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Connectedness, local connectedness, and components on bipolar soft generalized topological spaces

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Abstract

Connectedness represents the most significant and fundamental topological property. It highlights the main characteristics of topological spaces and distinguishes one topology from another. There is a constant study of bipolar soft generalized topological spaces ($\mathcal{BSGTS}$s) by presenting $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space in $\mathcal{BSGTS}$s as well as it is discussing some properties and results for these topics. Additionally, the notion of bipolar soft disjoint sets is put forward, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation set, $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-hereditary property. Moreover, there is an extensive study of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-locally connected space and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-component with some related properties and theorems following them, such as the concepts of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-locally connected spaces and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected are independent of each other; also determined the conditions under which the $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected subsets are $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-components.

Keywords: $\mathcal{BSGTS}$, $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-locally connected space, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-component.

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1. Introduction

For the time being, researchers, on a daily basis, tackle the complexities of modeling vague/uncertain problems in varied domains like medical science, economics, engineering, sociology and computer science. Due to the frequent failure of the classical methods in accounting such types of problems, there have been some suggested novel approfhes, namely soft set, rough set, fuzzy set and bipolar soft set. Soft set, the gist of this manuscript, first came into existence in 1999 by Molodtsov [25]. This date has indicated researchers' application of the soft sets to various domains; examples of which are computer sciences, medical science and decision-making problem [1, 15, 22, 32, 42]. That was followed by Maji [16] introduction of the basic concepts pertinent to the soft set theory. The difference, the union and the intersection operators between two soft sets and a complement of a soft set were defined by these researchers.

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In an attempt to get rid of the defects in Maji's work, some researchers and scholars tended to reformulate some of the mentioned operators and provided some new types such as those by Maji in a way that facilitated the preservation of some results and properties of crisp set theory in soft set theory. Further contributions came in the form of defining several types of soft equality relations as those presented in [2, 5–9, 11, 14, 23, 24, 28, 33, 34, 38, 40, 44, 45].

In 2011, soft sets and a general topology were hybridized by Çagman [16] and Shabir and Naz [40] so as to start the notion of soft topological spaces. Different methods were adopted to give definitions of a soft topology. Çagman's definition attended to a soft topology over an absolute soft set and different sets of parameters. As for Shabir and Naz, they formulated a soft topology over fixed sets of the universe and parameters. In 2013, Shabir and Naz [41] explained that the bipolar soft set structure has clearer and more general results than the soft set structure. A year later, in 2014, the notion of soft generalized topological space ($\mathcal{SGTS}$) was put forward by Thomas and John [42], who defined it as an initial universe with a fixed set of parameters containing the soft union of any soft sets and soft null sets. They investigated some types of soft spaces such as soft connected spaces, soft compact spaces and soft separation axioms via soft generalized open sets. Different studies have accounted for the concept of bipolar soft sets, see [4, 10, 17–21, 39, 43], yet mathematicians need to study the limit point concept so as to get more developments in mathematics. Bipolarity play a significant role in characterization between the positive and negative information for excellence which is to be sensible occurrence and fail it [40]. The idea of the bipolar soft set has been a main point of focus for researchers. Also, bipolar soft separation axioms were explored and studied in detail by Shabir and Bakhtawar, who in 2016, introduced and studied the concepts bipolar soft connected and bipolar soft disconnected spaces.

In [30, 31], Öztürk defined a bipolar soft points and introduced other results on bipolar soft topological spaces such as interior points, adherent points and neighborhood. So, Musa and Asaad defined new type of topology by using a hypersoft sets and introduces bipolar hypersoft topological spaces. Addition of that, they studied some notions of bipolar hypersoft topological spaces such as interior, closure, exterior and connectness in this topology (see [26, 27, 29]). In 2022, Saleh et al. ([35, 36]) initiated the study of $\mathcal{BSGTS}$s by defining it as a collection of $\tilde{\tilde{\mathfrak{g}}}$ of bipolar soft sets over the universe Ω. Accordingly, the basic notions of $\mathcal{BSGTS}$s, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open and $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed sets, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closure, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-interior, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-limit point, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-boundary and bipolar soft generalized topological subspace ($\mathcal{BSGTSS}$) were defined and their several properties were investigated.

In the present work, we initiate some new ideas in $\mathcal{BSGTS}$s such as $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected sets, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected spaces and $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected spaces. Then, we devoted towards the idea of $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation sets, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-hereditary property, some examples are given for the better understanding of these ideas. Furthermore, we study the concept of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected spaces and $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components. So, some results related to these concepts are exhibited. In addition, we explore the concept of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components that the family of all $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components forms a partition for $\mathcal{BSGTS}$. We give some properties of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components in $\mathcal{BSGTS}$s.

2. Preliminaries

In this section, we introduce some basic concepts about bipolar soft sets and bipolar soft points. Throughout the present paper, $\Upsilon(\Omega)$ be the class of all subsets of an initial universe Ω. Let ϖ be a set of parameters and $\rho, \eta \subseteq \varpi$. Let $\mathcal{BSS}(\Omega)$ be the set of all bipolar soft sets over Ω with parameters ϖ. We recall some definitions and results related to $\mathcal{BSS}$ in $\mathcal{BSGTS}$s.

Definition 2.1 ([22])**.** The **Not** set of a set of parameters $\rho = \{\sigma_1, \sigma_2, \dots, \sigma_n\}$ is denoted by $\neg\rho$ and is defined as $\neg\rho = \{\neg\sigma_1, \neg\sigma_2, \dots, \neg\sigma_n\}$ where $\neg\sigma_i = \text{Not } \sigma_i$ for all $i = 1, 2, \dots, n$.

Definition 2.2 ([41])**.** A triple (Λ, Θ, ρ) is said to be a bipolar soft set on Ω, denoted by $\mathcal{BSS}$, where Λ and Θ are mappings defined by $\Lambda : \rho \longrightarrow \Upsilon(\Omega)$ and $\Theta : \neg\rho \longrightarrow \Upsilon(\Omega)$ in which $\Lambda(\sigma) \cap \Theta(\neg\sigma) = \phi$ for all $\sigma \in \rho$

and $\neg\sigma \in \neg\eta$. In other words, a $\mathcal{BSS}$ (Λ, Θ, ρ) can be written as:

$$(\Lambda, \Theta, \rho) = \{(\sigma, \Lambda(\sigma), \Theta(\neg\sigma)) : \sigma \in \rho, \Lambda(\sigma) \cap \Theta(\neg\sigma) = \phi\}.$$

Definition 2.3 ([41])**.** Let $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \eta)$ be two $\mathcal{BSS}$s, then we say that $(\Lambda_1, \Theta_1, \rho)$ is a bipolar soft subset of $(\Lambda_2, \Theta_2, \eta)$, denoted by $(\Lambda_1, \Theta_1, \rho) \mathrel{\tilde{\tilde{\subseteq}}} (\Lambda_2, \Theta_2, \rho)$, if:

1. $\rho \subseteq \eta$;
2. $\Lambda_1(\sigma) \subseteq \Lambda_2(\sigma)$ and $\Theta_2(\neg\sigma) \subseteq \Theta_1(\neg\sigma)$ for all $\sigma \in \rho$ and $\neg\sigma \in \neg\rho$.

Similarly, we say that $(\Lambda_1, \Theta_1, \rho)$ is a bipolar soft superset of $(\Lambda_2, \Theta_2, \eta)$, denoted by $(\Lambda_1, \Theta_1, \rho) \mathrel{\tilde{\tilde{\supseteq}}} (\Lambda_2, \Theta_2, \eta)$, if $(\Lambda_2, \Theta_2, \eta)$ is a bipolar soft subset of $(\Lambda_1, \Theta_1, \rho)$.

Definition 2.4 ([41])**.** Two $\mathcal{BSS}$s $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \eta)$ are called equal, which is denoted by $(\Lambda_1, \Theta_1, \rho) = (\Lambda_2, \Theta_2, \eta)$, if $(\Lambda_1, \Theta_1, \rho) \mathrel{\tilde{\tilde{\subseteq}}} (\Lambda_2, \Theta_2, \rho)$ and $(\Lambda_2, \Theta_2, \rho) \mathrel{\tilde{\tilde{\subseteq}}} (\Lambda_1, \Theta_1, \rho)$.

Definition 2.5 ([41])**.** The bipolar soft complement of a $\mathcal{BSS}$ (Λ, Θ, ρ) is denoted by $(\Lambda, \Theta, \rho)^c$ and defined by $(\Lambda, \Theta, \rho)^c = (\Lambda^c, \Theta^c, \rho)$ where Λ^c and Θ^c are mappings having $\Lambda^c(\sigma) = \Theta(\neg\sigma)$ and $\Theta^c(\neg\sigma) = \Lambda(\sigma)$ for all $\sigma \in \rho$ and $\neg\sigma \in \neg\rho$.

Definition 2.6 ([41])**.** A $\mathcal{BSS}$ (Λ, Θ, ρ) is called a relative null $\mathcal{BSS}$ (with respect to the parameter set ρ), which is denoted by $(\Phi, \tilde{\tilde{\Omega}}, \rho)$, if $\Lambda(\sigma) = \phi$ and $\Theta(\neg\sigma) = \Omega$ for all $\sigma \in \rho$ and for all $\neg\sigma \in \neg\rho$. The relative null $\mathcal{BSS}$ (with respect to the universe set of parameter ϖ) is called the null $\mathcal{BSS}$ on Ω, denoted by $(\Phi, \tilde{\tilde{\Omega}}, \varpi)$. Obviously, a $\mathcal{BSS}$ $(\Lambda, \Theta, \varpi)$ is said to be non-null $\mathcal{BSS}$ if $\Lambda(\sigma) \neq \phi$ for some $\sigma \in \varpi$.

Definition 2.7 ([41])**.** A $\mathcal{BSS}$ (Λ, Θ, ρ) is called a relative absolute $\mathcal{BSS}$ (with respect to the parameter set ρ), which is denoted by $(\tilde{\tilde{\Omega}}, \Phi, \rho)$, if $\Lambda(\sigma) = \Omega$ and $\Theta(\neg\sigma) = \phi$ for all $\sigma \in \rho$ and for all $\neg\sigma \in \neg\rho$. The relative absolute $\mathcal{BSS}$ (with respect to the universe set of parameter ϖ) is called the absolute $\mathcal{BSS}$ over Ω, denoted by $(\tilde{\tilde{\Omega}}, \Phi, \varpi)$. Obviously, a $\mathcal{BSS}$ $(\Lambda, \Theta, \varpi)$ is said to be non-absolute $\mathcal{BSS}$ if $\Lambda(\sigma) \neq \Omega$ for some $\sigma \in \varpi$.

Definition 2.8 ([41])**.** Let $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \eta)$ be two $\mathcal{BSS}$s, then the bipolar soft intersection of these $\mathcal{BSS}$s is the $\mathcal{BSS}$ (χ, Ψ, κ), where $\kappa = \rho \cup \eta$ is a non-empty set and for all $\sigma \in \kappa$, we have

$$\chi(\sigma) = \begin{cases} \Lambda_1(\sigma), & \sigma \in \rho - \eta, \\ \Lambda_2(\sigma), & \sigma \in \eta - \rho, \\ \Lambda_1(\sigma) \cap \Lambda_2(\sigma), & \sigma \in \rho \cap \eta, \end{cases} \quad \text{and} \quad \Psi(\neg\sigma) = \begin{cases} \Lambda_1(\neg\sigma), & \neg\sigma \in \neg\rho - \neg\eta, \\ \Lambda_2(\neg\sigma), & \neg\sigma \in \neg\eta - \neg\rho, \\ \Lambda_1(\neg\sigma) \cup \Lambda_2(\neg\sigma), & \neg\sigma \in \neg\rho \cap \neg\eta. \end{cases}$$

It is denoted by $(\Lambda_1, \Theta_1, \rho) \mathbin{\tilde{\tilde{\cap}}} (\Lambda_2, \Theta_2, \eta) = (\chi, \Psi, \kappa)$.

Definition 2.9 ([41])**.** Let $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \eta)$ be two $\mathcal{BSS}$s, then the bipolar soft union of these $\mathcal{BSS}$s is the $\mathcal{BSS}$ (χ, Ψ, κ), where $\kappa = \rho \cup \eta$ is a non-empty set and for all $\sigma \in \kappa$, we have

$$\chi(\sigma) = \begin{cases} \Lambda_1(\sigma), & \sigma \in \rho - \eta, \\ \Lambda_2(\sigma), & \sigma \in \eta - \rho, \\ \Lambda_1(\sigma) \cup \Lambda_2(\sigma), & \sigma \in \rho \cap \eta, \end{cases} \quad \text{and} \quad \Psi(\neg\sigma) = \begin{cases} \Lambda_1(\neg\sigma), & \neg\sigma \in \neg\rho - \neg\eta, \\ \Lambda_2(\neg\sigma), & \neg\sigma \in \neg\eta - \neg\rho, \\ \Lambda_1(\neg\sigma) \cap \Lambda_2(\neg\sigma), & \neg\sigma \in \neg\rho \cap \neg\eta. \end{cases}$$

It is denoted by $(\Lambda_1, \Theta_1, \rho) \mathbin{\tilde{\tilde{\cup}}} (\Lambda_2, \Theta_2, \eta) = (\chi, \Psi, \kappa)$.

Definition 2.10 ([41])**.** The restricted union of two $\mathcal{BSS}$s $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \eta)$ over the common universe Ω is the $\mathcal{BSS}$ (χ, ψ, κ), where $\kappa = \rho \cap \eta$ is a non-empty set and for all $\sigma \in \kappa$, we have

$$\chi(\sigma) = \Lambda_1(\sigma) \cup \Lambda_2(\sigma) \text{ and } \psi(\neg\sigma) = \Theta_1(\neg\sigma) \cap \Theta_2(\neg\sigma).$$

It is denoted by $(\Lambda_1, \Theta_1, \rho) \,\widetilde{\widetilde{\cup}}_{\mathcal{R}}\, (\Lambda_2, \Theta_2, \eta) = (\chi, \psi, \kappa)$.

Definition 2.11 ([41])**.** The restricted intersection of two $\mathcal{BSS}$s $(\Lambda, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \eta)$ over the common universe Ω is the $\mathcal{BSS}$ (χ, ψ, κ), where $\kappa = \rho \cap \eta$ is a non-empty set and for all $\sigma \in \kappa$, we have

$$\chi(\sigma) = \Lambda_1(\sigma) \cap \Lambda_2(\sigma) \text{ and } \psi(\neg\sigma) = \Theta_1(\neg\sigma) \cup \Theta_2(\neg\sigma).$$

It is denoted by $(\Lambda_1, \Theta_1, \rho) \,\widetilde{\widetilde{\cap}}_{\mathcal{R}}\, (\Lambda_2, \Theta_2, \eta) = (\chi, \psi, \kappa)$.

Definition 2.12 ([18])**.** Let $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \eta)$ be two $\mathcal{BSS}$s, then the bipolar soft difference between these $\mathcal{BSS}$s is the $\mathcal{BSS}$ $(\Lambda, \Theta, \kappa)$, where $\kappa = \rho \cup \eta$, which is defined as:

$$(\Lambda_1, \Theta_1, \rho) \,\widetilde{\widetilde{\setminus}}\, (\Lambda_2, \Theta_2, \eta) = (\Lambda_1, \Theta_1, \rho) \,\widetilde{\widetilde{\cap}}\, (\Lambda_2, \Theta_2, \eta)^c.$$

Definition 2.13 ([41])**.** Let $\omega \in \Omega$. Then, a $\mathcal{BSS}$ $\big(\Lambda_\omega(\sigma), \Theta_\omega(\sigma), \rho\big)$ is defined by $\Lambda_\omega(\sigma)=\{\omega\}$ and $\Theta_\omega(\neg\sigma) = \Omega \setminus \{\omega\}$, for each $\sigma \in \rho$ and $\neg\sigma \in \neg\rho$.

Definition 2.14 ([35])**.** Let $\widetilde{\widetilde{\mathfrak{g}}}$ be the class of $\mathcal{BSS}$s, then $\widetilde{\widetilde{\mathfrak{g}}}$ is called a bipolar soft generalized topology ($\mathcal{BSGT}$) on Ω if the following conditions are satisfying:

1. $(\Phi, \widetilde{\widetilde{\Omega}}, \rho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$;
2. if $(\Lambda_j, \Theta_j, \rho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$ for all $j \in \mathcal{J}$, then $\widetilde{\widetilde{\bigcup}}_{j\in\mathcal{J}}(\Lambda_j, \Theta_j, \rho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$.

We called $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ by a bipolar soft generalized topological space ($\mathcal{BSGTS}$) over Ω. Every member of $\widetilde{\widetilde{\mathfrak{g}}}$ is said to be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set and its bipolar soft complement is said to be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closed set. Clearly, $(\Phi, \widetilde{\widetilde{\Omega}}, \rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open but $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$ need not to be $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open. If $(\widetilde{\widetilde{\Omega}}, \Phi, \rho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$, then we say that $\widetilde{\widetilde{\mathfrak{g}}}$ is strong $\mathcal{BSGT}$.

Definition 2.15 ([35])**.** Let $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ and $(\Lambda, \Theta, \rho) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$. We denote $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho)$ by $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closure of (Λ, Θ, ρ), which is

$$c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho) = \widetilde{\widetilde{\bigcap}}\{(\chi, \psi, \rho) : (\chi, \psi, \rho)^c \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}} \text{ and } (\chi, \psi, \rho) \,\widetilde{\widetilde{\supseteq}}\, (\Lambda, \Theta, \rho)\}.$$

Definition 2.16 ([30])**.** Let $(\Lambda, \Theta, \rho) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$ and θ be a non-empty subset of Ω. Then we denote $({}^\theta\Lambda, {}^\theta\Theta, \rho)$ by the sub bipolar soft set of (Λ, Θ, ρ) over θ, which is defined as follows

$${}^\theta\Lambda(\sigma) = \theta \cap \Lambda(\sigma) \text{ and } {}^\theta\Theta(\neg\sigma) = \theta \cap \Theta(\neg\sigma), \text{ for each } \sigma \in \rho \text{ and } \neg\sigma \in \neg\rho.$$

Proposition 2.17 ([30])**.** *Let $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ and θ be a non-empty subset of Ω. Then $\widetilde{\widetilde{\mathfrak{g}}}_\theta = \{({}^\theta\Lambda, {}^\theta\Theta, \rho) : (\Lambda, \Theta, \rho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}\}$ is a $\mathcal{BSGTS}$ on Ω.*

Theorem 2.18 ([35])**.** *Let $(\Omega, \widetilde{\mathfrak{g}}, \rho)$ be a $\mathcal{SGT}$. Then $\widetilde{\widetilde{\mathfrak{g}}}$ is the class including $\mathcal{BSS}$s (Λ, Θ, ρ) in which $(\Lambda, \rho) \in \widetilde{\mathfrak{g}}$ and $\Theta(\neg\rho) = \Omega \setminus \Lambda(\sigma)$ for all $\sigma \in \rho$ and $\neg\sigma \in \neg\rho$ defines a $\mathcal{BSGT}$ on Ω.*

Definition 2.19 ([39])**.** Two $\mathcal{BSS}$s $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are said to be disjoint $\mathcal{BSS}$s if $\Lambda_1(\sigma) \cap \Lambda_2(\sigma) = \phi$ for all $\sigma \in \rho$.

Proposition 2.20 ([39])**.** *Let $(\Theta, \Lambda, \sigma) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$. Then*

1. $(\Theta, \Lambda, \sigma) \,\widetilde{\widetilde{\cup}}\, (\Theta, \Lambda, \sigma)^c = (\chi, \Phi, \sigma)$*, where* $\chi(\rho) = \Theta(\rho) \cup \Theta^c(\rho) \subseteq \Omega$ *for each* $\rho \in \sigma$ *and* $\Phi(\neg\rho) = \Lambda(\neg\rho) \cap \Lambda^c(\neg\rho) = \phi$ *for each* $\neg\rho \in \neg\sigma$.

2. $(\Theta, \Lambda, \sigma) \,\tilde{\tilde{\cap}}\, (\Theta, \Lambda, \sigma)^c = (\Phi, \psi, \sigma)$, *where* $\Phi(\rho) = \Theta(\rho) \cap \Theta^c(\rho) = \phi$ *for each* $\rho \in \sigma$ *and* $\psi(\neg\rho) = \Lambda(\neg\rho) \cup \Lambda^c(\neg\rho) \subseteq \Omega$ *for each* $\neg\rho \in \neg\sigma$. *Further* $(\Theta, \Lambda, \sigma)$, $(\Theta, \Lambda, \sigma)^c$ *will always satisfy* $\Theta(\rho) \cup \Theta^c(\rho) = \Lambda(\neg\rho) \cup \Lambda^c(\neg\rho)$ *for all* $\rho \in \sigma$.
3. $(\Theta, \Lambda, \sigma) \,\tilde{\tilde{\cup}}\, (\tilde{\tilde{\Omega}}, \Phi, \sigma) = (\tilde{\tilde{\Omega}}, \Phi, \sigma)$ *and* $(\Theta, \Lambda, \sigma) \,\tilde{\tilde{\cap}}\, (\tilde{\tilde{\Omega}}, \Phi, \sigma) = (\Theta, \Lambda, \sigma)$.

Definition 2.21 ([36])**.** Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ over Ω and $\tilde{\tilde{\mathfrak{B}}} \,\tilde{\tilde{\subseteq}}\, \tilde{\tilde{\mathfrak{g}}}$. Then, $\tilde{\tilde{\mathfrak{B}}}$ is said to be a bipolar soft generalized basis for the $\mathcal{BSGT}$ $\tilde{\tilde{\mathfrak{g}}}$, denoted by, $\mathcal{BSGB}$ if every element in $\tilde{\tilde{\mathfrak{g}}}$ can be written as the bipolar soft union of elements of $\tilde{\tilde{\mathfrak{B}}}$.

Definition 2.22 ([36])**.** Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ and $(\Lambda, \Theta, \rho) \,\tilde{\tilde{\in}}\, \mathcal{BSS}(\Omega)$. Then the collection

$$\tilde{\tilde{\mathfrak{g}}}_{(\Lambda,\Theta,\rho)} = \{(\Lambda, \Theta, \rho) \,\tilde{\tilde{\cap}}\, (\Lambda_\gamma, \Theta_\gamma, \rho) \colon (\Lambda_\gamma, \Theta_\gamma, \rho) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\mathfrak{g}}}, \gamma \in \Gamma\}.$$

Then $(\Omega_{(\Lambda,\Theta,\rho)}, \tilde{\tilde{\mathfrak{g}}}_{(\Lambda,\Theta,\rho)}, \rho, \neg\rho)$ is called a bipolar soft generalized subspace of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ and denoted by $\mathcal{BSGTSS}$.

Definition 2.23 ([10])**.** Let $(\Lambda, \Theta, \rho) \,\tilde{\tilde{\in}}\, \mathcal{BSS}(\Omega)$. The $\mathcal{BSS}$ (Λ, Θ, ρ) is called a bipolar soft point ($\mathcal{BSP}$) if there exist $\pi, \upsilon \in \Omega$, $\sigma \in \rho$ and $\neg\sigma \in \neg\rho$ such that

$$\Lambda(\gamma) = \begin{cases} \{\pi\}, & \gamma = \sigma, \\ \phi, & \gamma \in \rho \setminus \{\sigma\}, \end{cases} \quad \text{and} \quad \Theta(\gamma') = \begin{cases} \Omega \setminus \{\pi, \upsilon\}, & \gamma' = \neg\sigma, \\ \Omega, & \gamma' \in \neg\rho \setminus \{\neg\sigma\}. \end{cases}$$

We denoted $\mathcal{BSP}(\Lambda, \Theta, \rho)$ briefly by π^σ_υ and denoted the family of all $\mathcal{BSP}$s over Ω briefly by $\mathcal{BSP}(\Omega)_{(\rho,\neg\rho)}$.

Definition 2.24 ([31])**.** Let π^σ_υ, $\pi'^{\sigma'}_{\upsilon'} \,\tilde{\tilde{\in}}\, \mathcal{BSP}(\Omega)_{(\rho,\neg\rho)}$ be two $\mathcal{BSP}$s. Then π^σ_υ and $\pi'^{\sigma'}_{\upsilon'}$ are called different $\mathcal{BSP}$s if $\pi \neq \pi'$ or $\sigma \neq \sigma'$.

Definition 2.25 ([31])**.** Let $(\Lambda, \Theta, \rho) \,\tilde{\tilde{\in}}\, \mathcal{BSS}(\Omega)$ and $\pi^\sigma_\upsilon \,\tilde{\tilde{\in}}\, \mathcal{BSP}(\Omega)_{(\rho,\neg\rho)}$. Then π^σ_υ is said to be contained in (Λ, Θ, ρ), which is denoted by $\pi^\sigma_\upsilon \,\tilde{\tilde{\in}}\, (\Lambda, \Theta, \rho)$, if $\pi \in \Lambda(\sigma)$ and $\upsilon \in \Omega \setminus \Theta(\neg\sigma)$.

Definition 2.26 ([31])**.** Let π^σ_υ, $\pi'^{\sigma'}_{\upsilon'} \,\tilde{\tilde{\in}}\, \mathcal{BSP}(\Omega)_{(\rho,\neg\rho)}$ be two $\mathcal{BSP}$s. Then π^σ_υ and $\pi'^{\sigma'}_{\upsilon'}$ are called not different $\mathcal{BSP}$s if $\pi = \pi'$ and $\sigma = \sigma'$. Clearly $\upsilon = \upsilon'$ or $\upsilon \neq \upsilon'$.

Definition 2.27 ([36])**.** Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ defined on Ω and $\pi^\sigma_\upsilon \,\tilde{\tilde{\in}}\, \mathcal{BSP}(\Omega)_{(\rho,\neg\rho)}$. Then π^σ_υ is said to be $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit point of (Λ, Θ, ρ) if for each $(\chi, \psi, \rho) \,\tilde{\tilde{\in}}\, \tilde{\tilde{\mathfrak{g}}}$ such that $\pi^\sigma_\upsilon \,\tilde{\tilde{\in}}\, (\chi, \psi, \rho)$, we have

$$(\Lambda, \Theta, \rho) \,\tilde{\tilde{\cap}}\, (\chi, \psi, \rho) \setminus \{\pi^\sigma_\upsilon\} \neq (\Phi, \Theta, \rho).$$

In the other words, π^σ_υ is called a $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit point of (Λ, Θ, ρ) if every $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-neighborhood of π^σ_υ contains at least one $\mathcal{BSP}$ of (Λ, Θ, ρ) other than π^σ_υ. The $\mathcal{BSS}$ of all $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit points of (Λ, Θ, ρ) is called $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*derived set of (Λ, Θ, ρ) and it is denoted by $\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho)$. Now we can extend the definition of the $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit point to $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-limit point by making the last concept contains $\mathcal{BS}\,\tilde{\tilde{\mathfrak{g}}}$-*limit points to gather with all $\mathcal{BSP}$s which is not belong to each $\mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi^\sigma_\upsilon)$. The set of all $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}$-limit points denoted by $\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho)$, i.e., the set of all $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}$-limit points can be defined as

$$\mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho) = \mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho) \,\tilde{\tilde{\cup}}\, \{\pi^\sigma_\upsilon \,\tilde{\tilde{\in}}\, \mathcal{BSP}(\Omega)_{(\rho,\neg\rho)} \colon \pi^\sigma_\upsilon \,\tilde{\tilde{\notin}}\, \mathfrak{N}_{\tilde{\tilde{\mathfrak{g}}}}(\pi^\sigma_\upsilon)\}.$$

Remark 2.28 ([36]). Clearly, for any $(\Lambda, \Theta, \rho) \,\tilde{\tilde{\in}}\, \mathcal{BSS}(\Omega)$, we have $\mathfrak{d}^*_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho) \,\tilde{\tilde{\subseteq}}\, \mathfrak{d}_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda, \Theta, \rho)$.

3. $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected sets

This section introduces and defines one of the most important property of $\mathcal{BSGTS}$s called the bipolar soft generalized connected set, denoted by, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set and some concepts of $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set.

Definition 3.1. Let $(\Omega,\widetilde{\widetilde{\mathfrak{g}}},\rho,\neg\rho)$ be a $\mathcal{BSGTS}$ over Ω. Two non-null $\mathcal{BSS}$s $(\Lambda_1,\Theta_1,\eta)$ and $(\Lambda_2,\Theta_2,\eta)$ are said to be $\widetilde{\widetilde{\mathfrak{g}}}$-separated bipolar soft sets ($\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s) if $(\Lambda_1,\Theta_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\rho)=(\Phi,\Theta,\rho)$ and $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho)=(\Phi,\Theta,\rho)$.

Proposition 3.2. *Any two $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s are disjoint $\mathcal{BSS}$s.*

Proof. This is straightforward. □

The converse of the above proposition does not hold, i.e., two disjoint $\mathcal{BSS}$s are not necessarily $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s, we can explain by the next example.

Example 3.3. Let $\Omega=\{\omega_1,\omega_2,\omega_3,\omega_4\}$, $\rho=\{\sigma_1,\sigma_2\}$ and $\widetilde{\widetilde{\mathfrak{g}}}=\{(\Phi,\widetilde{\widetilde{\Omega}},\rho),(\Lambda_1,\Theta_1,\rho),(\Lambda_2,\Theta_2,\rho),(\Lambda_3,\Theta_3,\rho)\}$ be a $\mathcal{BSGTS}$ over Ω, where $(\Lambda_1,\Theta_1,\rho)$,$(\Lambda_2,\Theta_2,\rho)$,$(\Lambda_3,\Theta_3,\rho)\ \widetilde{\widetilde{\in}}\ \mathcal{BSS}(\Omega)$, defined as follows

$$\begin{aligned}(\Lambda_1,\Theta_1,\rho)&=\{(\sigma_1,\{\omega_2,\omega_3\},\{\omega_1\}),(\sigma_2,\{\omega_3,\omega_4\},\{\omega_2\})\},\\(\Lambda_2,\Theta_2,\rho)&=\{(\sigma_1,\{\omega_1,\omega_3\},\{\omega_2,\omega_4\}),(\sigma_2,\{\omega_1,\omega_3\},\{\omega_2\})\},\\(\Lambda_3,\Theta_3,\rho)&=\{(\sigma_1,\{\omega_1,\omega_2,\omega_3\},\phi),(\sigma_2,\{\omega_1,\omega_3,\omega_4\},\{\omega_2\})\}.\end{aligned}$$

Now, suppose that (χ_1,ψ_1,ρ) and (χ_2,ψ_2,ρ) are disjoint $\mathcal{BSS}$s over Ω define by

$$\begin{aligned}(\chi_1,\psi_1,\rho)&=\{(\sigma_1,\{\omega_1,\omega_2,\omega_4\},\{\omega_3\}),(\sigma_2,\{\omega_1,\omega_2,\omega_4\},\{\omega_3\})\},\\(\chi_2,\psi_2,\rho)&=\{(\sigma_1,\{\omega_3\},\{\omega_2\}),(\sigma_2,\{\omega_3\},\{\omega_2\})\}.\end{aligned}$$

Then $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\rho)=c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\rho)=(\widetilde{\widetilde{\Omega}},\Phi,\rho)$ and $(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\rho)=(\chi_1,\psi_1,\rho)$, $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\chi_2,\psi_2,\rho)=(\chi_2,\psi_2,\rho)$. But $(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\chi_2,\psi_2,\rho)=(\Phi,\psi,\rho)$. Therefore, the two $\mathcal{BSS}$s (χ_1,ψ_1,ρ) and (χ_2,ψ_2,ρ) are disjoint $\mathcal{BSS}$s but they are not $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s.

Proposition 3.4. *If $(\Lambda_1,\Theta_1,\rho)$ and $(\Lambda_1,\Theta_1,\rho)$ are two $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s over Ω with $(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_1,\Theta_1,\rho)$ and $(\chi_2,\psi_2,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_2,\Theta_2,\rho)$. Then, (χ_1,ψ_1,ρ) and (χ_2,ψ_2,ρ) are also $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s over Ω.*

Proof. Suppose that $(\Lambda_1,\Theta_1,\rho)$ and $(\Lambda_1,\Theta_1,\rho)$ are two $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s over Ω, thus

$$(\Lambda_1,\Theta_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\rho)=c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\eta)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho)=(\Phi,\Theta,\rho).$$

Since $(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_1,\Theta_1,\rho)$ and $(\chi_2,\psi_2,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_2,\Theta_2,\rho)$, then $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\rho)$ and $c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\rho)$. Therefore,

$$(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_2,\psi_2,\rho)=c_{\widetilde{\widetilde{\mathfrak{g}}}}(\chi_1,\psi_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\chi_2,\psi_2,\rho)=(\Phi,\psi,\rho).$$

Hence, (χ_1,ψ_1,ρ) and (χ_2,ψ_2,ρ) are $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s over Ω. □

Theorem 3.5. *Two $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closed subsets $(\Lambda_1,\Theta_1,\rho)$ and $(\Lambda_2,\Theta_2,\rho)$ of $\mathcal{BSGTS}$ $(\Omega,\widetilde{\widetilde{\mathfrak{g}}},\rho,\neg\rho)$ over Ω are $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s if and only if they are disjoint $\mathcal{BSS}$s.*

Proof. The necessity of condition is obvious. For sufficiency, suppose that $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are both $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed and disjoint $\mathcal{BSS}$s. Then, $(\Lambda_1, \Theta_1, \eta) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \rho) = (\Phi, \Theta, \rho)$ and $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1, \Theta_1, \rho) = (\Lambda_1, \Theta_1, \rho)$, $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2, \Theta_2, \rho) = (\Lambda_2, \Theta_2, \rho)$ and so that

$$(\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2, \Theta_2, \rho) = c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1, \Theta_1, \eta) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \rho) = (\Phi, \Theta, \rho)$$

showing that $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s over Ω. □

Remark 3.6. If we take two $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open sets $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ which are also disjoint $\mathcal{BSS}$s, then they may not be $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s.

Example 3.7. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\rho = \{\sigma_1, \sigma_2\}$ and $\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho)\}$ be a $\mathcal{BSGTS}$ over Ω where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho)$ $\tilde{\tilde{\in}}$ $\mathcal{BSS}(\Omega)$, defined as follows

$$\begin{aligned}
(\Lambda_1, \Theta_1, \rho) &= \{(\sigma_1, \{\omega_2\}, \{\omega_1, \omega_4\}), (\sigma_2, \{\omega_2\}, \{\omega_1, \omega_4\})\}, \\
(\Lambda_2, \Theta_2, \rho) &= \{(\sigma_1, \{\omega_3\}, \{\omega_1, \omega_4\}), (\sigma_2, \{\omega_3\}, \{\omega_1, \omega_4\})\}, \\
(\Lambda_3, \Theta_3, \rho) &= \{(\sigma_1, \{\omega_2, \omega_3\}, \{\omega_1, \omega_4\}), (\sigma_2, \{\omega_2, \omega_3\}, \{\omega_1, \omega_4\})\}.
\end{aligned}$$

Clearly $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$ are both $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open and disjoint $\mathcal{BSS}$s but not $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s because $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1, \Theta_1, \rho) = c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2, \Theta_2, \rho) = (\tilde{\tilde{\Omega}}, \Phi, \rho)$. Implies that $(\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2, \Theta_2, \rho) = (\Lambda_1, \Theta_1, \rho)$, $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \rho) = (\Lambda_2, \Theta_2, \rho)$. Therefore, $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open and disjoint $\mathcal{BSS}$s but they are not $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s.

Definition 3.8. A $\mathcal{BS}$ subset (Λ, Θ, ρ) of $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ over Ω is said to be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set over Ω if there is no $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s of (Λ, Θ, ρ). Otherwise, a $\mathcal{BSS}$ (Λ, Θ, ρ) is said to be $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected set over Ω. The $\mathcal{BSS}$s $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are said to be $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnection $\mathcal{BSS}$s of (Λ, Θ, ρ).

Remark 3.9. The null $\mathcal{BSS}$ $(\Phi, \tilde{\tilde{\Omega}}, \rho)$ is trivially $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set over Ω. So, every singleton $\mathcal{BSS}$ such as π^{σ}_{υ} is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set over Ω because it can not expressed as a bipolar soft union of two non-null $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s.

Definition 3.10. Let π^{σ}_{υ}, $\pi'^{\sigma'}_{\upsilon'}$ $\tilde{\tilde{\in}}$ $\mathcal{BSP}(\Omega)_{(\rho, \neg\rho)}$ of a $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$. Then, π^{σ}_{υ} and $\pi'^{\sigma'}_{\upsilon'}$ are said to be $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected points if they are contained in $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set over Ω.

Proposition 3.11. *Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ over Ω and (Λ, Θ, ρ) be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set such that (Λ, Θ, ρ) $\tilde{\tilde{\subseteq}}$ $(\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cup}} (\Lambda_2, \Theta_2, \rho)$, where $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s. Then $(\Lambda, \Theta, \rho) \tilde{\tilde{\subseteq}} (\Lambda_1, \Theta_1, \rho)$ or $(\Lambda, \Theta, \rho) \tilde{\tilde{\subseteq}} (\Lambda_2, \Theta_2, \rho)$.*

Proof. From $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s, then $(\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cap}} c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2, \Theta_2, \rho) = (\Phi, \Theta, \rho)$ and $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \rho) = (\Phi, \Theta, \rho)$. Since $(\Lambda, \Theta, \rho) \tilde{\tilde{\subseteq}} (\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cup}} (\Lambda_2, \Theta_2, \rho)$, then $(\Lambda, \Theta, \rho) = (\Lambda, \Theta, \rho) \tilde{\tilde{\cap}} \big((\Lambda_1, \Theta_1, \rho) \tilde{\tilde{\cup}} (\Lambda_2, \Theta_2, \rho)\big) = \big((\Lambda, \Theta, \rho) \tilde{\tilde{\cap}} (\Lambda_1, \Theta_1, \rho)\big) \tilde{\tilde{\cup}} ((\Lambda, \Theta, \rho) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \rho))$. We claim that at least one of the $\mathcal{BSS}$s $\big((\Lambda, \Theta, \rho) \tilde{\tilde{\cap}} (\Lambda_1, \Theta_1, \rho)\big)$ and $\big((\Lambda, \Theta, \rho) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \rho)\big)$ is null $\mathcal{BSS}$. Now, suppose that if possible non of these $\mathcal{BSS}$s is null, thus,

$$(\Lambda, \Theta, \rho) \tilde{\tilde{\cap}} (\Lambda_1, \Theta_1, \rho) \neq (\Phi, \Theta, \rho) \text{ and } (\Lambda, \Theta, \rho) \tilde{\tilde{\cap}} (\Lambda_2, \Theta_2, \rho) \neq (\Phi, \Theta, \rho).$$

Hence,

$$\begin{aligned}
((\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_1,\Theta_1,\rho))\ \widetilde{\widetilde{\sqcap}}\ c_{\tilde{\tilde{\mathfrak{g}}}}\big((\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho)\big) & \\
\widetilde{\widetilde{\sqsubseteq}}\ \big((\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_1,\Theta_1,\rho)\big)\ \widetilde{\widetilde{\sqcap}}\ \big(c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\rho)\big) & \\
= \big((\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)\big)\ \widetilde{\widetilde{\sqcap}}\ \big((\Lambda_1,\Theta_1,\rho)\ \widetilde{\widetilde{\sqcap}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\rho)\big) & \\
= (\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Phi,\Theta,\rho) = (\Phi,\Theta,\rho). &
\end{aligned}$$

Similarly,

$$c_{\tilde{\tilde{\mathfrak{g}}}}\big((\Lambda,\Theta,\rho)\widetilde{\widetilde{\sqcap}}(\Lambda_1,\Theta_1,\rho)\big)\widetilde{\widetilde{\sqcap}}\big((\Lambda,\Theta,\rho)\widetilde{\widetilde{\sqcap}}(\Lambda_2,\Theta_2,\rho)\big) = (\Phi,\Theta,\rho).$$

Hence, $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_1,\Theta_1,\rho)$ and $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho)$ are $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s. Thus, (Λ,Θ,ρ) can be expressed as bipolar soft union of two $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s. Therefore, (Λ,Θ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected. Which is a contradiction. Hence, at least one of the $\mathcal{BSS}$s $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_1,\Theta_1,\rho)$ and $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho)$ is null $\mathcal{BSS}$. Now, if $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_1,\Theta_1,\rho) = (\Phi,\Theta,\rho)$, then $(\Lambda,\Theta,\rho) = (\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho)$, which implies that $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_2,\Theta_2,\rho)$. If $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho) = (\Phi,\Theta,\rho)$, then $(\Lambda,\Theta,\rho) = (\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_1,\Theta_1,\rho)$, which implies that $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_1,\Theta_1,\rho)$. Therefore, either $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_1,\Theta_1,\rho)$ or $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_2,\Theta_2,\rho)$. □

Proposition 3.12. *Let (Λ,Θ,ρ) be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected and $(\chi,\psi,\rho)\ \widetilde{\widetilde{\in}}\ \mathcal{BSS}(\Omega)$ such that $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\chi,\psi,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)$. Then (χ,ψ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. In particular, $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)$ also is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected.*

Proof. Assume that (χ,ψ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected. Then, there exist non-null $\mathcal{BSS}$s $(\Lambda_1,\Theta_1,\rho)$ and $(\Lambda_2,\Theta_2,\rho)$ such that

$$(\Lambda_1,\Theta_1,\rho)\widetilde{\widetilde{\sqcap}}c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_2,\Theta_2,\rho) = c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\rho)\widetilde{\widetilde{\sqcap}}(\Lambda_2,\Theta_2,\rho) = (\Phi,\Theta,\rho), \quad (\chi,\psi,\rho) = (\Lambda_1,\Theta_1,\rho)\widetilde{\widetilde{\sqcup}}(\Lambda_2,\Theta_2,\rho).$$

From $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\chi,\psi,\rho) = (\Lambda_1,\Theta_1,\rho)\ \widetilde{\widetilde{\sqcup}}\ (\Lambda_2,\Theta_2,\rho)$, it follows from Proposition 3.11 that $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_1,\Theta_1,\rho)$ or $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_2,\Theta_2,\rho)$. Let $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\Lambda_1,\Theta_1,\rho)$, thus, $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\rho)$, then,

$$c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)\widetilde{\widetilde{\sqcap}}(\Lambda_2,\Theta_2,\rho)\widetilde{\widetilde{\sqsubseteq}}c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda_1,\Theta_1,\rho)\widetilde{\widetilde{\sqcap}}(\Lambda_2,\Theta_2,\rho) = (\Phi,\Theta,\rho),$$

but $(\Phi,\Theta,\rho)\widetilde{\widetilde{\sqsubseteq}}c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)\widetilde{\widetilde{\sqcap}}(\Lambda_2,\Theta_2,\rho)$, therefore,

$$c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)\widetilde{\widetilde{\sqcap}}(\Lambda_2,\Theta_2,\rho) = (\Phi,\Theta,\rho).$$

So, $(\Lambda_1,\Theta_1,\rho)\ \widetilde{\widetilde{\sqcup}}\ (\Lambda_2,\Theta_2,\rho) = (\chi,\psi,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)$, then, $(\Lambda_2,\Theta_2,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\chi,\psi,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)$ implies that $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqcap}}\ (\Lambda_2,\Theta_2,\rho) = (\Lambda_2,\Theta_2,\rho)$. Hence, $(\Lambda_2,\Theta_2,\rho) = (\Phi,\Theta,\rho)$. This is a contradiction because $(\Lambda_2,\Theta_2,\rho)$ is non-null $\mathcal{BSS}$. Therefore, (χ,ψ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. Also, from $(\Lambda,\Theta,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ (\chi,\psi,\rho)\ \widetilde{\widetilde{\sqsubseteq}}\ c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)$, implies that $c_{\tilde{\tilde{\mathfrak{g}}}}(\Lambda,\Theta,\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. □

Proposition 3.13. *Let $\{(\Lambda_\gamma,\Theta_\gamma,\rho) : \gamma\in\Gamma\}$ be the collection of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected sets such that $\widetilde{\widetilde{\bigcap}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho) \neq (\Phi,\Theta,\rho)$. Then $\widetilde{\widetilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected.*

Proof. Suppose $(\chi,\psi,\rho)=\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)$ is not $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. Then, there exist two non-null disjoint $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open sets (χ_1,ψ_1,ρ) and (χ_2,ψ_2,ρ) such that $(\chi,\psi,\rho)=(\chi_1,\psi_1,\rho)\ \tilde{\tilde{\cup}}\ (\chi_2,\psi_2,\rho)$. For each $\gamma\in\Gamma$, $(\chi_1,\psi_1,\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)$ and $(\chi_2,\psi_2,\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)$ are disjoint $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open sets in $(\Lambda_\gamma,\Theta_\gamma,\rho)$ such that

$$((\chi_1,\psi_1,\rho)\tilde{\tilde{\cap}}(\Lambda_\gamma,\Theta_\gamma,\rho))\tilde{\tilde{\cup}}((\chi_2,\psi_2,\rho)\tilde{\tilde{\cap}}(\Lambda_\gamma,\Theta_\gamma,\rho))=((\chi_1,\psi_1,\rho)\tilde{\tilde{\cup}}(\chi_2,\psi_2,\rho))\tilde{\tilde{\cap}}(\Lambda_\gamma,\Theta_\gamma,\rho)=(\Lambda_\gamma,\Theta_\gamma,\rho).$$

Now, from $(\Lambda_\gamma,\Theta_\gamma,\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected sets, one of the $\mathcal{BSS}$s $(\chi_1,\psi_1,\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)$ and $(\chi_2,\psi_2,\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)$ is a null $\mathcal{BSS}$s, say, $(\chi_1,\psi_1,\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)=(\Phi,\psi,\rho)$. Then, $(\chi_2,\psi_2,\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)=(\Lambda_\gamma,\Theta_\gamma,\rho)$, which implies that $(\Lambda_\gamma,\Theta_\gamma,\rho)\ \tilde{\tilde{\subseteq}}\ (\chi_2,\psi_2,\rho)$ for all $\gamma\in\Gamma$ and hence $\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)\ \tilde{\tilde{\subseteq}}\ (\chi_2,\psi_2,\rho)$, that is, $(\chi_1,\psi_1,\rho)\ \tilde{\tilde{\cup}}\ (\chi_2,\psi_2,\rho)\ \tilde{\tilde{\subseteq}}\ (\chi_2,\psi_2,\rho)$. This given, $(\chi_1,\psi_1,\rho)=(\Phi,\psi,\rho)$. This is a contradiction because (χ_1,ψ_1,ρ) is non-null $\mathcal{BSS}$. Hence, (χ,ψ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. □

Proposition 3.14. *For any two $\mathcal{BSP}$s π^{σ}_{υ}, $\pi'^{\sigma'}_{\upsilon'}\ \tilde{\tilde{\in}}\ (\Lambda,\Theta,\rho)\ \tilde{\tilde{\in}}\ \mathcal{BSS}(\Omega)$ in a $\mathcal{BSGTS}$ $(\Omega,\tilde{\tilde{\mathfrak{g}}},\rho,\neg\rho)$ are contained in some $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set $(\chi,\psi,\rho)\ \tilde{\tilde{\subseteq}}\ (\Lambda,\Theta,\rho)$, then (Λ,Θ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected.*

Proof. Suppose that the given conditions are satisfied and (Λ,Θ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected set. Then, there exists a $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s $(\Lambda_1,\Theta_1,\rho)$ and $(\Lambda_2,\Theta_2,\rho)$ of (Λ,Θ,ρ). Therefore, there are two $\mathcal{BSP}$s π^{σ}_{υ}, $\pi'^{\sigma'}_{\upsilon'}$ such that $\pi^{\sigma}_{\upsilon}\ \tilde{\tilde{\in}}\ (\Lambda_1,\Theta_1,\rho)$ and $\pi'^{\sigma'}_{\upsilon'}\ \tilde{\tilde{\in}}\ (\Lambda_2,\Theta_2,\rho)$. By hypothesis, there is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set (χ,ψ,ρ) containing π^{σ}_{υ}, $\pi'^{\sigma'}_{\upsilon'}$ such that

$$(\chi,\psi,\rho)\tilde{\tilde{\subseteq}}(\Lambda,\Theta,\rho)=(\Lambda_1,\Theta_1,\rho)\tilde{\tilde{\cup}}(\Lambda_2,\Theta_2,\rho).$$

Thus, by Proposition 3.11, we have $(\chi,\psi,\rho)\ \tilde{\tilde{\subseteq}}\ (\Lambda_1,\Theta_1,\rho)$ or $(\chi,\psi,\rho)\ \tilde{\tilde{\subseteq}}\ (\Lambda_2,\Theta_2,\rho)$. This leads to

$$(\Lambda_1,\Theta_1,\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_2,\Theta_2,\rho)\neq(\Phi,\Theta,\rho).$$

This is a contradiction because $(\Lambda_1,\Theta_1,\rho)$ and $(\Lambda_2,\Theta_2,\rho)$ are $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s. Hence, (Λ,Θ,ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. □

Proposition 3.15. *Let $\{(\Lambda_\gamma,\Theta_\gamma,\rho):\gamma\in\Gamma\}$ be the collection of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected sets in which one of the members of this collection intersects every other member. So, $\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected.*

Proof. Let $(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)$ be a fixed member of the given family such that $(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)\ \tilde{\tilde{\cap}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)\neq(\Phi,\Theta\rho)$ for every $\gamma\in\Gamma$. Then, $(\chi_\gamma,\psi_\gamma,\rho)=(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)\ \tilde{\tilde{\cup}}\ (\Lambda_\gamma,\Theta_\gamma,\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set for every $\gamma\in\Gamma$, by Proposition 3.14. Now,

$$\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)=\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}((\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)\tilde{\tilde{\cup}}(\Lambda_\gamma,\Theta_\gamma,\rho))=(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)\tilde{\tilde{\cup}}(\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)).$$

Since $(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)$ is one of the family $\{(\Lambda_\gamma,\Theta_\gamma,\rho):\gamma\in\Gamma\}$ and

$$\tilde{\tilde{\bigcap}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)=\tilde{\tilde{\bigcap}}_{\gamma\in\Gamma}((\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)\tilde{\tilde{\cup}}(\Lambda_\gamma,\Theta_\gamma,\rho))=(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)\tilde{\tilde{\cap}}(\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho))\neq(\Phi,\tilde{\tilde{\Omega}},\rho).$$

From $(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)$ intersects every $(\Lambda_\gamma,\Theta_\gamma,\rho)$. Therefore, $(\Lambda_{\gamma_0},\Theta_{\gamma_0},\rho)\neq(\Phi,\tilde{\tilde{\Omega}},\rho)$. Hence, by Proposition 3.13, $\tilde{\tilde{\bigcup}}_{\gamma\in\Gamma}(\Lambda_\gamma,\Theta_\gamma,\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. □

Proposition 3.16. *If (Λ, Θ, ρ) is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected subset of a $\mathcal{BSGTS}$ $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ such that $(\Lambda, \Theta, \rho) \,\widetilde{\widetilde{\subseteq}}\, (\Lambda_1, \Theta_1, \rho) \,\widetilde{\widetilde{\cup}}\, (\Lambda_2, \Theta_2, \rho)$ where $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are both $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closed and non-null disjoint $\mathcal{BSS}$s, then, $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are $\widetilde{\widetilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s.*

Proof. From Proposition 3.11 and Theorem 3.5. □

Proposition 3.17. *For every two π^{σ}_{υ}, $\pi'^{\sigma'}_{\upsilon'} \,\widetilde{\widetilde{\in}}\, \mathcal{BSP}(\Omega)_{(\rho,\neg\rho)}$ of a $\mathcal{BSGTS}$ $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ are $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected, then $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ is $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected.*

Proof. Let π^{σ}_{υ} be a fixed $\mathcal{BSP}$ in a $\mathcal{BSGTS}$ $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$. Then, for each π^{σ}_{υ} bipolar soft different than $\pi'^{\sigma'}_{\upsilon'}$, we have a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set, say, (Λ, Θ, ρ) containing π^{σ}_{υ} and $\pi'^{\sigma'}_{\upsilon'}$. Since $\pi^{\sigma}_{\upsilon} \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\bigcap}}_{\pi^{\sigma}_{\upsilon} \widetilde{\widetilde{\in}} (\widetilde{\widetilde{\Omega}}, \Phi, \rho)} (\Lambda, \Theta, \rho)$, it follows from Proposition 3.13 that $\widetilde{\widetilde{\bigcup}}_{\pi^{\sigma}_{\upsilon} \widetilde{\widetilde{\in}} (\widetilde{\widetilde{\Omega}}, \Phi, \rho)} (\Lambda, \Theta, \rho) = (\widetilde{\widetilde{\Omega}}, \Phi, \rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected. □

4. $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected spaces

In this section, we discuss and explore some concepts of $\mathcal{BSGTS}$s such as bipolar soft generalized connected space (denoted by $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space) and some results of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space. Clearly, any $\mathcal{BSGTS}$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space if $(\widetilde{\widetilde{\Omega}}, \Phi, \rho) \,\widetilde{\widetilde{\notin}}\, \widetilde{\widetilde{\mathfrak{g}}}$. Thus, we assume that $(\widetilde{\widetilde{\Omega}}, \Phi, \rho) \,\widetilde{\widetilde{\in}}\, \widetilde{\widetilde{\mathfrak{g}}}$. Therefore, the $\mathcal{BSGTS}$ in this study is called strong $\mathcal{BSGTS}$ denoted by $\mathcal{SBSGTS}$.

Definition 4.1. Let $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ over Ω. A $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$ is a pair $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ of non-null disjoint $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open sets over Ω such that $\Lambda_1(\sigma) \cup \Lambda_1(\sigma) = \Omega$ for all $\sigma \in \rho$.

Definition 4.2. A $\mathcal{BSGTS}$ $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ is said to be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space if there are no a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$, i.e., there are no two non-null disjoint $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open sets, say, $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ such that $\Lambda_1(\sigma) \cup \Lambda_2(\sigma) = \Omega$ for all $\sigma \in \rho$. Otherwise, $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ is said to be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space. Observe that if $|\ \Omega\ |= 1$, there exist only two $\mathcal{BSGTS}$ in Ω (i.e., $(\Phi, \widetilde{\widetilde{\Omega}}, \rho), (\widetilde{\widetilde{\Omega}}, \Phi, \rho)$) are a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected. Therefore we especially assume $|\ \Omega\ |> 1$.

Example 4.3. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ be the universe set representing "watches shop". Let $\rho = \{\sigma_1, \sigma_2, \sigma_3\}$ = {classic watches, brand watches, expensive watches} and $\neg\rho = \{\neg\sigma_1, \neg\sigma_2, \neg\sigma_3\}$ ={modern watches, copy watches, cheap watches}. Let $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ be represents the preference of shopping for selection of watches by two men. Then the $\mathcal{BSGTS}$ over Ω generated by $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ is given by $\widetilde{\widetilde{\mathfrak{g}}} = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho)\}$, where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$ are defined as follows:

$$(\Lambda_1, \Theta_1, \rho) = \{(\sigma_1, \{\omega_1, \omega_3\}, \{\omega_2\}), (\sigma_2, \{\omega_2, \omega_3\}, \{\omega_1, \omega_4\}), (\sigma_3, \{\omega_1, \omega_2\}, \{\omega_3\})\},$$
$$(\Lambda_2, \Theta_2, \rho) = \{(\sigma_1, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2\}), (\sigma_2, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}), (\sigma_3, \{\omega_1, \omega_4\}, \phi\},$$
$$(\Lambda_3, \Theta_3, \rho) = \{(\sigma_1, \{\omega_1, \omega_3, \omega_4\}, \{\omega_2\}), (\sigma_2, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}), (\sigma_3, \{\omega_1, \omega_2, \omega_4\}, \phi)\}.$$

Then $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space since there does not exists a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$.

Example 4.4. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ be the universe set and $\rho = \{\sigma_1, \sigma_2\}$. Then the $\mathcal{BSGTS}$ $\widetilde{\widetilde{\mathfrak{g}}}$ over Ω is given by $\widetilde{\widetilde{\mathfrak{g}}} = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho), (\widetilde{\widetilde{\Omega}}, \Phi, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho)\}$, where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho) \,\widetilde{\widetilde{\in}}\, \mathcal{BSS}(\Omega)$ are defined as follows:

$$(\Lambda_1, \Theta_1, \rho) = \{(\sigma_1, \{\omega_1\}, \{\omega_2\}), (\sigma_2, \{\omega_1\}, \{\omega_2\})\},$$

$$(\Lambda_2, \Theta_2, \rho) = \{(\sigma_1, \{\omega_2, \omega_3\}, \{\omega_1\}), (\sigma_2, \{\omega_2, \omega_3\}, \{\omega_1\})\},$$
$$(\Lambda_3, \Theta_3, \rho) = \{(\sigma_1, \{\omega_1, \omega_3\}, \{\omega_2\}), (\sigma_2, \{\omega_1, \omega_3\}, \{\omega_2\})\}.$$

Therefore, $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space because $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ form a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$.

Theorem 4.5. *A $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ over Ω is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space if and only if there exist two $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed sets $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ with $\Theta_1(\neg\sigma) \neq \phi$, $\Theta_2(\neg\sigma) \neq \phi$ for some $\neg\sigma \in \neg\rho$, such that $\Theta_1(\neg\sigma) \cup \Theta_2(\neg\sigma)$ $= \Omega$ for all $\neg\sigma \in \neg\rho$ and $\Theta_1(\neg\sigma) \cap \Theta_2(\neg\sigma) = \phi$ for all $\neg\sigma \in \neg\rho$.*

Proof. Assume that $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space. Thus, there exists a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$, say, $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$. Then,

$$\Lambda_1(\sigma) \cup \Lambda_2(\sigma) = \Omega \text{ for all } \sigma \in \rho,$$
$$\Lambda_1(\sigma) \cap \Lambda_2(\sigma) = \phi \text{ for all } \sigma \in \rho,$$
$$\Lambda_1(\sigma) \neq \phi, \Lambda_2(\sigma) \neq \phi \text{ for some } \sigma \in \rho.$$

Since $\Lambda_1(\sigma) = \Theta_1^c(\neg\sigma)$ and $\Lambda_2(\sigma) = \Theta_2^c(\neg\sigma)$. Now, we get

$$\Theta_1^c(\neg\sigma) \cup \Theta_2^c(\neg\sigma) = \Omega \text{ for all } \sigma \in \rho,$$
$$\Theta_1^c(\neg\sigma) \cap \Theta_2^c(\neg\sigma) = \phi \text{ for all } \sigma \in \rho,$$
$$\Theta_1^c(\sigma) \neq \phi, \Lambda_2(\sigma) \neq \phi \text{ for some } \sigma \in \rho.$$

From, $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{\mathfrak{g}}}$, then $(\Lambda_1, \Theta_1, \rho)^c$ and $(\Lambda_2, \Theta_2, \rho)^c$ are $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed sets. Conversely, suppose there exist two $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed sets $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ with

$$\Theta_1(\neg\sigma) \cup \Theta_2(\neg\sigma) = \Omega \text{ for all } \neg\sigma \in \neg\rho,$$
$$\Theta_1(\neg\sigma) \cap \Theta_2(\neg\sigma) = \phi \text{ for all } \neg\sigma \in \neg\rho,$$
$$\Theta_1(\neg\sigma) \neq \phi, \Theta_2(\neg\sigma) \neq \phi \text{ for some } \neg\sigma \in \neg\rho.$$

Then $(\Lambda_1, \Theta_1, \rho)^c$ and $(\Lambda_2, \Theta_2, \rho)^c$ are $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open sets with

$$\Lambda_1^c(\sigma) = \Theta_1(\neg\sigma) \neq \phi \text{ and } \Lambda_2^c(\sigma) = \Theta_2(\neg\sigma) \neq \phi \text{ for some } \sigma \in \rho,$$
$$\Lambda_1^c(\sigma) \cup \Lambda_2^c(\sigma) = \Theta_1(\neg\sigma) \cup \Theta_2(\neg\sigma) = \Omega \text{ for all } \sigma \in \rho,$$
$$\Lambda_1^c(\sigma) \cap \Lambda_2^c(\sigma) = \Theta_1(\neg\sigma) \cap \Theta_2(\neg\sigma) = \phi \text{ for all } \sigma \in \rho.$$

Therefore, $(\Lambda_1, \Theta_1, \rho)^c$ and $(\Lambda_2, \Theta_2, \rho)^c$ form a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$. Hence, $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space. □

Proposition 4.6. *The bipolar soft intersection of two $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected spaces over a same universe is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space.*

Proof. Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ and $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ be two $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_i$-connected spaces over Ω, $i = 1, 2$ and $\tilde{\tilde{\mathfrak{g}}} = \tilde{\tilde{\mathfrak{g}}}_1 \tilde{\tilde{\cap}} \tilde{\tilde{\mathfrak{g}}}_2$. We have to show that the space $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. If we say that $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is not $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. Then there exist two $\mathcal{BSS}$s $(\Lambda_1, \Theta_1, \rho)$,$(\Lambda_2, \Theta_2, \rho)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{\mathfrak{g}}}$, which forms a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$ in $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$. From $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{\mathfrak{g}}}$, then $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{\mathfrak{g}}}_1$ and $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$ $\tilde{\tilde{\in}}$ $\tilde{\tilde{\mathfrak{g}}}_2$. This lead to $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ form a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_1$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$ in $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ and also $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ form a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_2$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$ in $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ which is the contradiction to given hypothesis. Therefore, $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space over Ω. □

Remark 4.7. The bipolar soft union of two $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected spaces over the same universe need not be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space.

Example 4.8. Let $\Omega = \{\omega_1, \omega_2\}$,$\rho = \{\sigma_1, \sigma_2\}$, $\widetilde{\widetilde{\mathfrak{g}}}_1 = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho), (\Lambda_1, \Theta_1, \rho)\}$ and $\widetilde{\widetilde{\mathfrak{g}}}_2 = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho), (\Lambda_2, \Theta_2, \rho)\}$, where

$$(\Lambda_1, \Theta_1, \rho) = \{(\sigma_1, \Phi, \Omega), (\sigma_2, \Omega, \Phi)\}, \qquad (\Lambda_2, \Theta_2, \rho) = \{(\sigma_1, \Omega, \Phi), (\sigma_2, \Phi, \Omega)\}.$$

Clearly $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ and $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ are $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected spaces over Ω where $\widetilde{\widetilde{\mathfrak{g}}} = \widetilde{\widetilde{\mathfrak{g}}}_1 \mathbin{\widetilde{\widetilde{\cup}}} \widetilde{\widetilde{\mathfrak{g}}}_2$. But we note that $\widetilde{\widetilde{\mathfrak{g}}}_1 \mathbin{\widetilde{\widetilde{\cup}}} \widetilde{\widetilde{\mathfrak{g}}}_2 = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho)\}$ is not a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected space over Ω because $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ form a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$ in $\widetilde{\widetilde{\mathfrak{g}}}_1 \mathbin{\widetilde{\widetilde{\cup}}} \widetilde{\widetilde{\mathfrak{g}}}_2$.

Proposition 4.9. *The bipolar soft union of two $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected spaces over a same universe is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space.*

Proof. Straightforward. □

Remark 4.10. The bipolar soft intersection of two $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected spaces over the same universe need not be a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space.

Example 4.11. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\rho = \{\sigma_1, \sigma_2\}$, $\widetilde{\widetilde{\mathfrak{g}}}_1 = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho),\ (\widetilde{\widetilde{\Omega}}, \Phi, \rho),\ (\Lambda_1, \Theta_1, \rho),\ (\Lambda_2, \Theta_2, \rho)\}$ and $\widetilde{\widetilde{\mathfrak{g}}}_2 = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho),\ (\widetilde{\widetilde{\Omega}}, \Phi, \rho),\ (\Lambda_3, \Theta_3, \rho),\ (\Lambda_4, \Theta_4, \rho)\}$, where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho)$, $(\Lambda_4, \Theta_4, \rho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$ defined as follows

$$\begin{aligned}
(\Lambda_1, \Theta_1, \rho) &= \{(\sigma_1, \{\omega_1\}, \{\omega_2\}), (\sigma_2, \{\omega_1, \omega_2\}, \{\omega_3\})\},\\
(\Lambda_2, \Theta_2, \rho) &= \{(\sigma_1, \{\omega_2, \omega_3\}, \phi), (\sigma_2, \{\omega_3\}, \{\omega_1\})\},\\
(\Lambda_3, \Theta_3, \rho) &= \{(\sigma_1, \{\omega_1, \omega_3\}, \{\omega_2\}), (\sigma_2, \{\omega_1, \omega_3\}, \{\omega_2\})\},\\
(\Lambda_4, \Theta_4, \rho) &= \{(\sigma_1, \{\omega_2\}, \{\omega_1\}), (\sigma_2, \{\omega_2\}, \{\omega_1\})\}.
\end{aligned}$$

Clearly $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ and $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ are $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected spaces over Ω where $\widetilde{\widetilde{\mathfrak{g}}} = \widetilde{\widetilde{\mathfrak{g}}}_1 \mathbin{\widetilde{\widetilde{\cap}}} \widetilde{\widetilde{\mathfrak{g}}}_2$. But we note that $\widetilde{\widetilde{\mathfrak{g}}}_1 \mathbin{\widetilde{\widetilde{\cap}}} \widetilde{\widetilde{\mathfrak{g}}}_2 = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho),\ (\widetilde{\widetilde{\Omega}}, \Phi, \rho)\}$ is not a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space over Ω because there is no two $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$ in $\widetilde{\widetilde{\mathfrak{g}}}_1 \mathbin{\widetilde{\widetilde{\cap}}} \widetilde{\widetilde{\mathfrak{g}}}_2$.

Theorem 4.12. *Let $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ over Ω and let $\mathcal{BSS}$s $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$ form a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$. If $(\Pi, \widetilde{\widetilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected subspace of $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$, then $\Pi \subseteq \Lambda_1(\sigma)$ for all $\sigma \in \rho$ or $\Pi \subseteq \Lambda_2(\sigma)$ for all $\sigma \in \rho$.*

Proof. Since $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ form a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$, thus

$$\begin{aligned}
&\Omega \cap (\Lambda_1(\sigma) \cup \Lambda_2(\sigma)) = \Omega \text{ for each } \sigma \in \rho,\\
&\Lambda_1(\sigma) \cap \Lambda_2(\sigma) = \phi \text{ for each } \sigma \in \rho,\\
&\Lambda_1(\sigma) \neq \phi, \Lambda_2(\sigma) \neq \phi \text{ for some } \sigma \in \rho.
\end{aligned}$$

From $\Pi \subseteq \Omega$, we get $({}^{\Pi}\Lambda_1, {}^{\Pi}\Theta_1, \rho)$ and $({}^{\Pi}\Lambda_2, {}^{\Pi}\Theta_2, \rho)$ are $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}_\Pi$-open in $(\Pi, \widetilde{\widetilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$. Then

$$\Pi \cap (\Lambda_1(\sigma) \cup \Lambda_2(\sigma)) = \Pi \ \text{ for each } \sigma \in \rho.$$

This implies

$$(\Pi \cap \Lambda_1(\sigma)) \cup (\Pi \cap \Lambda_2(\sigma)) = \Pi \ \text{ for each } \ \sigma \in \rho \ \text{ and } \ (\Pi \cap \Lambda_1(\sigma)) \cap (\Pi \cap \Lambda_2(\sigma)) = \phi \ \text{ for each } \ \sigma \in \rho.$$

Since $(\Pi, \tilde{\tilde{\mathfrak{g}}}_{\Pi}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_{\Pi}$-connected, so either $\Pi \cap \Lambda_1(\sigma) = \phi$ for all $\sigma \in \rho$ or $\Pi \cap \Lambda_2(\sigma) = \phi$ for all $\sigma \in \rho$. If $\Pi \cap \Lambda_1(\sigma) = \phi$ for each $\sigma \in \rho$, then $\Pi \cap \Lambda_2(\sigma) = \Pi$ for each $\sigma \in \rho$ and this implies $\Pi \subseteq \Lambda_2(\sigma) = \Pi$ for each $\sigma \in \rho$. If $\Pi \cap \Lambda_2(\sigma) = \phi$ for each $\sigma \in \rho$, then $\Pi \cap \Lambda_1(\sigma) = \Pi$ for each $\sigma \in \rho$ and this implies $\Pi \subseteq \Lambda_1(\sigma) = \Pi$ for each $\sigma \in \rho$. □

Remark 4.13. The converse of Theorem 4.12 does not hold in general, we can explain by the next example.

Example 4.14. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $\rho = \{\sigma_1, \sigma_2\}$ and $\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho), (\Lambda_4, \Theta_4, \rho), (\Lambda_5, \Theta_5, \rho), (\Lambda_6, \Theta_6, \rho)\}$ where

$$\begin{aligned}
(\Lambda_1, \Theta_1, \rho) &= \{(\sigma_1, \{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}), (\sigma_2, \{\omega_1\}, \{\omega_2, \omega_3, \omega_4\})\},\\
(\Lambda_2, \Theta_2, \rho) &= \{(\sigma_1, \{\omega_2\}, \{\omega_1, \omega_3, \omega_4\}), (\sigma_2, \{\omega_2\}, \{\omega_1, \omega_3, \omega_4\})\},\\
(\Lambda_3, \Theta_3, \rho) &= \{(\sigma_1, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}), (\sigma_2, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\})\},\\
(\Lambda_4, \Theta_4, \rho) &= \{(\sigma_1, \{\omega_1, \omega_3, \omega_4\}, \phi), (\sigma_2, \{\omega_1, \omega_3, \omega_4\}, \phi)\},\\
(\Lambda_5, \Theta_5, \rho) &= \{(\sigma_1, \{\omega_2, \omega_3, \omega_4\}, \phi), (\sigma_2, \{\omega_2, \omega_3, \omega_4\}, \phi)\},\\
(\Lambda_6, \Theta_6, \rho) &= \{(\sigma_1, \Omega, \phi), (\sigma_1, \Omega, \phi)\}.
\end{aligned}$$

Then $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BSGTS}$ over Ω. Also, note that $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_4, \Theta_4, \rho)$ form a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$. Now let $\Pi = \{\omega_1, \omega_2\}$, then $\tilde{\tilde{\mathfrak{g}}}_{\Pi} = \{(\Phi, \tilde{\tilde{\Pi}}, \rho), ({}^{\Pi}\Lambda_1, {}^{\Pi}\Theta_1, \rho), ({}^{\Pi}\Lambda_2, {}^{\Pi}\Theta_2, \rho), ({}^{\Pi}\Lambda_3, {}^{\Pi}\Theta_3, \rho), ({}^{\Pi}\Lambda_4, {}^{\Pi}\Theta_4, \rho), ({}^{\Pi}\Lambda_5, {}^{\Pi}\Theta_5, \rho), ({}^{\Pi}\Lambda_6, {}^{\Pi}\Theta_6, \rho)\}$ where

$$\begin{aligned}
({}^{\Pi}\Lambda_1, {}^{\Pi}\Theta_1, \rho) &= \{(\sigma_1, \{\omega_1\}, \{\omega_2\}), (\sigma_2, \{\omega_1\}, \{\omega_2, \})\},\\
({}^{\Pi}\Lambda_2, {}^{\Pi}\Theta_2, \rho) &= \{(\sigma_1, \{\omega_2\}, \{\omega_1\}), (\sigma_2, \{\omega_2\}, \{\omega_1\})\},\\
({}^{\Pi}\Lambda_3, {}^{\Pi}\Theta_3, \rho) &= \{(\sigma_1, \{\omega_1, \omega_2\}, \phi), (\sigma_2, \{\omega_1, \omega_2\}, \phi)\},\\
({}^{\Pi}\Lambda_4, {}^{\Pi}\Theta_4, \rho) &= \{(\sigma_1, \{\omega_1\}, \phi), (\sigma_2, \{\omega_1\}, \phi)\},\\
({}^{\Pi}\Lambda_5, {}^{\Pi}\Theta_5, \rho) &= \{(\sigma_1, \{\omega_2\}, \phi), (\sigma_2, \{\omega_2\}, \phi)\}.
\end{aligned}$$

Clearly $\Pi \subseteq \Lambda_3(\sigma)$ for each $\sigma \in \rho$. But $(\Pi, \tilde{\tilde{\mathfrak{g}}}_{\Pi}, \rho, \neg\rho)$ is not $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_{\Pi}$-connected space because $({}^{\Pi}\Lambda_1, {}^{\Pi}\Theta_1, \rho)$ and $({}^{\Pi}\Lambda_2, {}^{\Pi}\Theta_2, \rho)$ form a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_{\Pi}$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \rho)$.

Proposition 4.15. *Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ over Ω. If there exists a non-null, non-absolute $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-clopen set (Λ, Θ, ρ) over Ω with $\Lambda(\sigma) \cup \Lambda^c(\sigma) = \Omega$ for each $\sigma \in \rho$, then $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space.*

Proof. Since (Λ, Θ, ρ) is a non-null, non-absolute $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-clopen set over Ω, then $(\Lambda, \Theta, \rho)^c$ is a non-null, non-absolute $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-clopen set over Ω. Now, by given hypothesis and by Proposition 2.20, we have

$$\Lambda(\sigma) \cup \Lambda^c(\sigma) = \Omega \ \text{ for each } \ \sigma \in \rho \ \text{ and } \ \Theta(\neg\sigma) \cap \Theta^c(\neg\sigma) = \phi \ \text{ for each } \ \neg\sigma \in \neg\rho,$$

and

$$\Lambda(\sigma) \cap \Lambda^c(\sigma) = \phi \ \text{ for each } \ \sigma \in \rho \ \text{ and } \ \Theta(\neg\sigma) \cup \Theta^c(\neg\sigma) = \Omega \ \text{ for each } \ \neg\sigma \in \neg\rho.$$

Therefore, (Λ, Θ, ρ) and $(\Lambda, \Theta, \rho)^c$ form a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$. Hence, $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space. □

Remark 4.16. If there exists a non-null, non-absolute $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-clopen set, then $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ may not be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space.

Example 4.17. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\rho = \{\sigma_1, \sigma_2\}$, $\widetilde{\widetilde{\mathfrak{g}}}_1 = \{(\Phi, \widetilde{\widetilde{\Omega}}, \rho), (\widetilde{\widetilde{\Omega}}, \Phi, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho)\}$, where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho) \mathrel{\widetilde{\widetilde{\in}}} \mathcal{BSS}(\Omega)$ defined as follows

$$\begin{aligned}(\Lambda_1, \Theta_1, \rho) &= \{(\sigma_1, \{\omega_1, \omega_2\}, \{\omega_3\}), (\sigma_2, \{\omega_1\}, \{\omega_3\})\},\\ (\Lambda_2, \Theta_2, \rho) &= \{(\sigma_1, \{\omega_3\}, \{\omega_1, \omega_2\}), (\sigma_2, \{\omega_3\}, \{\omega_1\})\},\\ (\Lambda_3, \Theta_3, \rho) &= \{(\sigma_1, \Omega, \phi), (\sigma_2, \{\omega_1, \omega_3\}, \phi)\}.\end{aligned}$$

Clearly, $(\Lambda_1, \Theta_1, \rho)$ is non-null, non-absolute $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-clopen but $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ is not a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space because there does not exist $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$.

Proposition 4.18. *If $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space over Ω, then, the collection $\widetilde{\mathfrak{g}} = \{(\Lambda, \rho) : (\Lambda, \Theta, \rho) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\mathfrak{g}}}\}$ is a $\mathcal{S}$ $\widetilde{\mathfrak{g}}$-disconnected space over Ω.*

Proof. This is straightforward. □

Theorem 4.19. *Let $(\Omega, \widetilde{\mathfrak{g}}, \rho)$ be a $\mathcal{SGTS}$ over Ω and $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$ over Ω constructed from $(\Omega, \widetilde{\mathfrak{g}}, \rho)$ as in Theorem 2.18. If $(\Omega, \widetilde{\mathfrak{g}}, \rho)$ is a $\mathcal{S}$ $\widetilde{\mathfrak{g}}$-disconnected space over Ω, then $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space over Ω.*

Proof. Since the space $(\Omega, \widetilde{\mathfrak{g}}, \rho)$ is $\mathcal{S}$ $\widetilde{\mathfrak{g}}$-disconnected over Ω, then there exist non-null $\mathcal{S}$ $\widetilde{\mathfrak{g}}$-open sets, say, (Λ_1, ρ) and (Λ_2, ρ) over Ω such that,

$$(\widetilde{\Omega}, \rho) = (\Lambda_1, \rho) \widetilde{\cup} (\Lambda_2, \rho) \quad \text{and} \quad (\Phi, \rho) = (\Lambda_1, \rho) \widetilde{\cap} (\Lambda_2, \rho).$$

Further, $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are non-null $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open sets because $\Lambda_1(\sigma) \neq \phi$, $\Lambda_2(\sigma) \neq \phi$ where for all $\neg\sigma \in \neg\rho$, $\Theta_1(\neg\sigma) = \Omega \setminus \Lambda_1(\sigma)$ and $\Theta_2(\neg\sigma) = \Omega \setminus \Lambda_2(\sigma)$. So, from (Λ_1, ρ),$(\Lambda_2, \rho) \mathrel{\widetilde{\in}} \widetilde{\mathfrak{g}}$. Now, for each $\sigma \in \rho$ we have

$$\Lambda_1(\sigma) \cup \Lambda_2(\sigma) = \Omega \quad \text{and} \quad \Theta_1(\neg\sigma) \cap \Theta_2(\neg\sigma) = (\Omega \setminus \Lambda_1(\sigma)) \cap (\Omega \setminus \Lambda_2(\sigma)) = \phi.$$

Also, for each $\sigma \in \rho$ and $\neg\sigma \in \neg\rho$,

$$\Lambda_1(\sigma) \cap \Lambda_2(\sigma) = \phi, \quad \Theta_1(\neg\sigma) \cup \Theta_2(\neg\sigma) = (\Omega \setminus \Lambda_1(\sigma)) \cup (\Omega \setminus \Lambda_2(\sigma)) = \Omega.$$

Therefore, $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho) \mathrel{\widetilde{\widetilde{\in}}} \widetilde{\widetilde{\mathfrak{g}}}$ form a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-separation of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$. Hence, $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-disconnected space over Ω. □

Theorem 4.20. *Let $(\Pi, \widetilde{\widetilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ and $(\eta, \widetilde{\widetilde{\mathfrak{g}}}_\eta, \rho, \neg\rho)$ be two $\mathcal{BSGTSS}$s of $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$ and let $\Pi \subseteq \eta$. Then $(\Pi, \widetilde{\widetilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ is a $\mathcal{BSGTSS}$ of $(\eta, \widetilde{\widetilde{\mathfrak{g}}}_\eta, \rho, \neg\rho)$.*

Proof. From $\Pi \subseteq \eta$, so $\Pi = \Pi \cap \eta$. Moreover, each $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}_\Pi$-open set $({}^\Pi\Lambda, {}^\Pi\Theta, \rho)$ of $(\Pi, \widetilde{\widetilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ is of the form

$${}^\Pi\Lambda(\sigma) = \Pi \cap \Lambda(\sigma) \quad \text{and} \quad {}^\Pi\Theta(\neg\sigma) = \Pi \cap \Theta(\neg\sigma) \quad \text{for each} \quad \sigma \in \rho,$$

where (Λ, Θ, ρ) is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open set of $(\Omega, \widetilde{\widetilde{\mathfrak{g}}}, \rho, \neg\rho)$. Now for each $\sigma \in \rho$,

$$\Pi \cap \Lambda(\sigma) = (\Pi \cap \eta) \cap \Lambda(\sigma) \quad \text{and} \quad \Pi \cap \Theta(\neg\sigma) = (\Pi \cap \eta) \cap \Theta(\neg\sigma).$$

Then $\Pi \cap \Lambda(\sigma) = \Pi \cap \big(\eta \cap \Lambda(\sigma)\big)$ and $\Pi \cap \Theta(\neg\sigma) = \Pi \cap \big(\eta \cap \Theta(\neg\sigma)\big)$. Therefore

$$\Pi \cap \Lambda(\sigma) = \Pi \cap {}^\eta\Lambda(\sigma) \quad \text{and} \quad \Pi \cap \Theta(\neg\sigma) = \Pi \cap {}^\eta\Theta(\neg\sigma),$$

where $({}^\eta\Lambda, {}^\eta\Theta, \rho)$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}_\eta$-open set in $(\eta, \widetilde{\widetilde{\mathfrak{g}}}_\eta, \rho, \neg\rho)$. □

Theorem 4.21. *Let $\{(\Pi_\gamma, \tilde{\tilde{\mathfrak{g}}}_{\Pi_\gamma}, \rho, \neg\rho)\}_{\gamma\in\Gamma}$ be the collection of $\mathcal{BS}\tilde{\tilde{\mathfrak{g}}}_{\Pi_\gamma}$-connected subspaces of $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$. If $\cap_{\gamma\in\Gamma} \Pi_\gamma \neq \phi$, then $(\cup_{\gamma\in\Gamma} \Pi_\gamma, \tilde{\tilde{\mathfrak{g}}}_{\cup_{\gamma\in\Gamma}\Pi_\gamma}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_{\cup_{\gamma\in\Gamma}\Pi_\gamma}$-connected subspace of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$.*

Proof. Let $\{(\Pi_\gamma, \tilde{\tilde{\mathfrak{g}}}_{\Pi_\gamma}, \rho, \neg\rho)\}_{\gamma\in\Gamma}$ be the collection of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_{\Pi_\gamma}$-connected subspaces of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$, such that $\cap_{\gamma\in\Gamma} \Pi_\gamma \neq \phi$. Suppose that $\Pi = \cup_{\gamma\in\Gamma} \Pi_\gamma$ and suppose to the contrary $(\Pi, \tilde{\tilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ is not to be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_\Pi$-connected subspace of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$. Then there exist $({}^\Pi\Lambda_1, {}^\Pi\Theta_1, \rho)$, $({}^\Pi\Lambda_2, {}^\Pi\Theta_2, \rho)$ are a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_\Pi$-separation of $(\tilde{\tilde{\Pi}}, \Phi, \rho)$. So,

$$\begin{gathered} {}^\Pi\Lambda_1(\sigma) \cup {}^\Pi\Lambda_2(\sigma) = \Pi \quad \text{and} \quad {}^\Pi\Lambda_1(\sigma) \cap {}^\Pi\Lambda_2(\sigma) = \phi \ \text{ for all } \ \sigma \in \rho, \\ {}^\Pi\Lambda_1(\sigma) \neq \phi, {}^\Pi\Lambda_2(\sigma) \neq \phi \ \text{ for some } \ \sigma \in \rho. \end{gathered}$$

This implies that,

$$\begin{aligned} {}^\Pi\Lambda_1(\sigma) \cup {}^\Pi\Lambda_2(\sigma) = \Pi \cap \big(\Lambda_1(\sigma) \cup \Lambda_2(\sigma)\big) &= \Pi \ \text{ for all } \ \sigma \in \rho, \\ \big(\Pi \cap \Lambda_1(\sigma)\big) \cap \big(\Pi \cap \Lambda_2(\sigma)\big) = \Pi \cap (\Lambda_1(\sigma) \cap \Lambda_2(\sigma)) &= \phi \ \text{ for all } \ \sigma \in \rho, \\ \Pi \cap \Lambda_1(\sigma) \neq \phi \text{ and } \Pi \cap \Lambda_2(\sigma) \neq \phi \ \text{ for some } \ \sigma \in \rho. & \end{aligned}$$

Consider a fixed Π_γ. Then,

$$\begin{aligned} &\Pi_\gamma \cap \big(\Lambda_1(\sigma) \cup \Lambda_2(\sigma)\big) = \Pi_\gamma \ \text{ for each } \ \sigma \in \rho, \\ &\Pi_\gamma \cap \big(\Lambda_1(\sigma) \cap \Lambda_2(\sigma)\big) = \phi \ \text{ for each } \ \sigma \in \rho, \\ &\Pi_\gamma \cap \Lambda_1(\sigma) \neq \phi \ \text{ and } \ \Pi_\gamma \cap \Lambda_2(\sigma) \neq \phi \ \text{ for some } \ \sigma \in \rho. \end{aligned}$$

From $(\Pi_\gamma, \tilde{\tilde{\mathfrak{g}}}_{\Pi_\gamma}, \rho, \neg\rho)$ be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_{\Pi_\gamma}$-connected subspaces of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$, so, either $\Pi_\gamma \cap \Lambda_1(\sigma) = \phi$ for all $\sigma \in \rho$ or $\Pi_\gamma \cap \Lambda_2(\sigma) = \phi$ for all $\sigma \in \rho$.

Now, there are three cases:

1. $\Pi_\gamma \cap \Lambda_1(\sigma) = \phi$ for all $\sigma \in \rho$ and for all $\gamma \in \Gamma$;
2. $\Pi_\gamma \cap \Lambda_2(\sigma) = \phi$ for all $\sigma \in \rho$ and for all $\gamma \in \Gamma$;
3. for some $\gamma \in \Gamma$, $\Pi_\gamma \cap \Lambda_1(\sigma) = \phi$ and for other some $\gamma \in \Gamma$, $\Pi_\gamma \cap \Lambda_2(\sigma) = \phi$.

Case 1. If $\Pi_\gamma \cap \Lambda_1(\sigma) = \phi$ for all $\sigma \in \rho$ and for all $\gamma \in \Gamma$, then $(\cup_{\gamma\in\Gamma} \Pi_\gamma) \cap \Lambda_1(\sigma) = \phi$, that is $\Pi \cap \Lambda_1(\sigma) = \phi$ for all $\sigma \in \rho$. Hence this is a contradiction.

Case 2. If $\Pi_\gamma \cap \Lambda_2(\sigma) = \phi$ for all $\sigma \in \rho$ and for all $\gamma \in \Gamma$, then $(\cup_{\gamma\in\Gamma} \Pi_\gamma) \cap \Lambda_2(\sigma) = \phi$, that is $\Pi \cap \Lambda_2(\sigma) = \phi$ for all $\sigma \in \rho$. Hence this is also a contradiction.

Case 3. From $\cap_{\gamma\in\Gamma} \Pi_\gamma \neq \phi$, so there exist some $\omega \in \Pi_\gamma$ for all $\gamma \in \Gamma$. This implies $\omega \in \Lambda_1(\sigma) \cup \Lambda_2(\sigma)$ for all $\sigma \in \rho$. So either $\omega \in \Lambda_1(\sigma)$ or $\omega \in \Lambda_2(\sigma)$.

$$\text{If } \omega \in \Lambda_1(\sigma), \text{ then } \Pi_\gamma \cap \Lambda_1(\sigma) \neq \phi; \qquad \text{if } \omega \in \Lambda_2(\sigma), \text{ then } \Pi_\gamma \cap \Lambda_2(\sigma) \neq \phi.$$

So this is an impossible case. Therefore, $(\cup_{\gamma\in\Gamma} \Pi_\gamma, \tilde{\tilde{\mathfrak{g}}}_{\cup_{\gamma\in\Gamma}\Pi_\gamma}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_\Pi$-connected subspace of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$. □

Proposition 4.22. *Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ and $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ be two $\mathcal{BSGTS}$s over Ω. Then,*

1. *if $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_1$-connected such that $\tilde{\tilde{\mathfrak{g}}}_2 \,\tilde{\tilde{\subseteq}}\, \tilde{\tilde{\mathfrak{g}}}_1$, then $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_2$-connected;*
2. *if $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_1$-disconnected such that $\tilde{\tilde{\mathfrak{g}}}_1 \,\tilde{\tilde{\subseteq}}\, \tilde{\tilde{\mathfrak{g}}}_2$, then $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_2$-disconnected.*

Proof.

1. Assume that $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_1$-connected such that $\tilde{\tilde{\mathfrak{g}}}_2 \;\tilde{\tilde{\subseteq}}\; \tilde{\tilde{\mathfrak{g}}}_1$. Assume the contrary that $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_2$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$ in $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$. Since $\tilde{\tilde{\mathfrak{g}}}_2 \;\tilde{\tilde{\subseteq}}\; \tilde{\tilde{\mathfrak{g}}}_1$, then $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$ are $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_1$-separation of $(\tilde{\tilde{\Omega}}, \Phi, \rho)$ in $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$. This is a contradiction. Therefore, $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_2$-connected.

2. Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_1$-disconnected such that $\tilde{\tilde{\mathfrak{g}}}_1 \;\tilde{\tilde{\subseteq}}\; \tilde{\tilde{\mathfrak{g}}}_2$. Assume the contrary that $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_2$-connected. Since $\tilde{\tilde{\mathfrak{g}}}_1 \;\tilde{\tilde{\subseteq}}\; \tilde{\tilde{\mathfrak{g}}}_2$, then by (1), we get $(\Omega, \tilde{\tilde{\mathfrak{g}}}_1, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_1$-connected. This is a contradiction. Therefore, $(\Omega, \tilde{\tilde{\mathfrak{g}}}_2, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_2$-disconnected. □

Proposition 4.23. *Let $\left((\Lambda, \Theta, \rho), \tilde{\tilde{\mathfrak{g}}}_{(\Lambda,\Theta,\rho)}, \rho, \neg\rho\right)$ be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space, then (Λ, Θ, ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected.*

Proof. Let $\left((\Lambda, \Theta, \rho), \tilde{\tilde{\mathfrak{g}}}_{(\Lambda,\Theta,\rho)}, \rho, \neg\rho\right)$ be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space. Suppose (Λ, Θ, ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected, then there exist $\tilde{\tilde{\mathfrak{g}}}$-separated $\mathcal{BSS}$s, say, $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ of (Λ, Θ, ρ), thus by Proposition 3.5 that $(\Lambda_1, \Theta_1, \rho)$ and $(\Lambda_2, \Theta_2, \rho)$ are $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-separation of (Λ, Θ, ρ). This is a contradiction. Hence, (Λ, Θ, ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected. □

Definition 4.24. A property $\mathcal{P}$ of a $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is called be a bipolar soft generalized hereditary property ($\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-hereditary property) if every $\mathcal{BSGSS}$ $(\Pi, \tilde{\tilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ also has the property $\mathcal{P}$.

Proposition 4.25. *Let $(\Pi, \tilde{\tilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ be a $\mathcal{BSGTSS}$ of $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ over Ω and $({}^\Pi\Lambda, {}^\Pi\Theta, \rho)$ be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed set in Π. Then (Λ, Θ, ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed set in Ω.*

Proof. Assume that $({}^\Pi\Lambda, {}^\Pi\Theta, \rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_\Pi$-closed set in Π. Thus $({}^\Pi\Lambda, {}^\Pi\Theta, \rho)^c = ({}^\Pi\Theta, {}^\Pi\Lambda, \rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}_\Pi$-open set in Π, where (Θ, Λ, ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set in Ω. Hence $(\Theta, \Lambda, \rho)^c = (\Lambda, \Theta, \rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-closed set in Ω. □

Remark 4.26. The $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-contentedness (resp. $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-discontentedness) is not a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-hereditary property.

Example 4.27. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\rho = \{\sigma_1, \sigma_2\}$ and $\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho)\}$ where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho) \;\tilde{\tilde{\in}}\; \mathcal{BSS}(\Omega)$, defined as follows

$$\begin{aligned}
(\Lambda_1, \Theta_1, \rho) &= \{(\sigma_1, \{\omega_1\}, \{\omega_2, \omega_3\}), (\sigma_2, \{\omega_1\}, \{\omega_2, \omega_3\})\},\\
(\Lambda_2, \Theta_2, \rho) &= \{(\sigma_1, \{\omega_2\}, \{\omega_1, \omega_3\}), (\sigma_2, \{\omega_2\}, \{\omega_1, \omega_3\})\},\\
(\Lambda_3, \Theta_3, \rho) &= \{(\sigma_1, \{\omega_1, \omega_2\}, \{\omega_3\}), (\sigma_2, \{\omega_1, \omega_2\}, \{\omega_3\})\}.
\end{aligned}$$

Then $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space. Now let $\Pi = \{\omega_1, \omega_2\}$, then $\tilde{\tilde{\mathfrak{g}}}_\Pi = \{(\Phi, \tilde{\tilde{\Pi}}, \rho), ({}^\Pi\Lambda_1, {}^\Pi\Theta_1, \rho), ({}^\Pi\Lambda_2, {}^\Pi\Theta_2, \rho), ({}^\Pi\Lambda_3, {}^\Pi\Theta_3, \rho)\}$, such that

$$\begin{aligned}
({}^\Pi\Lambda_1, {}^\Pi\Theta_1, \rho) &= \{(\sigma_1, \{\omega_1\}, \{\omega_2\}), (\sigma_2, \{\omega_1\}, \{\omega_2, \})\},\\
({}^\Pi\Lambda_2, {}^\Pi\Theta_2, \rho) &= \{(\sigma_1, \{\omega_2\}, \{\omega_1\}), (\sigma_2, \{\omega_2\}, \{\omega_1\})\},\\
({}^\Pi\Lambda_3, {}^\Pi\Theta_3, \rho) &= \{(\sigma_1, \Pi, \phi), (\sigma_2, \Pi, \phi)\} = (\tilde{\tilde{\Pi}}, \Phi, \rho).
\end{aligned}$$

Clearly, $(\Pi, \tilde{\tilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected subspace of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$. While $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space.

Example 4.28. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\rho = \{\sigma_1, \sigma_2\}$ and $\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \rho), (\tilde{\tilde{\Omega}}, \Phi, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho)\}$ where $(\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$, defined as follows

$$(\Lambda_1, \Theta_1, \rho) = \{(\sigma_1, \{\omega_1\}, \{\omega_2\}), (\sigma_2, \{\omega_2\}, \{\omega_1, \omega_3\})\},$$
$$(\Lambda_2, \Theta_2, \rho) = \{(\sigma_1, \{\omega_2, \omega_3\}, \phi), (\sigma_2, \{\omega_1, \omega_3\}, \{\omega_2\})\}.$$

Then $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-disconnected space. Now let $\Pi = \{\omega_3\}$, then $\tilde{\tilde{\mathfrak{g}}}_\Pi = \{(\Phi, \tilde{\tilde{\Pi}}, \rho), ({}^\Pi\Lambda_1, {}^\Pi\Theta_1, \rho), ({}^\Pi\Lambda_2, {}^\Pi\Theta_2, \rho)\}$, such that

$$({}^\Pi\Lambda_1, {}^\Pi\Theta_1, \rho) = \{(\sigma_1, \phi, \phi), (\sigma_2, \phi, \Pi)\}, \quad ({}^\Pi\Lambda_2, {}^\Pi\Theta_2, \rho) = \{(\sigma_1, \Pi, \phi), (\sigma_2, \Pi, \phi)\}.$$

Clearly, $(\Pi, \tilde{\tilde{\mathfrak{g}}}_\Pi, \rho, \neg\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected subspace of $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$. While $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space.

5. $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected spaces and $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components

In this section, we introduce a new concept of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected spaces called $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected. We investigate some of its important properties. Moreover, we show that the concepts of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space and $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected space are independent of each other. Furthermore, we explore the concept of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components and we show that the family of all $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components forms a partition for $\mathcal{BSGTS}$. We present some properties of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components in $\mathcal{BSGTS}$s.

Definition 5.1. A $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is called a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected at $\pi_\upsilon^\sigma \tilde{\tilde{\in}} (\tilde{\tilde{\Omega}}, \Phi, \rho)$ if for every $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set (Λ, Θ, ρ) containing π_υ^σ, there is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open (χ, ψ, ρ) containing π_υ^σ such that $\pi_\upsilon^\sigma \tilde{\tilde{\in}} (\chi, \psi, \rho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \rho)$. A $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is said to be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected if it is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected at every $\mathcal{BSP}$ $\pi_\upsilon^\sigma \tilde{\tilde{\in}} (\tilde{\tilde{\Omega}}, \Phi, \rho)$. Otherwise, it is said to be $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally disconnected.

Remark 5.2. We recall that we called (Λ, Θ, ρ) a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open of a $\mathcal{BSP}$ π_υ^σ if it is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected and $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set of π_υ^σ, i.e., there exists a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set (χ, ψ, ρ) such that $\pi_\upsilon^\sigma \tilde{\tilde{\in}} (\chi, \psi, \rho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \rho)$.

Remark 5.3. $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected does not imply $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected as shown by the following example.

Example 5.4. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\rho = \{\sigma_1, \sigma_2\}$ and $\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \rho), (\tilde{\tilde{\Omega}}, \Phi, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho), (\Lambda_4, \Theta_4, \rho)\}$, where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho)$, $(\Lambda_4, \Theta_4, \rho) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$, defined as follows

$$(\Lambda_1, \Theta_1, \rho) = \{(\sigma_1, \{\omega_1, \omega_2\}, \phi)\}, \qquad (\Lambda_2, \Theta_2, \rho) = \{(\sigma_1, \{\omega_2, \omega_3\}, \phi)\},$$
$$(\Lambda_3, \Theta_3, \rho) = \{(\sigma_1, \Omega, \phi), (\sigma_2, \phi, \Omega)\}, \qquad (\Lambda_4, \Theta_4, \rho) = \{(\sigma_1, \phi, \Omega), (\sigma_2, \Omega, \phi)\}.$$

Then $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected space but not $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected.

Remark 5.5. $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected does not imply $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected as we explain by the next example.

Example 5.6. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\rho = \{\sigma_1, \sigma_2\}$ and $\tilde{\tilde{\mathfrak{g}}} = \{(\Phi, \tilde{\tilde{\Omega}}, \rho), (\tilde{\tilde{\Omega}}, \Phi, \rho), (\Lambda_1, \Theta_1, \rho), (\Lambda_2, \Theta_2, \rho), (\Lambda_3, \Theta_3, \rho), (\Lambda_4, \Theta_4, \rho)\}$ where $(\Lambda_1, \Theta_1, \rho)$, $(\Lambda_2, \Theta_2, \rho)$, $(\Lambda_3, \Theta_3, \rho)$, $(\Lambda_4, \Theta_4, \rho) \tilde{\tilde{\in}} \mathcal{BSS}(\Omega)$, defined as follows

$$(\Lambda_1, \Theta_1, \rho) = \{(\sigma_1, \{\omega_2\}, \{\omega_3\})\},$$
$$(\Lambda_2, \Theta_2, \rho) = \{(\sigma_1, \{\omega_1, \omega_2\}, \phi)\},$$
$$(\Lambda_3, \Theta_3, \rho) = \{(\sigma_1, \{\omega_2\}, \{\omega_3\}), (\sigma_2, \{\omega_2\}, \{\omega_1, \omega_3\})\},$$
$$(\Lambda_4, \Theta_4, \rho) = \{(\sigma_1, \{\omega_1, \omega_2\}, \phi), (\sigma_2, \{\omega_2\}, \{\omega_1, \omega_3\})\}.$$

Then $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space but not $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected because the $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set $(\Lambda_2, \Theta_2, \rho)$ containing $\omega_{1_{\omega_2}}^{\sigma_1}$, but there is no $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open subset of $(\Lambda_2, \Theta_2, \rho)$ containing $\omega_{1_{\omega_2}}^{\sigma_1}$.

Theorem 5.7. *A $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected at $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}} (\tilde{\tilde{\Omega}}, \Phi, \rho)$ if and only if every $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open containing π^{σ}_{υ} contains a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open of it.*

Proof.

Sufficiency. It comes from Definition 5.1.

Necessity. Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected at $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}} (\tilde{\tilde{\Omega}}, \Phi, \rho)$. Let (Λ, Θ, ρ) be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open containing π^{σ}_{υ}. So, there exists a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open, say, (χ, ψ, ρ) containing π^{σ}_{υ} such that every two $\mathcal{BSP}$s π^{σ}_{υ} and $\pi'^{\sigma'}_{\upsilon'}$ in (χ, ψ, ρ) are $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected in (Λ, Θ, ρ). For each $\mathcal{BSP}$s $\pi^{\sigma}_{\upsilon} \neq \pi'^{\sigma'}_{\upsilon'}$ in (χ, ψ, ρ), there exists a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set $(\chi_{\pi^{\sigma}_{\upsilon}}, \psi_{\pi^{\sigma}_{\upsilon}}, \rho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \rho)$ containing π^{σ}_{υ} and $\pi'^{\sigma'}_{\upsilon'}$. If putting (Γ, η, ρ) $= \tilde{\tilde{\bigcup}}_{\pi^{\sigma}_{\upsilon} \in (\chi, \psi, \rho)} (\chi_{\pi^{\sigma}_{\upsilon}}, \psi_{\pi^{\sigma}_{\upsilon}}, \rho)$. Now, $(\chi, \psi, \rho) \tilde{\tilde{\subseteq}} (\Gamma, \eta, \rho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \rho)$ by Proposition 3.13, (Γ, η, ρ) is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open of π^{σ}_{υ}. □

Definition 5.8. A $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component of $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ corresponding to π^{σ}_{υ} is the bipolar soft union of all $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected $(\Lambda, \Theta, \rho) \tilde{\tilde{\subseteq}} (\tilde{\tilde{\Omega}},\Phi,\rho)$ which contains π^{σ}_{υ}. It is denoted by $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\pi^{\sigma}_{\upsilon})$ that is

$$\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\pi^{\sigma}_{\upsilon}) = \tilde{\tilde{\bigcup}}\{(\Lambda, \Theta, \rho)\tilde{\tilde{\subseteq}}(\tilde{\tilde{\Omega}}, \Phi, \rho) : \pi^{\sigma}_{\upsilon}\tilde{\tilde{\in}}(\Lambda, \Theta, \rho) \text{ and } (\Lambda, \Theta, \rho) \text{ is } \mathcal{BS}\tilde{\tilde{\mathfrak{g}}}\text{-connected}\}.$$

Remark 5.9. For a $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$, we have

1. according to Proposition 3.13, every $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component of a $\mathcal{BSP}$ is a largest $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set containing this $\mathcal{BSP}$;
2. if $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected space, then $(\tilde{\tilde{\Omega}}, \phi, \rho)$ is only the $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component of each $\mathcal{BSP}$;
3. since the $\mathcal{BS}$ singleton set is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected, then the $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component is non-null $\mathcal{BSS}$.

Example 5.10. Consider the $\mathcal{BSGTS}$ in Example 5.4, we have the following:

$$\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_1}_{1_{\omega_2}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_1}_{1_{\omega_3}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_1}_{2_{\omega_1}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_1}_{2_{\omega_3}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_1}_{3_{\omega_1}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_1}_{3_{\omega_2}}) = (\Lambda_3, \Theta_3, \rho)$$

and

$$\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_2}_{1_{\omega_2}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_2}_{1_{\omega_3}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_2}_{2_{\omega_1}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_2}_{2_{\omega_3}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_2}_{3_{\omega_1}}) = \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\omega^{\sigma_2}_{3_{\omega_2}}) = (\Lambda_4, \Theta_4, \rho).$$

Theorem 5.11. *A $\mathcal{BSGTS}$ $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected if and only if the $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-components of $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open sets are $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open sets.*

Proof. Assume that the space $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-locally connected. Let (Λ, Θ, ρ) be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open and $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$ be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component of (Λ, Θ, ρ). If $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}} \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$ and since $\pi^{\sigma}_{\upsilon} \in (\Lambda, \Theta, \rho)$, there is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open set (χ, ψ, ρ) such that $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}} (\chi, \psi, \rho) \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \rho)$. Now, from $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component of π^{σ}_{υ} and (χ, ψ, ρ) is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected, we have $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}} (\chi, \psi, \rho)\tilde{\tilde{\subseteq}} \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$. This shows that $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$ is $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open.

Conversely, let $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}} (\tilde{\tilde{\Omega}}, \phi, \rho)$ be arbitrary and let (Λ, Θ, ρ) be a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-open set containing π^{σ}_{υ}. Suppose $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component of (Λ, Θ, ρ) such that $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}} \mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$. Now, $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}$ is a $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected open set with $\pi^{\sigma}_{\upsilon} \tilde{\tilde{\in}}$ $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}} \tilde{\tilde{\subseteq}} (\Lambda, \Theta, \rho)$. This proves the theorem. □

Theorem 5.12. *Let $(\Omega, \tilde{\tilde{\mathfrak{g}}}, \rho, \neg\rho)$ be a $\mathcal{BSGTS}$, then*

1. *each $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-component $\mathcal{C}_{\tilde{\tilde{\mathfrak{g}}}}(\pi^{\sigma}_{\upsilon})$ is a maximal $\mathcal{BS}$ $\tilde{\tilde{\mathfrak{g}}}$-connected set in $(\tilde{\tilde{\Omega}}, \phi, \rho)$;*

2. *the family of all distinct* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-components of a* $\mathcal{BSP}$*s of* $(\widetilde{\widetilde{\Omega}}, \phi, \rho)$ *forms a partition of* $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$*;*
3. *for any* $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$*-component* $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$*, we have* $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) = c_{\widetilde{\widetilde{\mathfrak{g}}}}\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$*.*

Proof.
1. Follows from the definition.
2. Let $\{\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) \colon \pi_{\upsilon}^{\sigma} \mathrel{\widetilde{\widetilde{\in}}} (\widetilde{\widetilde{\Omega}}, \Phi, \rho)\}$ be a family of all distinct $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-components of $(\widetilde{\widetilde{\Omega}}, \Phi, \rho)$. Clearly, $(\widetilde{\widetilde{\Omega}}, \Phi, \rho) = \widetilde{\widetilde{\bigcup}} \{\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) : \pi_{\upsilon}^{\sigma} \mathrel{\widetilde{\widetilde{\in}}} (\widetilde{\widetilde{\Omega}}, \Phi, \rho)\}$. Suppose that there are two distinct $\mathcal{BSP}$s π_{υ}^{σ} and $\pi_{\upsilon'}'^{\sigma'}$ such that $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) \mathrel{\widetilde{\widetilde{\cap}}} \mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon'}'^{\sigma'}) \neq (\Phi, \Theta, \rho)$. By Proposition 3.13 $(\Lambda, \Theta, \rho) = \mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) \mathrel{\widetilde{\widetilde{\cup}}} \mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon'}'^{\sigma'})$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set. This contradicts that $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$ and $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon'}'^{\sigma'})$ are the largest $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected sets containing π_{υ}^{σ} and $\pi_{\upsilon'}'^{\sigma'}$, respectively. Hence $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) \mathrel{\widetilde{\widetilde{\cap}}} \mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon'}'^{\sigma'}) = (\Phi, \Theta, \rho)$.

3. Since $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected and $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) \mathrel{\widetilde{\widetilde{\subseteq}}} c_{\widetilde{\widetilde{\mathfrak{g}}}}\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$, it follows from Proposition 3.12 that $c_{\widetilde{\widetilde{\mathfrak{g}}}}\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$ is a $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set as well. Since, $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$ is the largest $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected set containing π_{υ}^{σ} . Hence, $\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma}) = c_{\widetilde{\widetilde{\mathfrak{g}}}}\mathcal{C}_{\widetilde{\widetilde{\mathfrak{g}}}}(\pi_{\upsilon}^{\sigma})$. □

6. Conclusion

The fundamental concepts in the frame $\mathcal{BSGTS}$s, which are connected to $\mathcal{BSS}$s, are continued to be displayed and studied in this work. The definitions of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected sets and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected spaces were the major objectives of this study. The main definitions and outcomes are provided. On the other hand, we demonstrated the invalidity of a few $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-locally connected space and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-component features in $\mathcal{BSGTS}$. We defined them, demonstrated how the ideas of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-locally linked spaces and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected are distinct, and established the circumstances in which the $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-connected subsets are $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-components. The results showed that many of these ideas' traditional features are still applicable for bipolar soft systems.

For the future work, we can deeply study the concepts of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-locally connected spaces and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-components with respect to the ordinary points in place of $\mathcal{BSP}$s. Moreover, we predict that some properties of these concepts will lead to different results. In addition, we will investigate different types of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$- spaces such as $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-compactness and study different types of $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$- mappings such as $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-continuous, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-open, $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-closed, and $\mathcal{BS}$ $\widetilde{\widetilde{\mathfrak{g}}}$-homeomorphism via $\mathcal{BSP}$s. In addition, the direction of this research work can be extended to hypersoft sets and bipolar hypersoft sets [26].

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