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Lie group method for unsteady heat flow of a third-grade fluid between two parallel heated plates

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Abstract

In this paper, we use the classical Lie group method, to investigate the symmetries of the heat transfer flow of a third-grade fluid. This approach allows one to reduce the coupled partial differential equations governing the problem, to a system of nonlinear ordinary differential equations. Point symmetries of such systems are used to construct some classes of solutions. By using travelling wave solutions, we studied the influence of third-grade fluid parameters on the flow.

Keywords: Lie group method, optimal system, third-grade fluid, similarity reduction.

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1. Introduction

Non-Newtonian fluid flows have been the subject of many investigations, whether theoretical or experimental, because they are present in nature and different industrial activities alike, such as cosmetic industries, bioengineering and nuclear industries etc. Several models have been developed to investigate the complex behaviour of this class of fluids. Among these models are the well-known differential type fluids [19], which were first proposed by Rivlin and Ericksen in 1955. In particular, the subclass of fluids of the third-grade model is adequate for describing both the effect of shear thinning and thickening phenomena. Due to its importance in engineering applications and physical sciences, several authors have been interested in the study of third-grade fluid flows in various configurations and subject to diverse physical hypotheses. In what follows, we will give a brief overview of some of the previous works concerning this topic concerning both numerical and analytical approaches.

Szeri and Rajagopal [22] was the first to study the flow of a third-grade fluid between heated parallel plates by considering thermal effects. They particularly studied the case of constant heat flux at the walls, and through a similarity transformation and with a purely numerical approach, they calculated the Nusselt number as a function of the three physical parameters mentioned in their model.

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Akyildiz [6] resumed the same problem addressed by the previous authors. He found a similar solution in good agreement with the numerical solution obtained by Szeri and Rajagopal. Siddiqui et al. [20] developed approximate analytical solutions for the fluid velocity and the temperature distribution by using the homotopy perturbation method. The effect of the third-grade fluid parameters on this heat transfer flow has been reported. The heat transfer flow of a third-grade fluid with the porous medium through parallel plates considering Vogel's temperature dependent model based viscosity was reported in the study by Akinshilo [5]. He developed an analytical solution through the Adomian decomposition method, which allowed him to examine the effects of thermal fluidic parameters such as the pressure gradient, the heat generation parameter and the porosity term on the flow and the heat transfer.

The heat transfer flow of a third-grade fluid past an infinite porous plate embedded in a porous medium is investigated by Khani et al. [15]. These authors employed an analytical approach to solve, momentum and energy equations. From there, they studied the influence of thermophysical and hydromechanical parameters on velocity and heat transfer. Akgul and Pakdemirli [3] have considered the electro-osmotic flow of a third-grade fluid between micro-parallel plates. Approximate analytical solutions are obtained by perturbation techniques. Effects of physical parameters, such as electro-kinetic parameter, Joule heating parameter and viscosity index on the velocity and temperature profiles are depicted. Hayat et al. [14] have examined the problem of the rotating flow and heat transfer analysis of a third-grade fluid between two stationary porous plates. They have found an analytical solution for velocity and temperature. The significant contributions of the parameters of the model to the heat transfer flow have been pointed out.

Hayat et al. [13] considered the peristaltic flow of third-grade fluid in a curved channel with heat and mass transfer. An analytical solution has been found under the assumption of a small Deborah number. This study allowed them to describe the behaviour of the various parameters on the velocity field, temperature distribution, concentration and heat transfer coefficient. Adesanya and Falade [2] have analyzed the flow and heat transfer of hydromagnetic third-grade fluid between horizontal parallel plates saturated with porous materials. The governing equations of momentum and energy balance have been treated analytically using the regular perturbation method. The effect of various physical parameters on velocity and temperature profiles is investigated. Makinde and Chinyoka [16] examined the effects of a transverse magnetic field and variable viscosity on the unsteady heat transfer flow of a reactive third-grade electrically conductive fluid placed between two parallel plates.

In the present work, analysis has been carried out to study the heat flow of a third-grade fluid between two parallel plates using an analytical method. In this kind of problem, two analytical approaches are often used, namely those that can be assimilated to a perturbation method, or those derived from Lie group methods. For our part, we have applied the latter procedure to approach the problem. Indeed, in the last decade, much interest has been devoted to the application of Lie transformation group theory to study various non-Newtonian fluid flows. A non-exhaustive list dealing with this type of problem can be found in references [8–23]. Lie group analysis also called the Lie symmetry method is based on symmetry and invariance principles. The method consists in finding transformations which leave a given problem invariant. By applying this theory, we came to a set of determining equations whose resolution leads to the determination of the symmetry groups often generated by translations, scaling, Galilean transformations, etc. Once these groups are defined, we have reduced the given system to a lower system involving fewer dependent or independent variables, and further find a general solution in quadrature. Furthermore, in some cases, by combining groups of translations in space and time, we looked for so-called travelling wave solutions. This theory has received particular attention from authors such as Ovsiannikov [18], Bluman [10], Olver [17], and Stephani [21] who have contributed to building the foundation as well by applying this theory in various fields, from mathematics to biology.

The objective of this work is to study the heat transfer of a third-grade fluid between two parallel plates by the Lie group analysis. Exact solutions are investigated using symmetry reductions; the case of a travelling wave solution is also obtained. Finally, based on the Newton-Raphson technique, a numerical resolution of these reduced equations has been carried out. The velocity profiles and temperature distribution are given and the influence of some physical parameters on these flows has been studied and shown graphically.

This paper is organized as follows. Section 2 contains the basic equations for heat transfer in the flow of a third-grade fluid. Section 3 deals with the unsteady flow of a third-grade fluid over the heated parallel plate. We reformulated the problem as a system of partial differential equations of order three. Next, in Section 4, we determined the symmetry groups that leave the system invariant giving rise to the so-called similarity solutions. In Section 5, the numerical solution and the influence of physical parameters are given for the case of travelling wave solutions. In the last Section 6, we gave some conclusions about the Lie symmetry group method.

2. Governing equations

Assuming incompressible laminar flow, the equations of motion are the continuity, the conservation of momentum, and energy equations:

$$\operatorname{div}(\mathbf{u}) = 0, \tag{2.1}$$

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{f} + \operatorname{div}(\boldsymbol{\sigma}), \tag{2.2}$$

$$\rho c_p\frac{D\theta}{Dt} = k\Delta\theta + \boldsymbol{\sigma} : \mathbf{L}, \tag{2.3}$$

where $\mathbf{u}$ is the velocity vector, $\mathbf{f}$ is the body force per unit mass, $\boldsymbol{\sigma}$ is the Cauchy stress tensor, θ is the temperature, ρ is the density of the fluid, k is the thermal conductivity, c_p is the specific heat at constant pressure, $\mathbf{L}$ is the gradient of $\mathbf{u}$, $\boldsymbol{\sigma} : \mathbf{L}$ is the double dot product of $\boldsymbol{\sigma}$ by $\mathbf{L}$, and $\frac{D}{Dt}$ denotes the material derivative.

The Cauchy stress tensor $\boldsymbol{\sigma}$ for in incompressible homogeneous third-grade fluid is given by:

$$\boldsymbol{\sigma} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\operatorname{tr}\mathbf{A}_2)\mathbf{A}_1, \tag{2.4}$$

where p is the scalar pressure, μ is the coefficient of viscosity, tr denotes the trace of a second-order tensor, α_1, α_2 are the normal stress moduli, β_1, β_2, β_3 are the material constants, $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ are the three Rivlin-Ericksen tensors [19] defined by

$$\mathbf{A}_1 = \mathbf{L}^T + \mathbf{L}, \quad \mathbf{A}_n = \frac{D\mathbf{A}_{n-1}}{Dt} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{n-1}, \quad n = 2, 3,$$

where $\mathbf{L}^T$ is the transpose of $\mathbf{L}$. Furthermore, a complete thermodynamic analysis of the third-grade fluid was developed by Fosdick and Rajagopal [12]. These authors performed that equation (2.4) to be compatible with thermodynamics, then the following restrictions must hold:

$$\mu \geqslant 0, \quad \alpha_1 \geqslant 0, \quad |\alpha_1 + \alpha_2| \leqslant \sqrt{24\mu\beta_3}, \quad \beta_1 = \beta_2 = 0, \quad \beta_3 \geqslant 0.$$

Thus equation (2.4) takes the simplified form:

$$\boldsymbol{\sigma} = -p\mathbf{I} + (\mu + \beta_3(\operatorname{tr}\mathbf{A}_2))\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2.$$

3. Formulation of the problem

Consider the unsteady and laminar flow of a third-grade fluid between two infinite horizontal parallel plates located at $y = \pm a$ planes (see Figure 1). The plates are stationary and the flow is driven by a constant non zero pressure gradient in the streamwise direction. Assuming that the plates are kept at two constant temperatures θ_1 and θ_2, $\theta_2 > \theta_1$.

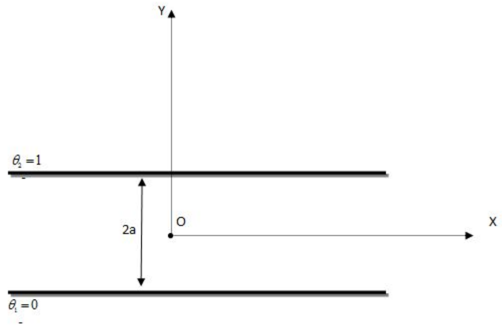

Figure 1: Configuration of the problem.

We shall seek the velocity and temperature as:

$$\mathbf{u} = (v(y,t), 0, 0), \qquad \theta = \theta(y,t).$$

This ensures that the equation of continuity (2.1) is identically satisfied. In the absence of body forces, the momentum and the energy equations (2.2) and (2.3) take the form

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 v}{\partial y^2} + 6\beta_3 \left(\frac{\partial v}{\partial y}\right)^2 \frac{\partial^2 v}{\partial y^2} + \alpha_1 \frac{\partial^3 v}{\partial t \partial y^2}, \quad -\frac{\partial p}{\partial y} + (4\alpha_1 + 2\alpha_2) \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y^2} = 0, \tag{3.1}$$

$$-\frac{\partial p}{\partial z} = 0,$$

$$\rho c_p \frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial y^2} + \mu \left(\frac{\partial v}{\partial y}\right)^2 + 2\beta_3 \left(\frac{\partial v}{\partial y}\right)^4 + \alpha_1 \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial t \partial y}. \tag{3.2}$$

From (3.1) we can write

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial \bar{p}_1}{\partial x} + \mu \frac{\partial^2 v}{\partial y^2} + 6\beta_3 \left(\frac{\partial v}{\partial y}\right)^2 \frac{\partial^2 v}{\partial y^2} + \alpha_1 \frac{\partial^3 v}{\partial t \partial y^2}, \tag{3.3}$$

$$-\frac{\partial \bar{p}_1}{\partial y} = 0, \tag{3.4}$$

where

$$\bar{p}_1 = p - (2\alpha_1 + \alpha_2) \left(\frac{\partial v}{\partial y}\right)^2.$$

Eliminating the pressure gradient between equations (3.3) and (3.4) finally yields

$$\rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial y^2} + 6\beta_3 \left(\frac{\partial v}{\partial y}\right)^2 \frac{\partial^2 v}{\partial y^2} + \alpha_1 \frac{\partial^3 v}{\partial t \partial y^2}. \tag{3.5}$$

Let us introduce the following variables:

$$\alpha^* = \frac{\alpha_1}{\rho}, \quad \beta^* = \frac{6\beta_3}{\rho}, \quad \mu^* = \frac{\mu}{\rho}, \quad \tilde{\alpha} = \frac{\alpha_1}{\rho c_p}, \quad \tilde{\beta} = \frac{2\beta_3}{\rho c_p}, \quad \tilde{k} = \frac{k}{\rho c_p}, \tilde{\mu} = \frac{\mu}{\rho c_p}.$$

Then equation (3.5) becomes:

$$\frac{\partial v}{\partial t} = \mu^* \frac{\partial^2 v}{\partial y^2} + \beta^* \left(\frac{\partial v}{\partial y}\right)^2 \frac{\partial^2 v}{\partial y^2} + \alpha^* \frac{\partial^3 v}{\partial t \partial y^2}, \tag{3.6}$$

and equation (3.2) reduces to:

$$\frac{\partial \theta}{\partial t} = \tilde{k} \frac{\partial^2 \theta}{\partial y^2} + \tilde{\mu} \left(\frac{\partial v}{\partial y}\right)^2 + \tilde{\beta} \left(\frac{\partial v}{\partial y}\right)^4 + \tilde{\alpha} \left(\frac{\partial v}{\partial y}\right) \left(\frac{\partial^2 v}{\partial t \partial y}\right). \tag{3.7}$$

4. Lie group analysis of equations (3.6) and (3.7)

4.1. Lie algebra and symmetry groups

In this section, we performed a Lie group analysis for equations (3.6)-(3.7) and we obtained its infinitesimal generator and commutation table of Lie algebra.

First of all, let us consider a one parameter Lie group infinitesimal transformation:

$$\bar{y} = y + \epsilon\xi(y,t,\nu,\theta) + O(\epsilon^2), \qquad \bar{t} = t + \epsilon\tau(y,t,\nu,\theta) + O(\epsilon^2),$$
$$\bar{\nu} = \nu + \epsilon\phi(y,t,\nu,\theta) + O(\epsilon^2), \qquad \bar{\theta} = \theta + \epsilon\gamma(y,t,\nu,\theta) + O(\epsilon^2),$$

where ϵ is a small parameter. The infinitesimal vector associated with the above group of transformations can be written as

$$\mathbf{V} = \xi(y,t,\nu,\theta)\frac{\partial}{\partial y} + \tau(y,t,\nu,\theta)\frac{\partial}{\partial t} + \phi(y,t,\nu,\theta)\frac{\partial}{\partial \nu} + \gamma(y,t,\nu,\theta)\frac{\partial}{\partial \theta}, \tag{4.1}$$

where $\xi(y,t,\nu,\theta)$, $\tau(y,t,\nu,\theta)$, $\phi(y,t,\nu,\theta)$, and $\gamma(y,t,\nu,\theta)$ are coefficient functions of the infinitesimal generator to be determined.

The equations (3.6)-(3.7) can be written in an equivalent form:

$$\Delta_1 = \frac{\partial \nu}{\partial t} - \mu^* \frac{\partial^2 \nu}{\partial y^2} - \beta^* \left(\frac{\partial \nu}{\partial y}\right)^2 \frac{\partial^2 \nu}{\partial y^2} - \alpha^* \frac{\partial^3 \nu}{\partial t \partial y^2} = 0,$$
$$\Delta_2 = \frac{\partial \theta}{\partial t} - \tilde{k}\frac{\partial^2 \theta}{\partial y^2} - \tilde{\mu}\left(\frac{\partial \nu}{\partial y}\right)^2 - \tilde{\beta}\left(\frac{\partial \nu}{\partial y}\right)^4 - \tilde{\alpha}\left(\frac{\partial \nu}{\partial y}\right)\left(\frac{\partial^2 \nu}{\partial t \partial y}\right) = 0.$$

The vector field (4.1) generates a one-parameter symmetry group of the equations (3.6)-(3.7), if and only if the invariance conditions holds,

$$Pr^{(3)}\mathbf{V}\,(\Delta_i)\,|_{\Delta_i=0} = 0; \text{ for } i = 1,2,$$

where $Pr^{(3)}\mathbf{V}$ is the third order prolonged infinitesimal generator $\mathbf{V}$ given by the formula:

$$Pr^{(3)}(\mathbf{V}) = \mathbf{V} + \sigma_1^y \frac{\partial}{\partial \nu_y} + \sigma_1^t \frac{\partial}{\partial \nu_t} + \sigma_2^y \frac{\partial}{\partial \theta_y} + \sigma_2^t \frac{\partial}{\partial \theta_t} + \sigma_1^{yy}\frac{\partial}{\partial \nu_{yy}} + \sigma_2^{yy}\frac{\partial}{\partial \theta_{yy}} + \sigma_1^{ty}\frac{\partial}{\partial \nu_{ty}} + \sigma_1^{tyy}\frac{\partial}{\partial \nu_{tyy}},$$

The coefficient functions of the extended infinitesimals σ_j^i, $i = y,t$; $j = 1,2$, are explicitly given by

$$\sigma_1^y = D_y(\phi) - \nu_y D_y(\xi) - \nu_t D_y(\tau), \qquad \sigma_1^t = D_t(\phi) - \nu_y D_t(\xi) - \nu_t D_t(\tau),$$
$$\sigma_2^y = D_y(\gamma) - \theta_y D_y(\xi) - \theta_t D_y(\tau), \qquad \sigma_1^t = D_t(\gamma) - \theta_y D_t(\xi) - \theta_t D_t(\tau),$$
$$\sigma_1^{yy} = D_y\left(\sigma_1^y\right) - \nu_{yy} D_y(\xi) - \nu_{yt} D_y(\tau), \qquad \sigma_2^{yy} = D_y\left(\sigma_2^y\right) - \theta_{yy} D_y(\xi) - \theta_{yt} D_y(\tau),$$
$$\sigma_1^{ty} = D_t\left(\sigma_1^y\right) - \nu_{ty} D_t(\xi) - \nu_{tt} D_t(\tau), \qquad \sigma_2^{tyy} = D_t\left(\sigma_1^{yy}\right) - \nu_{tyy} D_t(\xi) - \nu_{tt} D_t(\tau).$$

The total derivatives D_t and D_y operators are defined as

$$D_t = \frac{\partial}{\partial t} + \nu_t \frac{\partial}{\partial \nu} + \nu_{tt}\frac{\partial}{\partial \nu_t} + \theta_t \frac{\partial}{\partial \theta} + \theta_{tt}\frac{\partial}{\partial \theta_t} + \nu_{ty}\frac{\partial}{\partial \nu_y} + \theta_{ty}\frac{\partial}{\partial \theta_y} + \cdots,$$
$$D_y = \frac{\partial}{\partial y} + \nu_y \frac{\partial}{\partial \nu} + \nu_{yy}\frac{\partial}{\partial \nu_y} + \theta_y \frac{\partial}{\partial \theta} + \theta_{yy}\frac{\partial}{\partial \theta_y} + \nu_{ty}\frac{\partial}{\partial \nu_t} + \theta_{ty}\frac{\partial}{\partial \theta_t} + \cdots.$$

The invariance condition yields the following equations:

$$\begin{aligned} &\sigma_1^t - \left(\mu^* + \beta^*(\nu_y)^2\right)\sigma_1^{yy} - 2\beta^*\nu_y\nu_{yy}\sigma_1^y - \alpha^*\sigma_1^{tyy} = 0,\\ &\sigma_2^t - \left(\tilde{\alpha}\nu_{ty} + 4\tilde{\beta}(\nu_y)^3 + 2\tilde{\mu}\nu_y\right)\sigma_1^y - \tilde{\alpha}\nu_y\sigma_1^{ty} - \tilde{k}\sigma_2^{yy} = 0. \end{aligned} \tag{4.2}$$

After the substitution of σ_j^i, $i = y, t$; $j = 1, 2$, in (4.2), and with the help of a symbolic software, we get the following determining equations

$$\begin{cases} \xi_y = \xi_t = \xi_\nu = \xi_\theta = 0, \ \tau_y = \tau_t = \tau_\nu = \tau_\theta = 0, \\ \varphi_y = \varphi_t = \varphi_\nu = \varphi_\theta = 0, \ \gamma_{yy} - \frac{1}{\tilde{k}}\gamma_t = 0, \\ \gamma_\nu = \gamma_\theta = 0. \end{cases} \tag{4.3}$$

By solving above equations (4.3), we find

$$\xi = C_1, \quad \tau = C_2, \quad \varphi = C_3, \quad \gamma = \alpha(y, t)\, C_4,$$

where C_1, C_2, C_3 and C_4 are arbitrary constants and $\alpha(y, t)$ is an arbitrary solution satisfying the following equation:

$$\frac{\partial^2 \alpha(y, t)}{\partial y^2} - \frac{1}{\tilde{k}} \frac{\partial \alpha(y, t)}{\partial t} = 0. \tag{4.4}$$

Hence the Lie algebra of infinitesimal symmetries of equations (3.6)-(3.7) is spanned by the following vector generators:

$$\mathbf{V}_1 = \frac{\partial}{\partial y}, \quad \mathbf{V}_2 = \frac{\partial}{\partial t}, \quad \mathbf{V}_3 = \frac{\partial}{\partial \nu}, \quad \mathbf{V}_4 = \alpha(y, t)\frac{\partial}{\partial \theta}.$$

Then all of the infinitesimal generators of equations (3.6)-(3.7) can be expressed as follows

$$\mathbf{V} = C_1\mathbf{V}_1 + C_2\mathbf{V}_2 + C_3\mathbf{V}_3 + C_4\mathbf{V}_4,$$

with C_1, C_2, C_3, C_4 being arbitrary constants.

It is easy to verify that $\mathcal{L} = \{\mathbf{V}_1, \mathbf{V}_1, \mathbf{V}_3\}$ is closed under the Lie bracket operation. We observed that the three dimensional Lie algebra $\mathcal{L}$ is Abelian. Furthermore, we can compute the adjoint representations of the vector fields. We have $\mathrm{Ad}(\exp(\epsilon\mathbf{V}_i))\mathbf{V}_i = \mathbf{V}_i$, for $i = 1, 2, 3$.

Based on these representations of the vector fields, we obtained the optimal system of order one, two and three as Table 1.

Table 1: Optimal system of dimensions 1, 2, and 3.

Dimension	Subalgebra
1	$\mathbf{V}_1 + a\mathbf{V}_2 + b\mathbf{V}_3, \quad a, \ b \in \mathbb{R}$
	$\mathbf{V}_2 + a\mathbf{V}_3, \quad a \ \in \mathbb{R}$
	$\mathbf{V}_3$
2	$\langle \mathbf{V}_1 + b\mathbf{V}_3, \mathbf{V}_2 + b\mathbf{V}_3 \rangle, \quad a, \ b \in \mathbb{R}$
	$\langle \mathbf{V}_1 + a\mathbf{V}_2, \mathbf{V}_3 \rangle, \quad a \ \in \mathbb{R}$
	$\langle \mathbf{V}_2, \mathbf{V}_3 \rangle,$
3	$\langle \mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3 \rangle,$

To get the one parameter group, we should solve the Lie equations:

$$\frac{d\bar{y}}{d\epsilon} = \xi\left(\bar{y}, \bar{t}, \bar{\nu}, \bar{\theta}\right), \quad \frac{d\bar{t}}{d\epsilon} = \tau\left(\bar{y}, \bar{t}, \bar{\nu}, \bar{\theta}\right), \quad \frac{d\bar{\nu}}{d\epsilon} = \varphi\left(\bar{y}, \bar{t}, \bar{\nu}, \bar{\theta}\right), \quad \frac{d\bar{\theta}}{d\epsilon} = \gamma\left(\bar{t}, \bar{y}, \bar{\nu}, \bar{\theta}\right),$$

subject to the initial conditions

$$\bar{y}|_{\epsilon=0} = y, \quad \bar{t}|_{\epsilon=0} = t, \quad \bar{\nu}|_{\epsilon=0} = \nu, \quad \bar{\theta}|_{\epsilon=0} = \theta.$$

By solving this system of ordinary differential equations, we obtained the following groups of symmetry generated by $\mathbf{V}_i$ for $i = 1, 2, 3, 4$:

$$G_1: \quad (y, t, \nu, \theta) \to (y + \epsilon, t, \nu, \theta), \qquad G_2: \quad (y, t, \nu, \theta) \to (y, t + \epsilon, \nu, \theta),$$

$$G_3: \quad (y, t, \nu, \theta) \to (y, t, \nu + \epsilon, \theta), \qquad G_4: \quad (y, t, \nu, \theta) \to (y, t, \nu, \theta + \alpha(y, t)\,\epsilon),$$

where ϵ is any real number and $\alpha(y, t)$ is an arbitrary solution of equation (4.4).

We can see that G_1 is a space translation, G_2 is a time translation and G_3 is a dependent variable translation. The last group G_4 denotes an infinite dimensional system.

In the preceding, we obtained the symmetry groups of equations (3.6)-(3.7). Now we will consider the exact solutions of equations (3.6)-(3.7) based on the symmetry group analysis. Since each $G_i\,(i=1,\dots,4)$ is a symmetry group, it implies that if $v=f(y,t)$ and $\theta=g(y,t)$ are the known solution of equations (3.6)-(3.7), then by using the above groups $G_i\,(i=1,\dots,4)$, the corresponding new solutions v^i and θ^i $(i=1,\dots,4)$ can be obtained respectively as follows:

$$v^{(1)}=f(y-\epsilon,t),\quad \theta^{(1)}=g(y-\epsilon,t),\qquad v^{(2)}=f(y,t-\epsilon),\quad \theta^{(2)}=g(y,t-\epsilon),$$
$$v^{(3)}=f(y,t)+\epsilon,\quad \theta^{(3)}=g(y,t),\qquad v^{(4)}=f(y,t),\quad \theta^{(4)}=g(y,t)+\alpha(y,t)\,\epsilon.$$

4.2. Exact solutions

Travelling wave solutions form a special class of solutions that are invariant under a linear combination of time-translation and space-translation symmetry generators, i.e., of the form V_1+cV_2 with constant wave speed c.

In the following, we will seek an exact travelling wave solution using the symmetry approach. In the previous, we obtained the infinitesimal generators $\mathbf{V}_i\ (i=1,2,3,4)$. In the following, we will get similarity variables and their reduction equations, and obtain travelling wave solutions or similarity solutions. We will consider the following cases.

Case 1: For the infinitesimal generator $\mathbf{V}_1+a\mathbf{V}_2+b\mathbf{V}_3$, the similarity variables are $r=t-ya$, $F(r)=v-by$, $G(r)=\theta$ and the group invariant solution is $v=F(r)+by$, $\theta=G(r)$. Substituting the group invariant solutions into equations (3.6)-(3.7), we obtained the following reduction equation:

$$\begin{cases} \dfrac{d^3F}{dr^3}=\dfrac{1}{a^2\alpha^*}\begin{pmatrix} \frac{dF}{dr}-a^2\mu^*\frac{d^2F}{dr^2}-a^4\beta^*\frac{d^2F}{dr^2}\left(\frac{dF}{dr}\right)^2 \\ +2a^3b\beta^*\frac{d^2F}{dr^2}\frac{dF}{dr}-a^2b^2\beta^*\frac{d^2F}{dr^2}\end{pmatrix}, \\ \dfrac{d^2G}{dr^2}=\dfrac{1}{a^2\tilde{k}}\begin{pmatrix} \frac{dG}{dr}-a^2\tilde{\mu}\left(\frac{dG}{dr}\right)^2+2ab\tilde{\mu}\frac{dF}{dr} \\ -b^2\tilde{\mu}-a^4\tilde{\beta}\left(\frac{dF}{dr}\right)^4+4a^3b\tilde{\beta}\left(\frac{dF}{dr}\right)^3 \\ -6a^2b^2\tilde{\beta}\left(\frac{dF}{dr}\right)^2+4ab^3\tilde{\beta}\frac{dF}{dr} \\ -b^4\tilde{\beta}-a^2\tilde{\alpha}\frac{d^2F}{dr^2}\frac{dF}{dr}+ab\tilde{\alpha}\frac{d^2F}{dr^2}\end{pmatrix}. \end{cases}$$

It may be impossible to find the exact solutions to such problems. But, one can find its travelling wave solutions, by putting $a=c$ and $b=0$,

$$\begin{cases} \frac{d^3F}{dr^3}=\frac{1}{c^2\alpha^*}\left(\frac{dF}{dr}-c^2\mu^*\frac{d^2F}{dr^2}-c^4\beta^*\frac{d^2F}{dr^2}\left(\frac{dF}{dr}\right)^2\right), \\ \frac{d^2G}{dr^2}=\frac{1}{c^2\tilde{k}}\left(\frac{dG}{dr}-c^2\tilde{\mu}\left(\frac{dG}{dr}\right)^2-c^2\tilde{\alpha}\frac{d^2F}{dr^2}\frac{dF}{dr}\right). \end{cases}\tag{4.5}$$

In Section 5, these systems will be solved by using the Newton-Raphson technique.

Case 2: For the infinitesimal generator $\mathbf{V}_2+a\mathbf{V}_3$, the similarity variables are $r=y$, $F(r)=v-at$, $G(r)=\theta$ and the group invariant solution is $v=F(r)+at$, $\theta=G(r)$.

Substituting these expressions into equations (3.6)-(3.7), we get the following reduction equation:

$$\begin{cases} \mu^*\frac{d^2F}{dr^2}+\beta^*\left(\frac{dF}{dr}\right)^2\frac{d^2F}{dr^2}=a, \\ \tilde{k}\frac{d^2G}{dr^2}+\tilde{\mu}\left(\frac{dF}{dr}\right)^2+\tilde{\beta}\left(\frac{dF}{dr}\right)^4=0. \end{cases}\tag{4.6}$$

We can solve (4.6), to get the exact solutions by putting a=0, we have two different solutions. From reduction equations (4.5), we have two different solutions. The first is

$$\begin{cases} F(r)=c_1r+c_2, \\ G(r)=-\frac{1}{2}\frac{(c_1^2\tilde{\mu}+\tilde{\beta}c_1^4)r^2}{\tilde{k}}+c_3r+c_4, \end{cases}\quad\text{or}\quad \begin{cases} v(y,t)=c_1y+at+c_2, \\ \theta(t,y)=-\frac{1}{2}\frac{(c_1^2\tilde{\mu}+\tilde{\beta}c_1^4)y^2}{\tilde{k}}+c_3y+c_4, \end{cases}$$

and the second is

$$\begin{cases} F(r) = \pm\frac{\sqrt{-\beta^*\mu^*}r}{\beta^*} + c_1 r, \\ G(r) = \frac{1}{2}\frac{\left(\frac{\tilde{\mu}\mu^*}{\beta^*} - \frac{\tilde{\beta}(\mu^*)^2}{(\beta^*)^2}\right)r^2}{\tilde{k}} + c_2 r + c_3, \end{cases} \quad \text{or} \quad \begin{cases} v(y,t) = \pm\frac{\sqrt{-\beta^*\mu^*}y}{\beta^*} + at + c_1, \\ \theta(y,t) = \frac{1}{2}\frac{\left(\frac{\tilde{\mu}\mu^*}{\beta^*} - \frac{\tilde{\beta}(\mu^*)^2}{(\beta^*)^2}\right)y^2}{\tilde{k}} + c_2 y + c_3, \end{cases}$$

where c_1, c_2, c_3, and c_4 are arbitrary constants.

Case 3: For the infinitesimal generator of the dependent variable translation $\mathbf{V}_3$, we can see obviously that if $v = v(y,t)$ is a solution of equations (3.6)-(3.7), then $v = v(y,t) + C_1$ and $\theta = C_2$ (C_1, C_2 are arbitrary constants), are a trivial solution of equations (3.6)-(3.7).

5. Results and discussion

In this section, we will give a numerical solution to case 1. The reduced equations (4.5) is a system of coupled nonlinear ordinary differential equations. To solve this system, we adopted the Newton-Raphson method. The third-order differential system (4.5) can be reformulated as a system of five first-order equations by using the substitution:

$$F = y_1, \quad \frac{dF}{dr} = y_2, \quad \frac{d^2F}{dr^2} = y_3, \quad G = y_4, \quad \frac{dG}{dr} = y_5.$$

Then the system (4.5) is transformed as follows:

$$\begin{cases} \frac{dy_1}{dr} = y_2, \\ \frac{dy_2}{dr} = y_3, \\ \frac{dy_3}{dr} = \frac{1}{\alpha^* c^2}\left(-\mu^* c^2 y_3 - \beta^* c^4 (y_2)^2 y_3 + y_2\right), \\ \frac{dy_4}{dr} = y_5, \\ \frac{dy_5}{dr} = \frac{1}{\tilde{k}}\left(-\tilde{\mu}c^2(y_2)^2 - \tilde{\beta}c^4(y_2)^4 + y(5) - \tilde{\alpha}c^2 y_2 y_3\right). \end{cases}$$

Subject to the following conditions:

$$y_1(-1) = 0, \quad y_1(1) = 1, \quad y_4(-1) = 0, \quad y_4(1) = 1.$$

These four conditions correspond to the following boundary conditions imposed on the two parallel plates: $v(-1) = 0$, $v(1) = 1$, $\theta(-1) = 0$, and $\theta(1) = 1$.

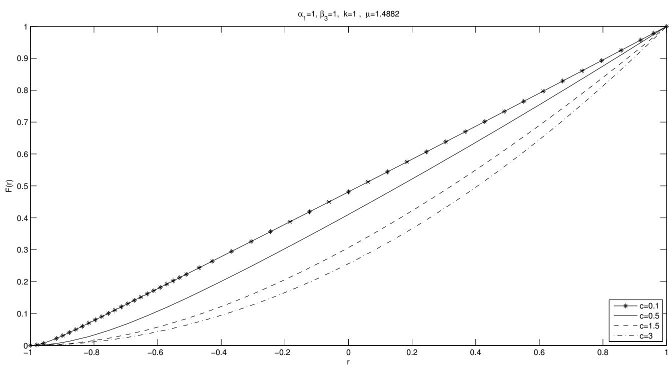

Figure 2: Travelling wave solutions of a velocity profile for various values of wave speed c.

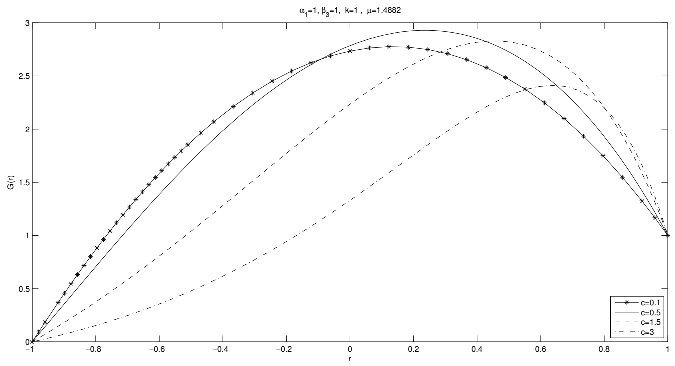

Figure 3: Travelling wave solutions of temperature profile for various values of wave speed c.

In Figure 2, we observed that increasing the values of c leads to a rapid decrease in the velocity profile. This is due to the inertia forces, i.e., the resistance of the fluid in motion. Figure 3, illustrates the same behaviour on the temperature distribution.

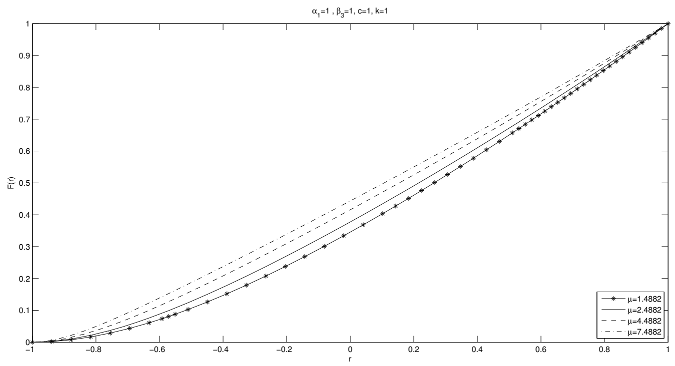

Figure 4: Travelling wave solutions of a velocity profile for various values of viscosity μ.

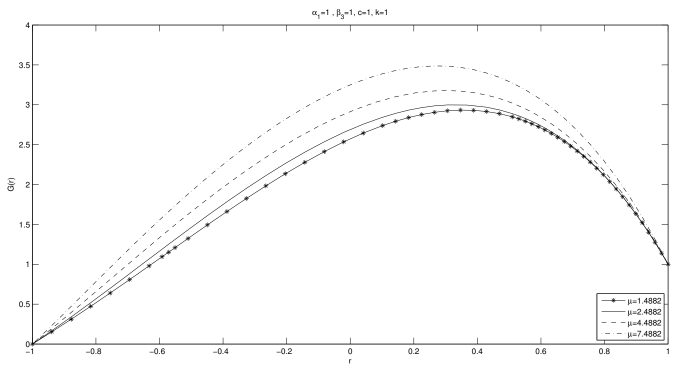

Figure 5: Travelling wave solutions of temperature profile for various values of viscosity μ.

In Figures 4-5, we showed that an increase in the viscosity parameter increases both the velocity and the temperature profiles.

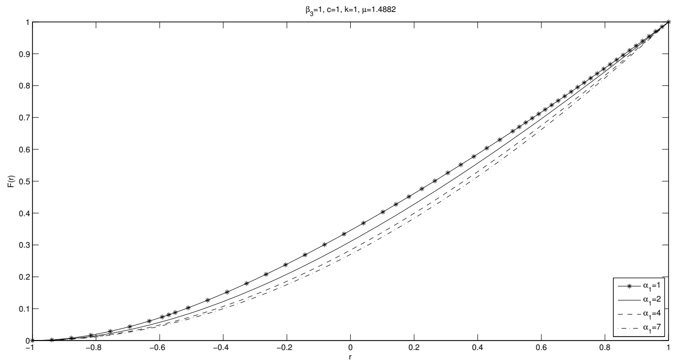

Figure 6: Travelling wave solutions of a velocity profile for various values of third-grade parameter α_1.

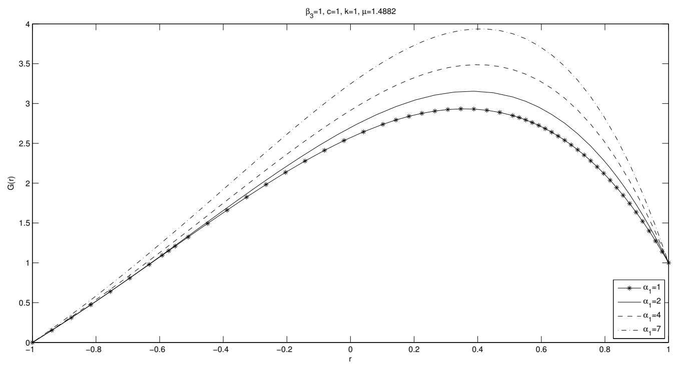

Figure 7: Travelling wave solutions of temperature profile for various values of third-grade parameter α_1.

In Figure 6, we have seen that the velocity decreases with increasing values of the parameter α_1. In Figure 7, it is observed that an increase in the parameter α_1 leads to a rapid increase in temperature.

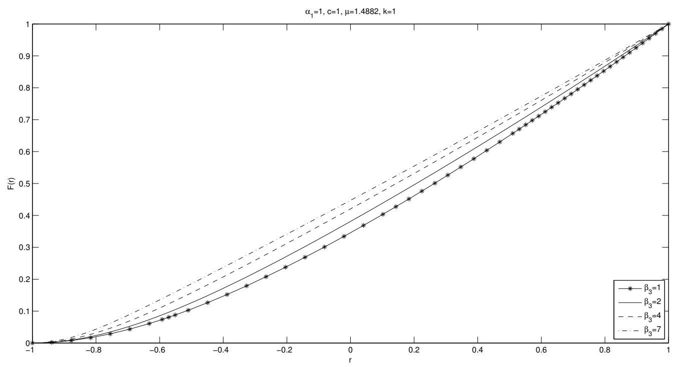

Figure 8: Travelling wave solutions of a velocity profile for various values of third-grade parameter β_3.

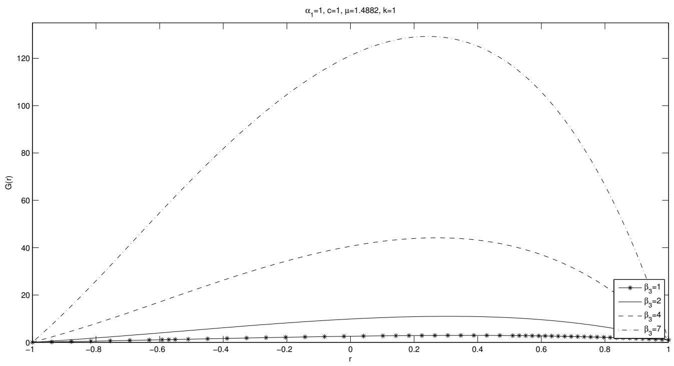

Figure 9: Travelling wave solutions of temperature profile for various values of third-grade parameter β_3.

In Figure 8, we have seen that the velocity profile increases as the third-grade parameter β_3 increases in magnitude. Given that, it is clear that β_3 reflects the property of shear thinning. On the other hand, we observed a rapid decrease of the temperature with decreasing values of the parameter β_3, as shown in Figure 9.

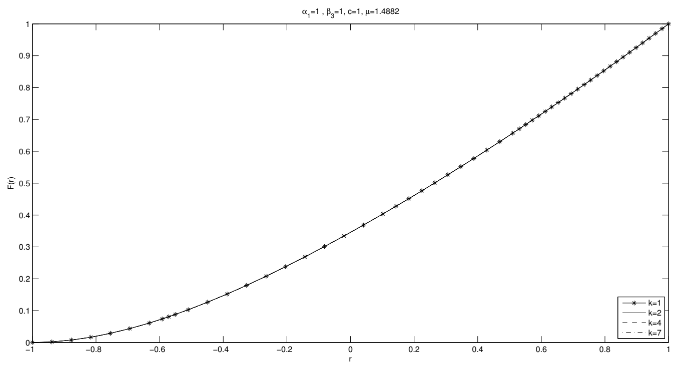

Figure 10: Travelling wave solutions of a velocity profile for various values of thermal conductivity k.

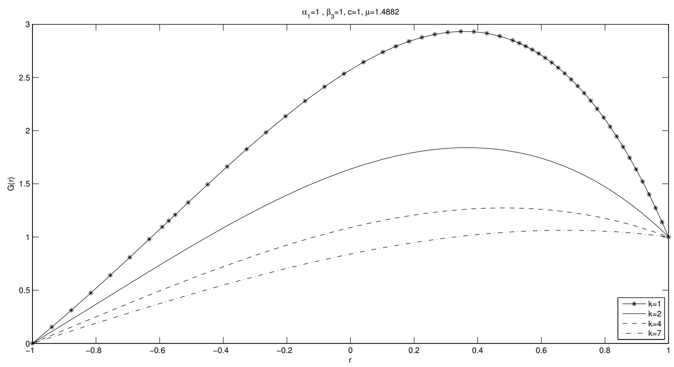

Figure 11: Travelling wave solutions of temperature profile for various values of thermal conductivity k.

In Figure 10, we did not observe a consequent variation velocity profile, for various values of the parameter k. On the other hand, in Figure 11, an increasing temperature with decreasing values of the parameter k is observed. All previous figures are by those related in the works [7, 11, 22].

6. Conclusion

In this paper, we applied the Lie group method to study the heat transfer of third-grade fluid between two parallel plates. The explicit exact solutions are obtained by similarity reductions. Furthermore, by using travelling wave solutions, we were able to highlight the influence of certain physical parameters on this heat flow. More generally, the Lie group analysis has shown to be an effective method for studying coupled highly nonlinear partial differential equations.

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