J. Math. Computer Sci., 30 (2023), 272–280

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

q-Janowski type close-to-convex functions associated with a convolution operator

Syed Ghoos Ali Shah[a,*], Sa'ud Al-Sa'di[b,*], Saqib Hussain[a], Shahbaz Khan[a], Maslina Darus[c]

[a]Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus 22060, Pakistan.

[b]Department of Mathematics, Faculty of Science, The Hashemite University, P.O. Box 330127, Zarqa 13133, Jordan.

[c]Department of Mathematical Sciences, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia.

Abstract

In this paper, we will discuss some generalized sub-classes of analytic function related with close-to-convex functions in conic domains by using q -calculus. We investigate some important properties such as necessary and sufficient conditions, coefficient estimates, convolution results, linear combination, weighted mean, arithmetic mean, radii of star likeness and growth and distortion for these classes. It is important to mention that our results are a generalization of several existing results.

Keywords: Analytic functions, subordination, Noor integral operator, q-conic domain, q-Janowski functions, close-to-convex functions.

2020 MSC: 30C45, 30C50.

1. Introduction

A function $\hat{g}$ is analytic at a point ξ_0 if $\hat{g}^{'}(\xi)$ exists at ξ_0 as well as in some neighborhood of ξ_0. A function $\hat{g}(\xi)$ is analytic in a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$. In most of the cases it is much harder to use an arbitrary domain. Riemann mapping theorem allows us to replace any arbitrary domain with open unit disk which is given by

$$\mathbb{U} = \{\xi \in \mathbb{C} : |\xi| < 1\}.$$

An analytic function $\hat{g}$ is univalent in $\mathbb{U}$, if $\hat{g}(\xi_1) = \hat{g}(\xi_2)$, then $\xi_1 = \xi_2$. A function $\hat{g}(\xi)$ is said to be the class $\mathfrak{A}$ if it has a Taylor series of the form

$$\hat{g}(\xi) = \xi + \sum_{t=2}^{\infty} a_t \xi^t, \quad \xi \in \mathbb{U}. \tag{1.1}$$

*Corresponding author

Email addresses: alishahsyedghoos@gmail.com (Syed Ghoos Ali Shah), saud@hu.edu.jo (Sa'ud Al-Sa'di), saqib_math@yahoo.com (Saqib Hussain), shahbazkhan8040@gmail.com (Shahbaz Khan), maslina@ukm.edu.my (Maslina Darus)

doi: 10.22436/jmcs.030.03.06

Received: 2022-04-24 *Revised:* 2022-08-31 *Accepted:* 2022-12-02

A collection of functions of the form (1.1), which are analytic and univalent in $\mathbb{U}$ are placed in the class $\mathfrak{S}$. An analytic function $p(\xi)$ having positive real part, i.e., $\operatorname{Re}\{p(\xi)\} > 0$ and $p(0) = 1$ is placed in class $\mathfrak{P}$. Or equivalently

$$p \in \mathfrak{P} : p(\xi) = 1 + \sum_{t=1}^{\infty} a_t \xi^t \iff \operatorname{Re}\{p(\xi)\} > 0, \quad \xi \in \mathbb{U}. \tag{1.2}$$

The class of normalized convex functions is given by

$$C = \left\{ \hat{g} : \hat{g} \in \mathfrak{S}; \operatorname{Re}\left(\frac{\left(\xi \hat{g}'(\xi)\right)'}{\hat{g}(\xi)} \right) > 0, \quad \xi \in \mathbb{U} \right\}.$$

Similarly, the class of normalized starlike functions concerning origin is defined as:

$$\mathcal{S}^* = \left\{ \hat{g} : \hat{g} \in \mathfrak{S}; \operatorname{Re}\left(\frac{\xi \hat{g}'(\xi)}{\hat{g}'(\xi)} \right) > 0, \quad \xi \in \mathbb{U} \right\},$$

for details, see [1–4].

In 1952, Kaplan [9] introduced the class KC of close-to-convex functions. A function is of the form (1.2) is in KC if and only if there exists $\hat{h}(\xi) \in \mathcal{S}^*$ such that $\operatorname{Re}\left(\frac{\xi \hat{g}'(\xi)}{\hat{h}(\xi)}\right) > 0$. Let $\hat{g}(\xi)$ is of the form (1.1) and $\hat{h}(\xi)$ is of the form

$$\hat{h}(\xi) = \xi + \sum_{t=2}^{\infty} b_t \xi^t, \ \xi \in \mathbb{U}.$$

Then the Hadamard product (convolution) of $\hat{g}$ and $\hat{h}$ is defined as:

$$\left(\hat{g} * \hat{h}\right)(\xi) = \xi + \sum_{t=2}^{\infty} a_t b_t \xi^t = \left(\hat{h} * \hat{g}\right)(\xi).$$

The q-derivative of a function $\hat{g}$ belonging to $\mathfrak{A}$ defined as:

$$D_q \hat{g}(\xi) = \frac{\hat{g}(q\xi) - \hat{g}(\xi)}{\xi(q-1)}, \qquad \text{for } \xi \neq 0, \tag{1.3}$$

where $q \in (0,1)$ and $\xi \in \mathbb{U}$, see [5] for more details. For $\xi = 0$, (1.3) can be written as $\hat{g}'(0)$ provided that the derivative exist. By using (1.1) and (1.3), the Maclaurin's series representation of $D_q \hat{g}$ is given by

$$D_q \hat{g}(\xi) = 1 + \sum_{t=0}^{\infty} [t, q]\, a_t \xi^{t-1}, \ t \in \mathbb{N}.$$

It can be noted from (1.3) that

$$\lim_{q \to 1^-} \left(D_q \hat{g}(\xi)\right) = \lim_{q \to 1^-} \left(\frac{\hat{g}(q\xi) - \hat{g}(\xi)}{\xi(q-1)} \right) = \hat{g}'(\xi), \quad \text{where } [t, q] = \frac{1 - q^t}{1 - q}.$$

For any non negative integer t, the q-number shift factorial is given by

$$[t, q]! = \begin{cases} 1, & t = 0, \\ [1, q]\,[2, q] \cdots [t, q], & t \in \mathbb{N}, \end{cases}$$

see [4]. For $y > 0$, the q-genralized Pochhammer symbol is defined as:

$$[y, q]_t = \begin{cases} 1, & t = 0, \\ [y, q]\,[y+1, q] \cdots [y+t-1, q], & t \in \mathbb{N}. \end{cases}$$

For $\mu > -1$, we defined a function $\mathfrak{F}^{-1}_{q,1+\mu}(\xi)$ such that

$$\mathfrak{F}_{q,1+\mu}(\xi) * \mathfrak{F}^{-1}_{q,1+\mu}(\xi) = \xi D_q \hat{g}(\xi),$$

where

$$\mathfrak{F}_{q,1+\mu}(\xi) = \xi + \sum_{t=2}^{\infty} \left(\frac{[1+\mu, q]_{t-1}}{[t-1, q]!} \xi^t \right), \quad \text{for } \xi \in \mathbb{U}.$$

The study of operators plays an important role in the geometric function theory. Many differential and integral operators can be written in terms of convolution of certain analytic functions. In [4], q-analogue of Noor integral operator $\mathfrak{I}^{\mu}_q : \mathfrak{A} \to \mathfrak{A}$ is define as:

$$\mathfrak{I}^{\mu}_q \hat{g}(\xi) = \hat{g}(\xi) * \mathfrak{F}^{-1}_{q,1+\mu}(\xi) = \xi + \sum_{t=2}^{\infty} \psi_{t-1} a_t \xi^t, \tag{1.4}$$

where

$$\psi_{t-1} = \frac{[t, q]!}{[1+\mu, q]_{t-1}}.$$

From (1.4) we can easily obtain the following identity

$$[1+\mu, q]\, \mathfrak{I}^{\mu}_q \hat{g}(\xi) = [\mu, q]\, \mathfrak{I}^{\mu+1}_q \hat{g}(\xi) + q^{\mu} \xi D_q \left(\mathfrak{I}^{\mu+1}_q \hat{g}(\xi) \right).$$

It can be seen that $\mathfrak{I}^0_q \hat{g}(\xi) = \xi D_q \hat{g}(\xi)$, $\mathfrak{I}^1_q \hat{g}(\xi) = \hat{g}(\xi)$, and

$$\lim_{q \to 1^-} \left(\mathfrak{I}^{\mu}_q \hat{g}(\xi) \right) = \xi + \sum_{t=2}^{\infty} \frac{t!}{(1+\mu)_{t-1}} a_t \xi^t. \tag{1.5}$$

From (1.5), we can observe that by applying limit $q \to 1$, the operator defined in (1.4) reduces to well known Noor integral operator, see [13, 15].

In [7, 8], Kanas and Waniowska introduced the concept of a conic domain Ξ_l for $l \geqslant 0$ as:

$$\Xi_l = \left\{ U + iV : U > l\sqrt{V^2 + (U-1)^2} \right\}.$$

This domain merely represents the right half plane for $l = 0$, a hyperbola for $0 < l < 1$, parabola for $l = 1$ and ellipse for $l > 1$. The extremal functions ϖ_l for this conic region Ξ_l is given by

$$\varpi_l(\xi) = \begin{cases} \frac{1+\xi}{1-\xi}, & l = 0, \\ 1 + \left\{ \frac{2}{\pi^2} \left(\log \frac{\sqrt{\xi}+1}{1-\sqrt{\xi}} \right)^2 \right\}, & l = 1, \\ 1 + \frac{2}{1-l^2} \sinh^2 \left[\left(\frac{2}{\pi} \arccos l \right) \left(\arctan h\sqrt{\xi} \right) \right], & 0 < l < 1, \\ 1 + \frac{1}{l^2-1} \sin \left[\frac{\pi}{2R(n)} \int_0^{\frac{U(\xi)}{\sqrt{n}}} \left(\frac{1}{\sqrt{1-n^2y^2}\sqrt{1-x^2}} \right) dx \right] + \frac{1}{l^2-1}, & l > 1, \end{cases}$$

where $U(\xi) = \frac{\xi - \sqrt{n}}{1-\sqrt{n}\xi}$, for all $\xi \in \mathbb{U}$, $0 < l < 1$ and $l = \cosh\left[\frac{\pi R'(n)}{4R(n)}\right]$, where $R(n)$ is Legendre's complete elliptic integral of the first kind and $R'(n)$ is complementary integral of $R(n)$, for more details, see [7]. If we take $\varpi_l(\xi) = 1 + \delta(l)\xi + \delta_1(l)\xi^2 + \cdots$, then

$$\delta(l) = \begin{cases} \frac{8(\arccos l)^2}{\pi^2(1-l^2)}, & 0 \leqslant l < 1, \\ \frac{8}{\pi^2}, & l = 1, \\ \frac{\pi^2}{4\sqrt{n}(l^2-1)(1+n)R^2(n)}, & l > 1. \end{cases} \tag{1.6}$$

Let $\delta_1(l) = \delta_2(l)\delta(l)$, where

$$\delta_2(l) = \begin{cases} \frac{2+\left(\frac{2}{\pi}\arccos l\right)^2}{3}, & 0 \leqslant l < 1, \\ \frac{2}{3}, & l = 1, \\ \frac{4R^2(n)\left(1+n^2+6n\right)-\pi^2}{24(1+n)\sqrt{n}R^2(n)}, & l > 1. \end{cases}$$

Definition 1.1 ([6]). Let p be a analytic function with $p(0) = 1$. Then $p \in \mathfrak{P}(\lambda, M)$ if and only if

$$p(\xi) \prec \frac{\lambda\xi + 1}{M\xi + 1}, \quad \text{where } -1 \leqslant M < \lambda \leqslant 1.$$

In [6] it was shown that $p \in \mathfrak{P}(\lambda, M)$ if and only if there exists a function $p \in \mathfrak{P}$ such that

$$\frac{(1+\lambda)p(\xi) - (\lambda - 1)}{(1+M)p(\xi) - (M-1)} \prec \frac{\lambda\xi + 1}{M\xi + 1}.$$

Definition 1.2 ([11]). Let a function $\hat{g} \in \mathfrak{A}$ considered in the class $k - ST_q(N, O)$ if and only if

$$\mathrm{Re}\left[\frac{(OL_1 - L_2)\left(\frac{\xi D_q(\hat{g}(\xi))}{\hat{g}(\xi)}\right) - (NL_1 - L_2)}{(OL_1 + L_2)\left(\frac{\xi D_q(\hat{g}(\xi))}{\hat{g}(\xi)}\right) - (NL_1 + L_2)}\right] > k\left|\frac{(OL_1 - L_2)\left(\frac{\xi D_q(\hat{g}(\xi))}{\hat{g}(\xi)}\right) - (NL_1 - L_2)}{(OL_1 + L_2)\left(\frac{\xi D_q(\hat{g}(\xi))}{\hat{g}(\xi)}\right) - (NL_1 + L_2)} - 1\right|, \tag{1.7}$$

where $k \geqslant 0$, $-1 \leqslant O < N \leqslant 1$, $L_1 = q + 1$ and $L_2 = 3 - q$. Or equivalently $\frac{\xi D_q(\hat{g}(\xi))}{\hat{g}(\xi)} \in k - P_q(N, O)$. One can observe that, for $q \to 1$, the class $k - ST_q(N, O)$ reduces to well known class defined in [14].

2. Set of lemmas

Lemma 2.1 ([18]). *Suppose* $d(\xi) = 1 + \sum_{t=1}^{\infty} c_t\xi^t \prec 1 + \sum_{t=1}^{\infty} C_t\xi^t = \mathbb{H}(\xi)$. *If* $\mathbb{H}(\mathbb{U})$ *is convex and* $\mathbb{H}(\xi) \in \mathfrak{A}$, *then*

$$|C_1| \geqslant |c_t|, \quad \textit{for } 1 \leqslant t.$$

Lemma 2.2 ([11]). *Suppose* $1 + \sum_{t=1}^{\infty} c_t\xi^t = d(\xi) \in k - ST_q(N, O)$, *then*

$$\frac{L_1(\lambda - M)}{4}\delta(l) = |\delta(l, \lambda, M)| \geqslant |c_t|,$$

where $\delta(l)$ *is given by* (1.6).

Lemma 2.3 ([11]). *If* $d(\xi) = \xi + \sum_{t=1}^{\infty} b_t\xi^t \in k - ST_q(N, O)$ *for* $\xi \in \mathbb{U}$ *and* $k \geqslant 0$, *then*

$$|b_t| \leqslant \prod_{p=0}^{t-2}\left[\frac{|(N-O)\delta(l)L_1 - 4O[p,q]|}{4[p+1,q]q}\right],$$

where $\delta(l)$ *is given by* (1.6).

Lemma 2.4. *If* $d \in \mathcal{S}^*$, $\mathcal{G} \in \mathfrak{G}$ *and* $\hat{g} \in C$, *then*

$$\frac{\hat{g}(\xi) * d(\xi)\,\mathcal{G}(\xi)}{\hat{g}(\xi) * d(\xi)} \in c\bar{o}\left(\mathcal{G}(\mathcal{U})\right), \quad \text{for all } \xi \in \mathbb{U},$$

where $c\bar{o}\left(\mathcal{G}(\mathbb{U})\right)$ *is the closed convex hull* $\mathcal{G}(\mathbb{U})$.

Lemma 2.5. *[[11]] A function* $\hat{g} \in \mathfrak{A}$ *will be in the class* $k - ST_q(N, O)$, *if*

$$\sum_{t=2}^{\infty} \{2(k+1)L_2 q\,[t-1,q] + |(OL_1 + L_2)\,[t,q] - (NL_1 + L_2)|\}\,|a_t| < L_1\,|O - N|\,.$$

Motivated by the work of Mahmood et al. [11], Noor and Malik [14] and Arif et al. [4], we define new subclasses of Janowski type q-starlike functions associated with q-conic domain as following.

Definition 2.6. A function $\hat{g}(\xi) \in \mathfrak{A}$ is apparently in the function class $k - UK_q(\mu, \lambda, M, N, O)$ if and only if there exists $\hat{h}(\xi) \in k - ST_q(\mu, N, O)$ such that

$$\mathrm{Re}\left[\frac{(ML_1 - L_2)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (\lambda L_1 - L_2)}{(ML_1 + L_2)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (\lambda L_1 + L_2)}\right] > k\left|\frac{(ML_1 - L_2)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (\lambda L_1 - L_2)}{(ML_1 + L_2)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (\lambda L_1 + L_2)} - 1\right|,$$

where $k \geqslant 0$, $-1 \leqslant O < N \leqslant 1$, $-1 \leqslant M < \lambda \leqslant 1$, $\mu > -1$, $L_1 = q + 1$ and $L_2 = 3 - q$. Or equivalently, $\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)} \in k - P_q(\lambda, M)$.

It is noted that for $\mu = 1$, the function class $k - UK_q(\mu, \lambda, M, N, O)$ reduces to well known class $k - UK_q(\lambda, M, N, O)$ introduced by Naeem et al. [12], for $\mu = 1$ along with $q \to 1$, the class $k - UK_q(\mu, \lambda, M, N, O)$ bring to well-known class interpreted, see details in [10], $0 - UK_{q \to 1}(1, \lambda, M, N, O) = K(\lambda, M, N, O)$ studied by Srivastava et al. [21], $k - UK_{q\to 1}(1, 1, -1, 1, -1) = k - UK$ is the class of k-uniformly close-to-convex investigated by Acu et al. [1] and $0 - UK_{q\to 1}(1, 1, -1, 1, -1) = K$ the class of close-to-convex, see [9, 17, 19, 20, 22–24] for more details.

All around the article, we will presume that $\mu > -1$, $k \geqslant 0$, $-1 \leqslant O < N \leqslant 1$, $L_1 = 1 + q$ and $L_2 = 3 - q$, if not mentioned.

3. Main results

3.1. *Necessary and sufficient conditions*

Here we have listed the necessary and sufficient conditions.

Theorem 3.1. *A function* $\hat{g}(\xi) \in \mathfrak{A}$ *and of the type* (1.1) *is considered in the class* $k - UK_q(\mu, \lambda, M, N, O)$, *if it satisfies the following criterion*

$$\sum_{t=2}^{\infty} \left[\{2L_2(1+k)\,|b_t - [t,q]\,a_t| + |(ML_1 + L_2)\,[t,q]\,a_t - (\lambda L_1 + L_2)|\}\,\psi_{t-1}\right] \leqslant L_1\,|M - \lambda|\,. \tag{3.1}$$

Proof. Assuming that relation (3.1) holds, then it is enough to prove that

$$k\left|\frac{(L_1 M - L_2)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (L_1\lambda - L_2)}{(L_2 + ML_1)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (L_2 + \lambda L_1)} - 1\right| - \mathrm{Re}\left[\frac{(L_1 M - L_2)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (L_1\lambda - L_2)}{(L_2 + L_1 M)\left(\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right) - (L_2 + \lambda L_1)} - 1\right] < 1,$$

we have,

$$k\left|\frac{(ML_1-L_2)\left(\frac{\xi D_q(\mathfrak{I}_q^{\mu}\hat{g}(\xi))}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right)-(\lambda L_1-L_2)}{(ML_1+L_2)\left(\frac{\xi D_q(\mathfrak{I}_q^{\mu}\hat{g}(\xi))}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right)-(\lambda L_1+L_2)}-1\right|-\mathrm{Re}\left[\frac{(ML_1-L_2)\left(\frac{\xi D_q(\mathfrak{I}_q^{\mu}\hat{g}(\xi))}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right)-(\lambda L_1-L_2)}{(ML_1+L_2)\left(\frac{\xi D_q(\mathfrak{I}_q^{\mu}\hat{g}(\xi))}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right)-(\lambda L_1+L_2)}-1\right]$$

$$\leqslant (k+1)\left|\frac{(ML_1-L_2)\left(\frac{\xi D_q(\mathfrak{I}_q^{\mu}\hat{g}(\xi))}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right)-(\lambda L_1-L_2)}{(ML_1+L_2)\left(\frac{\xi D_q(\mathfrak{I}_q^{\mu}\hat{g}(\xi))}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)}\right)-(\lambda L_1+L_2)}-1\right|$$

$$=(k+1)\,2L_2\left|\frac{\sum_{t=2}^{\infty}\{b_t-[t,q]\,a_t\}\psi_{t-1}\xi^t}{L_1\,(M-\lambda)\,\xi+\sum_{t=2}^{\infty}\{(ML_1+L_2)\,[t,q]\,a_t-(\lambda L_1+L_2)\,b_t\}\psi_{t-1}\xi^t}\right|$$

$$\leqslant\frac{(k+1)\,2L_2\sum_{t=2}^{\infty}\{|(b_t-[t,q]\,a_t)\psi_{t-1}a_t|\}}{L_1\,|M-\lambda|-\sum_{t=2}^{\infty}|\{(ML_1+L_2)\,[t,q]\,a_t-(\lambda L_1+L_2)\,b_t\}\psi_{t-1}|}.$$

The right-hand side of the above inequality is bounded above by 1 if

$$2\,(1+k)\,L_2\sum_{t=2}^{\infty}\{|(b_t-[t,q]\,a_t)\psi_{t-1}a_t|\}<L_1\,|M-\lambda|-\sum_{t=2}^{\infty}|\{(ML_1+L_2)\,[t,q]\,a_t-(\lambda L_1+L_2)\,b_t\}\psi_{t-1}|,$$

which reduces to

$$\sum_{t=2}^{\infty}[\{2L_2(k+1)\,|b_t-[t,q]\,a_t|+|(L_2+L_1M)\,[t,q]\,a_t-(L_2+\lambda L_1)\,b_t|\}\psi_{t-1}]\leqslant L_1\,|M-\lambda|.$$

□

For $\mu=1$, the above Theorem reduces to the result proved by Naeem et al. in [12].

Corollary 3.2. *Let* $\hat{g}\in\mathfrak{A}$. *Then* $\hat{g}\in k-UK_q(\lambda,M,N,O)$, *if*

$$\sum_{t=2}^{\infty}\{2L_2(k+1)\,|b_t-[t,q]\,a_t|+|(ML_1+L_2)\,[t,q]\,a_t-(\lambda L_1+L_2)\,b_t|\}\leqslant L_1\,|M-\lambda|.$$

For $\mu=1$ and $q\to1$, above theorem reduces to following corollary proved by Mahmood et al., for details see [10].

Corollary 3.3. *Let* $\hat{g}\in\mathfrak{A}$. *Then* $\hat{g}\in k-UK(\lambda,M,N,O)$, *if the following holds true*

$$\sum_{t=2}^{\infty}\{2(k+1)\,|b_t-ta_t|+|(M+1)\,ta_t-(\lambda+1)\,b_t|\}\leqslant|M-\lambda|.$$

3.2. Coefficient bound for the class $k-UK_q(\mu,\lambda,M,N,O)$

Theorem 3.4. *If* $\hat{g}\in k-UK_q(\mu,\lambda,M,N,O)$ *and* $\hat{h}\in k-ST_q(\mu,N,O)$, *then for* $(t\in\mathbb{N}\setminus\{1\})$,

$$|a_t|\leqslant\frac{1}{[t,q]}\prod_{m=0}^{t-2}\left(\frac{|(N-O)\,L_1\delta_l\psi_{m-1}-4qO\psi_m\,[m,\ q]|}{4q\,[m+1,q]\,\psi_{m+1}}\right)$$
$$+\frac{\delta_l L_1\,(\lambda-M)}{4\,[t,q]\,|\psi_{t-1}|}\sum_{j=1}^{t-1}\left[\prod_{m=0}^{j-2}\left(\frac{|(N-M)\,L_1\delta_l\psi_{m-1}-4qO\psi_m\,[m,q]|}{4\psi_{m+1}\,[m+1,q]\,q}\right)\right].$$

Proof. Let us take

$$\frac{\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\mathfrak{I}_q^{\mu}\hat{h}(\xi)} = \hat{J}(\xi), \tag{3.2}$$

where $\hat{J} \in k-P_q(\lambda, M)$ and $\hat{h} \in k-ST_q(\mu, N, O)$. Now from (3.2), we have

$$\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right) = \hat{J}(\xi)\,\mathfrak{I}_q^{\mu}\hat{h}(\xi),$$

which implies that

$$\xi + \sum_{t=2}^{\infty}\psi_{t-1}[t,q]\,a_t\xi^t = \left(\xi + \sum_{t=2}^{\infty}\psi_{t-1}b_t\xi^t\right)\left(1+\sum_{t=1}^{\infty}c_t\xi^t\right),$$

by equating ξ^t coefficients,

$$[t,q]\,\psi_{t-1}a_t = \psi_{t-1}b_t + \sum_{j=1}^{t-1}b_jc_{t-j}\psi_{j-1}, a_1 = 1, b_1 = 1.$$

This implies that

$$[t,q]\,|\psi_{t-1}|\,|a_t| \leqslant |\psi_{t-1}|\,|b_t| + \sum_{j=1}^{t-1}\left|b_j\right|\left|c_{t-j}\right|\left|\psi_{j-1}\right|. \tag{3.3}$$

Since $\hat{h} \in k-ST_q(\mu, N, O)$, therefore by using Lemma 2.2 on (3.3), we get

$$[t,q]\,|\psi_{t-1}|\,|a_t| \leqslant |\psi_{t-1}|\,|b_t| + \frac{\delta_l L_1(\lambda - M)}{4}\sum_{j=1}^{t-1}\left|b_j\right|\left|\psi_{j-1}\right|.$$

Again $\hat{h} \in k-ST_q(\mu, N, O)$, therefore by using (1.7), we get

$$\begin{aligned}|a_t| \leqslant{} & \frac{1}{[t,\ q]}\prod_{m=0}^{t-2}\left(\frac{|(N-O)\,L_1\delta_l\psi_{m-1} - 4qO\,[m,\ q]\,\psi_m|}{4q\psi_{m+1}\,[m+1,q]}\right)\\ &+\frac{\delta_l L_1(\lambda - M)}{4\,[t,q]\,|\psi_{t-1}|}\sum_{j=1}^{t-1}\left[\prod_{m=0}^{j-2}\left(\frac{|(N-M)\,L_1\delta_l\psi_{m-1} - 4qO\psi_m\,[m,q]|}{4\psi_{m+1}\,[m+1,q]\,q}\right)\right], \quad (t \in \mathbb{N}\setminus 1),\end{aligned}$$

as required. □

For $\mu = 1$, Theorem 3.4 reduces to the following results studied by Naeem et al. in [12].

Corollary 3.5. *If* $\hat{g} \in k-UK_q(\lambda, M, N, O)$ *and* $\hat{h} \in k-ST_q(N, O)$, *then*

$$\begin{aligned}|a_t| \leqslant{} & \frac{1}{[t,q]}\prod_{m=0}^{t-2}\left(\frac{|(N-O)\,L_1\delta_l - 4qO\,[m,q]|}{4q\,[m+1,q]}\right)\\ &+\frac{\delta_l L_1(\lambda - M)}{4\,[t,q]}\sum_{j=1}^{t-1}\prod_{m=0}^{j-2}\left(\frac{|(N-M)\,L_1\delta_l - 4qO\,[m,q]|}{4q\,[m+1,q]}\right), \quad (t \in \mathbb{N}\setminus\{1\}).\end{aligned}$$

Further if $q \to 1$ in corollary 3.5, then we have following results studied by Mahmood et al. in [10].

Corollary 3.6. *If* $\hat{g} \in k-UK(\lambda, M, N, O)$ *and* $\hat{h} \in k-ST(N, O)$, *then*

$$|a_t| \leqslant \frac{1}{t}\prod_{m=0}^{t-2}\left(\frac{|(N-O)\,\delta_l - 2Om|}{2\,(m+1)}\right) + \frac{\delta_l\,(\lambda - M)}{2t}\sum_{j=1}^{t-2}\prod_{m=0}^{j-2}\left(\frac{|(N-M)\,\delta_l - 2Om|}{2\,(m+1)}\right), \quad (t \in \mathbb{N}\setminus\{1\}).$$

If we take $\lambda = N = 1$, $M = O = -1$ in Corollary 3.6, then we have following results studied by Noor et al. in [16].

Corollary 3.7. *If* $\hat{g} \in k-UK(1, -1, 1, -1) = k-UK$, *then*

$$|a_t| \leqslant \frac{(\delta\,(l))_{t-1}}{t!} + \frac{\delta\,(l)}{t}\sum_{j=0}^{t-1}\left(\frac{(\delta\,(l))_{j-1}}{(j-1)!}\right), \ (t \in \mathbb{N}\setminus\{1\}).$$

Further if we take $k = 0$ in Corollary 3.7, then we have following well-known results studied by Kaplan in [9].

Corollary 3.8. *If* $\hat{g} \in 0-UK(1, -1, 1, -1) = K$, *then* $|a_t| \leqslant t, \ (t \in \mathbb{N}\setminus\{1\})$.

3.3. *Convolution property*

Following is the convolution property of our defined class.

Theorem 3.9. *If* $\hat{g} \in k-UK_q(\mu, \lambda, M, N, O)$ *and* $\chi \in C$, *then* $\hat{g} * \chi \in k-UK_q(\mu, \lambda, M, N, O)$.

Proof. As $\hat{g} \in k-UK_q(\mu, \lambda, M, N, O)$, there exist $\hat{h} \in k-ST_q(\mu, N, O)$, such that $\frac{\xi D_q(\mathfrak{I}_q^{\mu}\hat{g}(\xi))}{\mathfrak{I}_q^{\mu}\hat{g}(\xi)} \in k-P_q(\mu, \lambda, M)$. It follows from Lemma 2.4 that $\chi * \hat{h} \in k-ST_q(\mu, N, O)$. Now consider

$$\frac{\xi D_q\left(\chi\,(\xi) * \mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)}{\left(\chi\,(\xi) * \mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)} = \frac{\chi\,(\xi) * \left(\xi D_q\left(\mathfrak{I}_q^{\mu}\hat{g}(\xi)\right)\right)}{\mathfrak{I}_q^{\mu}\hat{g}(\xi) * \chi\,(\xi)} = \frac{\chi\,(\xi) * \mathcal{F}\,(\xi)\,\mathfrak{I}_q^{\mu}\hat{h}(\xi)}{\chi\,(\xi) * \mathfrak{I}_q^{\mu}\hat{h}(\xi)},$$

where $\mathcal{F} \in k-ST_q(\mu, \lambda, M)$. We arrive at our needed result using Lemma 2.5. □

For $\mu = 1$, the above theorem reduces to the Corollary, a result demonstrated by Naeem et al., for details see [12].

Corollary 3.10. *If* $\hat{g} \in k-UK_q(\lambda, M, N, O)$ *and* $\chi \in C$, *then* $\hat{g} * \chi \in k-UK_q(\lambda, M, N, O)$.

For $\mu = 1$ and $q \to 1$, the above theorem reduces to the Corollary, a result demonstrated by Mahmood et al., for details see [10].

Corollary 3.11. *If* $\hat{g} \in k-UK(\lambda, M, N, O)$ *and* $\chi \in C$, *then* $\hat{g} * \chi \in k-UK(\lambda, M, N, O)$.

4. Conclusions

By using the q-analogue of the Noor integral operator, we studied various properties such as necessary and sufficient conditions, coefficient bounds, convolution properties, linear combinations, weighted means, arithmetic means, distortion and covering theorems and radii of starlikenss etc for a newly defined class of analytic functions. We also pointed out many special cases in the form of corollaries by specializing the parameters.

Acknowledgment

The authors would like to thank the editor and the anonymous reviewers for their constructive comments and suggestions, which helped us to improve the manuscript considerably.

References

[1] M. Acu, *On a subclass of* n*-uniformly close-to-convex functions*, Gen. Math., **14** (2006), 55–64. 1, 2.6
[2] B. Ahmad, S. K. Ntouyas, *Boundary value problems for* q*-difference inclusions*, Abstr. Appl. Anal., **2011** (2011), 15 pages.
[3] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. Math., **17** (1915), 12–22.
[4] M. Arif, M. Ul Haq, J.-L. Liu, *A Subfamily of univalent functions associated with* q*-analogue of Noor integral operator*, J. Funct. Spaces, **2018** (2018), 5 pages. 1, 1, 2
[5] F. H. Jackson, *On* q*-functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, **46** (1909), 253–281. 1
[6] W. Janowski, *Some extremal problems for certain families of analytic functions I*, Ann. Polon. Math., **28** (1973), 297–326. 1.1, 1
[7] S. Kanas, A. Wisniowska, *Conic regions and k-uniform convexity*, J. Comput. Appl. Math., **105** (1999), 327–336. 1, 1
[8] S. Kanas, A. Wisniowska, *Conic domains and starlike functions*, Rev. Roumaine Math. Pures Appl., **45** (2002), 647–657. 1
[9] W. Kaplan, *Close-to-convex Schlicht functions*, Michigan Math. J., **1** (1952), 169–185. 1, 2.6, 3.2
[10] S. Mahmood, M. Arif, S. N. Malik, *Janowski type close-to-convex functions associated with conic regions*, J. Inequal. Appl., **2017** (2017), 14 pages. 2.6, 3.1, 3.2, 3.3
[11] S. Mahmood, M. Jabeen, S. N. Malik, H. M. Srivastava, R. Manzoor, S. M. J. Riaz, *Some coefficient inequalities of* q*-starlike functions associated with conic domain defined by* q*-derivative*, J. Funct. Spaces, **2018** (2018), 13 pages. 1.2, 2.2, 2.3, 2.5, 2
[12] M. Naeem, S. Hussain, S. Khan, T. Mahmood, M. Darus, Z. Shareef, *Janowski type* q*-convex and* q*-close-to-convex functions associated with* q*-conic domain*, Mathematics, **8** (2000), 14 pages. 2.6, 3.1, 3.2, 3.3
[13] K. I. Noor, *On new classes of integral operators*, J. Natur. Geom, **16** (1999), 71–80. 1
[14] K. I. Noor, S. N. Malik, *On coefficient inequalities of functions associated with conic domains*, Comput. Math. Appl., **62** (2011), 2209–2217. 1.2, 2
[15] K. I. Noor, M. A. Noor, *On integral operators*, J. Math. Anal. Appl., **238** (1999), 341–352. 1
[16] K. I. Noor, H. Owa, H. M. Srivistava, *On some subclasses of close-to-convex functions*, in: Univalent functions, fractional calculus, and their applications, **1989** (1989), 179–186. 3.2
[17] K. I. Noor, D. K. Thomas, *Quasi-convex univalent functions*, Int. J. Math. Math. Sci., **3** (1980), 255–266. 2.6
[18] W. Rogosinski, *On the coefficients of subordinate functions*, Proc. London Math. Soc. (2), **48** (1943), 48–82. 2.1
[19] S. G. A. Shah, S. Hussain, A. Rasheed, Z. Shareef, M. Darus, *Application of quasisubordination to certain classes of meromorphic functions*, J. Funct. Spaces, **2020** (2020), 8 pages. 2.6
[20] S. G. A. Shah, S. Noor, S. Hussain, A. Tasleem, A. Rasheed, M. Darus, *Analytic functions related with starlikeness*, Math. Probl. Eng., **2021** (2021), 5 pages. 2.6
[21] H. M. Srivastava, M. Tahir, B. Khan, Q. Z. Ahmad, N. Khan, *Some general classes of* q*-starlike functions associated with the Janowski functions*, Symmetry, **11** (2019), 12 pages. 2.6
[22] K. G. Subramanian, T. V. Sudharsan, H. Silverman, *On uniformly close-to-convex functions and uniformly quasiconvex functions*, Int. J. Math. Math. Sci., **2003** (2003), 3053–3058. 2.6
[23] C. L. Yu, J. Wang, *Existence of solutions for nonlinear second-order* q*-difference equations with first-order* q*-derivatives*, Adv. Difference Equ., **2013** (2013), 11 pages.
[24] W.-X. Zhou, H.-Z. Liu, *Existence solutions for boundary value problem of nonlinear fractional* q*-difference equations*, Adv. Difference Equ., **2013** (2013), 13 pages. 2.6