}$. The set T - Cov(X) of all T -covers of X is preordered by $V \leq U$ (i.e. For each $V \in V$ there exists $U \in U$ such that $V \leq U$).Further let $U \wedge V := \{U \wedge V \mid U \in U, V \in V\}$.

Proposition 3.2 : Let T be a frame. Then, for every T -covers U and V, $U \wedge V$ is again a T -cover and it is the meet of U and V in $(T\text{ - Cov}(X), \leq)$.

Proof: It suffices to see that

$$(\bigvee_{U \in U, V \in V} (U \wedge V))(x) = \bigvee_{U \in U, V \in V} (U(x) \wedge V(x)) = (\bigvee_{U \in U} U(x)) \wedge (\bigvee_{V \in V} V(x)) = 1.$$

The second assertion follows immediately.

For each $U \in U$, let

$$st(U, U) := \vee \{V \in U \mid V \wedge U \neq 0\}$$

and

$$U^* := \{st(U, U) \mid U \in U\}.$$

Clearly $U \leq U^*$, since $U \leq st(U, U)$ for every $U \in U$. Therefore U^* is a T -cover whenever U is a T -cover.

We say that a pair (X, μ) consisting of a set X and a non-empty family of T –covers of X is a covering T - uniform space[9] whenever the following conditions are satisfied:

(CU1) $U \leq V, U \in \mu \Rightarrow V \in \mu$.

(CU2) For every $U, V \in \mu$, $U \wedge V \in \mu$.

(CU3) For each $U \in \mu$ there exists $V \in \mu$ such that $V^* \leq U$.

A non-void family μ is said to be a weak covering T -uniform space on L if there hold CU_1, CU_2 and $(CU_{3'})$.

$(CU_{3'})$:for each $U \in \mu$ there is a $V \in \mu$ such that

$$V^{(2)} = \{U \vee U' \mid U, U' \in v, U \wedge U' \neq 0\} \leq U$$

A non-void family μ of T -covers of X is said to be a T -valued uniformly basis resp. (a weak T -valued uniformity basis) if it satisfies (CU3) resp (CU3').

A map $f : (X, \mu) \to (Y, V)$ is uniform homomorphism if, for every $V := (V_i)_I \in V$, $f^{-1}[V] := (V_i. f)_I \in \mu$. The resulting category will be denoted by T -Unif. Of course, for T = 2, this is just the category of (covering) uniform spaces of Tukey [13].

Now let $(X, \mu) \in$ T - Unif and define $L_\mu := \{U \in T^X \mid U(x) \neq 0 \Rightarrow \exists U \in \mu : st(\hat{x}, U) \leq U\}$,where $\hat{x} : X \to T$ is defined by $\hat{x}$ (y) = 1 if y = x and $\hat{x}$ (y) = 0 otherwise. Note that, when T = 2, L_μ is just the crisp topology τ_μ induced by the (classical) uniformity μ on X.

Proposition 3.3 : L_μ is a subframe of T^X.

In conclusion, (X, L_μ) is T -Top.

Definition 3.4 : For $\mu \subset T$ - cov(X) put

$$\tilde{\mu} = \{ U \mid U_1, \ldots, U_k \in \mu,\ U_1 \wedge U_2 \wedge \ldots \wedge U_k < U\}$$

Lemma 3.5 : We have $(U_1 \wedge \ldots \wedge U_n)^{(2)} < U_1^{(2)} \wedge \ldots \wedge U_n^{(2)}, (U_1 \wedge \ldots \wedge U_n)^* < U_1^* \wedge \ldots \wedge U_n^*$

Proof : Obviously, it suffices to prove the statement for n = 2.

(I) $(U_1 \wedge U_2)^{(2)} < U_1^{(2)} \wedge U_2^{(2)}$

Let

$U_1, V_1 \in U_1$, $U_2, V_2 \in U_2$ $(U_1 \wedge U_2) \vee (V_1 \wedge V_2) \in (U_1 \wedge U_2)^{(2)}$ then $(U_1 \wedge U_2) \vee (V_1 \wedge V_2) \neq 0$.

$$U_1 \wedge V_1 \neq 0 \neq U_2 \wedge V_2 \text{ and hence } U_1 \vee V_1 \in U_1^{(2)},\ U_2 \vee V_2 \in U_2^{(2)}. \text{Then}$$

$$(U_1 \wedge U_2) \vee (V_1 \wedge V_2) \leq (U_1 \vee V_1) \wedge (U_2 \vee V_2).$$

Hence $(U_1 \wedge U_2)^{(2)} < U_1^{(2)} \wedge U_2^{(2)}$

(II) $(U_1 \wedge U_2)^* < U_1^* \wedge U_2^*$

$$(U_1 \wedge U_2)^* = \{st(U_1 \wedge U_2, U_1 \wedge U_2)\ U_1 \vee U_2 \in (U_1 \wedge U_2)\}$$

$$st(U_1 \wedge U_2, U_1 \wedge U_2) \leq st(U_1 \wedge U_2, U_1) \wedge st(U_1 \wedge U_2, U_2) \leq st(U_1, U_1) \wedge st(U_2, U_2)$$

Theorem 3.6 : If μ is a T -valued uniform basis, $\tilde{\mu}$ is a T -valued uniformity. If μ is a weak T - valued uniform basis, $\tilde{\mu}$ is a weak T -valued uniformity.

Proof: the condition (CU1, CU2) are obviously satisfied.

(CU_3, CU'_3) : Let $U_1 \wedge U_2 \ldots \wedge U_k < U$, $U_i \in \mu$. Choose $V_i \in \mu$ such that $V_i^* < U_i$ resp. $V_i^2 < U_i$. Put $V = V_1 \wedge V_2 \ldots \wedge V_k$. By 3.5 we have $V^* < U$ resp $V^{(2)} < U$.

Definition 3.7 : For a complemented $\alpha \in T^X$ and $U \in T$ - cov(X) put $U \text{o} \alpha = \{U_i \wedge \alpha \mid U_i \in U\} \vee \{U_i \wedge \overline{\alpha} \mid U_i \in U\}$. For $\mu \subset T$ - cov(X) define $\mu^0 = \{ U_{o\alpha|}\ U \in \mu, \alpha \text{ complemented}\}$.

Proposition 3.8 : If μ is a T -valued uniform basis (resp weak T -valued uniform basis), μ^0 is a T - valued uniform basis (resp weak T -valued uniform basis).

Proof : It is trivial $(U\text{o}\alpha)^* < U^*\text{o}\alpha$ resp $(U\text{o}\alpha)^{(2)} < U^{(2)}\text{o}\alpha$ for $U \in \mu$, let $U\text{o}\alpha \in \mu^0$ then $U \in \mu$ choose $V \in \mu$ such that $V^* < U$ resp $V^{(2)} < U$. We consider $U\text{o}\alpha \in \mu^0$ such that $(V\text{o}\alpha)^* < V^*\text{o}\alpha < U\text{o}\alpha$ resp $(V\text{o}\alpha)^{(2)} < V^{(2)}\text{o}\alpha < U\text{o}\alpha$.

The Functor

θ: **Hutt-Unif** $\rightarrow$ T - **Unif** :

Finally, we want to study that the functors θ between the categories Hutton uniform spaces and T - valued uniform spaces. For each entourage e of T^X let U_e be the cover of all e-small elements of T^X i.e.

$$U_e = \{U \in T^X \mid U \leq e(V), U \wedge V \neq \underline{0}\}.$$

Proposition 3.9 : Let D be Hutton uniformity on X, then $\mu_D = \{ U_e \mid e \in D\}$ is a T -valued uniformity on X .

Proof :

CU_1, CU_2) Consider $U_e, U_f \in \mu_D$ and let $g \in D$ such that $g \leq e \wedge f$ then it is obvious that $U_g \leq U_e \wedge U_f$.

CU3) Let $U_e \in \mu_D$ and take $f \in D$ such that $f^3 \leq e$ we claim $U^*_f \leq U_e$ consider $st(U, U_f) \in U^*_f$. It suffices to show that $st(U, U_f)$ is e-small. So, consider $V \in T^X$ such that $V \wedge st(U, U_f) \neq 0$ then there is $W \in U_f$ with $W \wedge U \neq 0$ and $V \wedge W \neq 0$

then f -smallness of U, W then $U \leq f(W) \leq f^2(V)$. Therefore for every $W' \in U_f$ such that $W' \wedge U \neq 0$ we have $W' \leq f(U) \leq f^3(V) \leq e(V)$ .Then $st(U, U_f)$ is e-small.

In the sequel, if D is a Hutton uniformity, θ(D) denotes the uniformity generated by U_D.

Proposition 3.10 : Let $\boldsymbol{f}$:(X, D) $\to$ (X' , D') be a Hutton uniform homomorphism then $\boldsymbol{f}$: (X, θ (D)) $\to$ (X' , θ (D')) is a covering T -uniform homomorphism.

Proof : Let $U \in \theta(D)$ and $e \in D$ such that $U_{e^3} \leq U$ there exists $g \in D'$ with. $g.\vec{f} \leq \vec{f}.e.$

We show that $U_g \leq \vec{f}[U_{e^3}]$ Let U be a non-zero g-small element of T^X.Since $\vec{f}[U_{e^3}]$ is a cover of $T^{X'}$, there exists $V \in U_e$ satisfying $U \wedge \vec{f}(V) \neq 0$ .Consequently $U \leq g.\vec{f}(V) \leq \vec{f}.e(V)$. But as can be easily proved, the fact V is e-small implies that $e(V)$ is e^3-small. In conclusion

$$U_g \leq \vec{f}[U_{e^3}] \leq \vec{f}[U] \text{and } \vec{f}[U] \in \theta(D')$$

4. Conclusion

From a categorical point of view there is a concrete isomorphism between the category of Hutton uniform spaces and the category of covering fuzzy uniform spaces (which are concrete categories over the category of sets). Informally, this means that the description of these "structured sets", although distinct, are essentially the same and we may substitute one structure for the other with no problem.

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