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On the existence and stability of Caputo Volterra-Fredholm systems

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Abstract

In this paper, we discuss several problems related to the neutral fractional Volterra-Fredholm integro-differential systems in Banach spaces. Existence of the Schaefer's fixed point and Ulam-Hyers-Rassias stability properties for the fixed point problem will be discussed. Some results are presented, under appropriate conditions, and some open questions are pointed out. Our results extend recent results given for ψ-fractional derivative.

Keywords: ψ-Caputo fractional derivative, integro-differential equation, fixed point technique, stability problem.

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1. Introduction

Fractional calculus transacts with integrals and derivatives of arbitrary order complex order. The record of fractional calculus started on September 1695, when Leibnitz characterized a derivative of order $\beta = 1/2$ (see [20]). In the 19th century, Riemann and Liouville defined the concept of fractional differentiation. Recently, a growing number of monographs are focusing their awareness on fractional calculus, and they have shown that fractional systems can retain information that is missing in integral order systems [1, 4, 12, 13, 21]. Authors pay more concern to fractional calculus due to its considerable applications in many areas such as signal processing [9], visco-elastic materials [6], economics [7], solid mechanics [26], control theory [19], oscillation of earthquakes [15], electrode-electrolyte polarization [10], and continuum and statistical mechanics [22]. Nowadays, researchers focus on Caputo's derivative, which is more useful in real-life applications [5, 8, 24, 28].

Fractional differential equations (FDEs) have newly confirmed to be important aid in modelling of numerable phenomena in different fields of science and engineering. Viscoelasticity, electrochemistry, control, porous media, electromagnetic, and other difficulties have a variety of applications [25, 27]. In latterly, there has been a dramatic increase in the number of ODE and PDE incorporating both Riemann-Liouville and Caputo fractional derivatives; see the monographs of Hilfer [16], Podlubny [25] and Samko et al. [27]. The theoretical study of these kinds of differential equations is significant for the applicability on the reality. As a result, many researchers have investigated the existence of a solution to FDEs is

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the first action for finding the analytic solution. Many natural phenomena can be formulated by BVPs of FDEs. We mention here some works on FDEs with boundary conditions (see [5–7, 11, 14, 17–19, 22, 24, 28] and references therein). By creating various fractional integral inequalities with applying the nonlinear alternative Leray-Schauder type, Aghajani et al. investigated the solvability of a huge group of nonlinear fractional integro-differential equations. Balachandran and Kiruthika analysed the presence of results of nonlinear fractional integro-differential equations of Sobolev type with nonlocal condition in Banach spaces [2]. Very recently, Almeida [3] introduced the ψ fractional derivative with respect to some other function. For more information on ψ-type derivatives see [29, 30].

In the present study, we assume that the ψ-Caputo fractional impulsive Volterra-Fredholm integro differential system of boundary-type with following form

$$\begin{aligned} {}^{c}\mathbf{D}^{\beta;\psi}\Xi(\vartheta) &= \mathbf{H}\left(\vartheta, \Xi(\vartheta), \int_0^{\vartheta} \mathfrak{p}(\vartheta, s, \Xi(s))ds, \int_0^{\tau} \mathfrak{B}(\vartheta, s, \Xi(s))ds\right), \\ \Delta\Xi|_{\vartheta=\vartheta_j} &= \mathbb{I}_j\left(\Xi\left(\vartheta_k^-\right)\right), \ \vartheta = \vartheta_j, j = 1, 2, \ldots, n, \vartheta \in \mathbb{F} := [0, \tau], \\ \mathfrak{l}\Xi(0) + \mathfrak{m}\Xi(\tau) &= c, \end{aligned} \tag{1.1}$$

where ${}^{c}\mathbf{D}^{\beta;\psi}$ is the ψ-Caputo fractional derivative of order $\beta \in (0,1)$. Let the continuous functions be $\mathbf{H} : \mathbb{F} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, $\mathfrak{p} : \Delta \times \mathbb{E} \to \mathbb{E}$ and the real constants be $\mathfrak{l}, \mathfrak{m}, \mathfrak{n}$ with $\mathfrak{l} + \mathfrak{m} \neq 0$. Now $\Delta = \{(\vartheta, s) : 0 \leqslant s \leqslant \vartheta \leqslant \tau\}$. For the purpose of brevity, we make use of

$$\mathbb{P}\Xi(\vartheta) = \int_0^{\vartheta} \mathfrak{p}(\vartheta, s, \Xi(s))ds, \quad \mathbb{B}\Xi(\vartheta) = \int_0^{\tau} \mathfrak{B}(\vartheta, s, \Xi(s))ds.$$

The remainder of the paper is organized as follows. In Section 2 theoretical notions linked to fractional calculus and the preliminaries are presented. Section 3 is focused on existence results of the fractional system (1.1). In Section 4 our findings are expanded to include the concept of Ulam-Hyers-Rassias stability. Finally in Section 5 abstract and filter systems are provided for the illustration of the obtained theory.

2. Preliminaries

We now provide some fundamental theories, lemmas, and facts to discuss our main results. All continuous functions from $\mathbb{F}$ into $\mathbb{E}$ are represented in the Banach space by $C(\mathbb{F}, \mathbb{E})$ and

$$\|\Xi\|_{\infty} := \sup\{|\Xi(\vartheta)| : \Xi \in C(\mathbb{F}, \mathbb{E}), \vartheta \in \mathbb{F}\}.$$

Definition 2.1 ([25])**.** A family $\mathcal{A}$ in $C(\mathbb{F}, \mathbb{E})$ is equicontinuous at t in $\mathbb{F}$ if for each $\varepsilon > 0$ there exists $\delta(\varepsilon, t) > 0$ such that, for each $s \in \mathbb{F}$ with $\|t - s\| < \delta(\varepsilon, t)$, we have $\|f(t) - f(s)\| < \varepsilon$, uniformly with respect to $f \in \mathcal{A}$.

Definition 2.2. Let $\beta > 0, \mathbb{F} = [0, t]$ is either a finite or infinite interval, and $\mathbf{H}$ is an integrable function defined on $\mathbb{F}$ with an increasing function $\psi \in C^1(\mathbb{F}, \mathbb{E})$ such that $\psi'(\vartheta) \neq 0$ for all $\vartheta \in \mathbb{F}$. The followings are fractional integrals and fractional derivatives of a function $\mathbf{H}$ with respect to another function ψ,

$$\begin{aligned} I^{\beta;\psi}\mathbf{H}(\vartheta) &:= \frac{1}{\mu(\beta)} \int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}\mathbf{H}(s)ds, \\ \mathbf{D}^{\beta;\psi}\mathbf{H}(\vartheta) &:= \left(\frac{1}{\psi'(t)}\frac{d}{d\vartheta}\right)^r I^{r-\beta;\psi}\mathbf{H}(\vartheta) = \frac{1}{\mu(r-\beta)}\left(\frac{1}{\psi'(\vartheta)}\frac{d}{d\vartheta}\right)^r \int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{r-\beta-1}\mathbf{H}(s)ds, \end{aligned}$$

respectively, where $r = [\beta] + 1$.

Gronwall's lemma for ψ-fractional derivative is generalized as follows. It plays a vital role in Ulam-Hyers stability's proof.

Lemma 2.3 ([30]). *Let $\Upsilon, \mathbf{L} : [0,t] \to [0,\infty)$ be continuous functions where $t \leqslant \infty$. If $\mathbf{L}$ is nondecreasing, $k \geqslant 0$ and $0 < \beta < 1$ are constants such that*

$$\Upsilon(\vartheta) \leqslant \mathbf{L}(\vartheta) + k\int_0^{\vartheta} \psi'(\vartheta)(\psi(\vartheta)-\psi(s))^{\beta-1}\Upsilon(s)ds, \vartheta \in [0,t],$$

then

$$\Upsilon(\vartheta) \leqslant \mathbf{L}(\vartheta) + \int_0^{\vartheta}\left(\sum_{n=1}^{\infty}\frac{(k\mu(\beta))^r}{\mu(n\beta)}(\psi(\vartheta)-\psi(s))^{n\beta-1}\mathbf{L}(s)\right)ds, \vartheta \in [0,t].$$

Remark 2.4 ([30]). According to Lemma 2.3's hypothesis, for a nondecreasing function $\mathbf{L}(\vartheta)$ on $[0,t]$, we have

$$\Upsilon(\vartheta) \leqslant \mathbf{L}(\vartheta)E_{\beta;\psi}\left(k\mu(\beta)(\psi(\vartheta))^{\beta}\right).$$

Theorem 2.5 (Schaefer's fixed point theorem, [25]). *Let completely continuous operator be $\mathcal{P} : C(\mathbb{F},\mathbb{E}) \to C(\mathbb{F},\mathbb{E})$. If the set*

$$\kappa = \{\Xi \in C(\mathbb{F},\mathbb{E}) : \Xi = \mu\mathcal{P}(u) \ \textit{for any} \ \ \mu \in (0,t)\}$$

is bounded, then at least, $\mathcal{P}$ has a fixed point.

3. Existence results

In this section, by using the Schaefer's fixed point theorem, we study the existence of solutions of problem (1.1).

Definition 3.1. $u \in C^1(\mathbb{F},\mathbb{E})$ is a function and is said to be a solution of (1.1) if u fulfilled

$${}^{c}\mathbf{D}^{\beta;\psi}\Xi(\vartheta) = \mathbf{H}(t,\Xi(\vartheta))$$

on $\mathbb{F}$, $\Delta\Xi|_{\vartheta=\vartheta_k} = \mathbb{I}_k\left(\Xi\left(\vartheta_j^-\right)\right), \vartheta = \vartheta_j, j = 1,2,\ldots n$ with the condition $l\Xi(0) + m\Xi(t) = n$.

The following lemma is required to obtain the existence of solutions for the equation (1.1).

Lemma 3.2. *Let $\beta \in (0,1), \mathbf{H}, \psi : \mathbb{F} \to \mathbb{E}$ and $\mathfrak{p}, \mathfrak{B} : \Delta \times \mathbb{E} \to \mathbb{E}$ be a continuous functions. Ξ is the solution to the fractional integral equation*

$$\Xi(\vartheta) = \Xi_0 + \frac{1}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds$$

if and only if Ξ is the solution to the IVP for the ψ-fractional differential system

$${}^{c}\mathbf{D}^{\beta;\psi}\Xi(\vartheta) = \mathbf{H}(\vartheta,\Xi(\vartheta),\mathbb{P}\Xi(\vartheta),\mathbb{B}\Xi(\vartheta)), \vartheta \in [0,t], \beta \in (0,1), \Xi(0) = \Xi_0.$$

We have the following lemma as a result of Lemma 3.2 which will be applied in the later.

Lemma 3.3. *Let $\beta \in (0,1), \mathbf{H}, \psi : I \to \mathbb{E}$ and $\mathfrak{p}, \mathfrak{B} : \Delta \times \mathbb{E} \to \mathbb{E}$ be a continuous functions. The fractional impulsive integral equation ψ has a solution called Ξ as*

$$\begin{aligned}\Xi(\vartheta) =& \frac{1}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds - \frac{1}{l+m}\\ &\times\left[\frac{m}{\mu(\beta)}\int_0^{t}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds - n\right] + \sum_{0<\vartheta_j<\vartheta}\mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right)\end{aligned}$$

if and only if Ξ is the solution to the BVP for the ψ-fractional differential system

$$\begin{aligned} {}^{c}\mathbf{D}^{\beta;\psi}\Xi(\vartheta) &= \mathbf{H}(\vartheta,\Xi(\vartheta),\mathbb{P}\Xi(\vartheta),\mathbb{B}\Xi(s)),\ \vartheta\in\mathbb{F}:=[0,t],\beta\in(0,1),\\ \Delta\Xi|_{\vartheta=\vartheta_j} &= \mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right),\quad \vartheta=\vartheta_j,\quad j=1,2,\dots,n,\\ \mathfrak{l}\Xi(0)+\mathfrak{m}\Xi(t) &= \mathfrak{n}. \end{aligned}$$

We impose the following assumptions.

(H1) $\mathbf{H}:\mathbb{F}\times\mathbb{E}\times\mathbb{E}\times\mathbb{E}\to\mathbb{E}$ *is a continuous function.*
(H2) $\mathbf{K}>0$ *is constant such that*

$$|\mathbf{H}(\vartheta,\mathcal{U})-\mathbf{H}(\vartheta,\overline{\mathcal{U}})|\leqslant\mathbf{K}|\mathcal{U}-\overline{\mathcal{U}}|,\ \forall\,\vartheta\in\mathbb{F},\ \forall\,\mathcal{U},\overline{\mathcal{U}}\in\mathbb{E}.$$

(H3) $\mathbf{K}>0$ *is constant such that*

$$|\mathbf{H}(\vartheta,\mathcal{U},\overline{\mathcal{U}},\overline{\overline{\mathcal{U}}})|\leqslant\mathbf{K}\ \textit{for each}\ \ \vartheta\in\mathbb{F},\ \forall\,\mathcal{U},\overline{\mathcal{U}},\overline{\overline{\mathcal{U}}}\in\mathbb{E}.$$

(H4) $\mathfrak{p},\mathfrak{B}:\Delta\times\mathbb{E}\to\mathbb{E}$ *are continuous functions and* $\mathbb{P}_1,\mathbb{B}_1>0$ *are constants such that*

$$\begin{aligned} |\mathfrak{p}(\vartheta,s,\mathcal{U})-\mathfrak{p}(\vartheta,s,\overline{\mathcal{U}})| &\leqslant\mathbb{P}_1|\mathcal{U}-\overline{\mathcal{U}}|,\quad\forall\,\mathcal{U},\overline{\mathcal{U}}\in\mathbb{E},\\ |\mathfrak{B}(\vartheta,s,\mathcal{U})-\mathfrak{B}(\vartheta,s,\overline{\mathcal{U}})| &\leqslant\mathbb{B}_1|\mathcal{U}-\overline{\mathcal{U}}|,\quad\forall\,\mathcal{U},\overline{\mathcal{U}}\in\mathbb{E}. \end{aligned}$$

(H5) *There exists $\rho>0$ that says*

$$\|I_k(x)-I_k(y)\|\leqslant\rho\|x-y\|,\forall\,x,y\in X,\ \textit{with}\ \ k=1,2,\dots,m.$$

Also, there exists $\rho_1>0$ that says

$$\|I_k(x)\|\leqslant\rho_1,\ \forall\,x\in X,\ \textit{and}\ \ k=1,2,\dots,m.$$

Theorem 3.4. *Suppose* (H1), (H3), (H4), *and* (H5) *hold. Then there exists at least one solution on $\mathbb{F}$ to the* BVP (1.1).

Proof. Schaefer's fixed point theorem will be used to verify that $\mathcal{P}$ defined by

$$\begin{aligned} \mathcal{P}\Xi(\vartheta) &= \frac{1}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds-\frac{1}{\mathfrak{l}+\mathfrak{m}}\\ &\quad\times\left[\frac{\mathfrak{m}}{\mu(\beta)}\int_0^{t}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds-\mathfrak{n}\right]+\sum_{0<\vartheta_j<\vartheta}\mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right), \end{aligned}$$

has a fixed point. Proof will be presented in stages.
Claim 1: Let $\mathcal{P}$ be a continuous operator and $\{\Xi_n\}$ be a sequence such that $\Xi_n\to\Xi$ in Φ. Then for each $\vartheta\in\mathbb{F}$, we find

$$\begin{aligned} |\mathcal{P}\left(\Xi_{\mathbf{n}}\right)(\vartheta)-\mathcal{P}(\Xi)(\vartheta)| &\leqslant\frac{1}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\Big|\mathbf{H}\left(s,\Xi_n(s),\mathbb{P}_{\Xi_n}(s),\mathbb{B}_{\Xi_n}(s)\right)\\ &\quad-\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))\Big|ds+\frac{|\mathfrak{m}|}{|\mathfrak{l}+\mathfrak{m}|\mu(\beta)}\int_0^{t}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\\ &\quad\times|\mathbf{H}\left(s,\Xi_n(s),\mathbb{P}_{\Xi_n}(s),\mathbb{B}_{\Xi_n}(s)\right)-\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))|\,ds\\ &\quad+\sum_{0<\vartheta_j<\vartheta}\left\|\mathbb{I}_j\left(\Xi_n\left(\vartheta_j^-\right)\right)-\mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right)\right\| \end{aligned}$$

$$\begin{aligned}
&\leqslant \frac{1}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\sup_{s\in\mathbb{F}}\Big|\mathbf{H}\left(s,\Xi_n(s),\mathbb{P}\Xi_n(s),\mathbb{B}\Xi_n(s)\right)\\
&\quad -\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))\Big|ds+\frac{|m|}{|l+m|\mu(\beta)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\\
&\quad \times\sup_{s\in\mathbb{F}}|\mathbf{H}\left(s,\Xi_n(s),\mathbb{P}\Xi_n(s),\mathbb{B}\Xi_n(s)\right)-\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))|\,ds\\
&\quad +\sum_{0<\vartheta_j<\vartheta}\left\|\mathbb{I}_j\left(\Xi_n\left(\vartheta_j^-\right)\right)-\mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right)\right\|\\
&\leqslant \frac{1}{\mu(\beta)}\left[\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}ds+\frac{|m|}{|l+m|}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds\right]\\
&\quad \times\left\|\mathbf{H}\left(\cdot,\Xi_n(\cdot),\mathbb{P}_{\Xi_n}(\cdot),\mathbb{B}_{\Xi_n}(\cdot)\right)-\mathbf{H}(\cdot,\Xi(\cdot),\mathbb{P}\Xi(\cdot),\mathbb{B}\Xi(\cdot))\right\|_{\infty}+\|\Xi_n-\Xi\|\,\rho\\
&\leqslant \frac{(\psi(t))^{\beta}}{\mu(\beta+1)}\left(1+\frac{|m|}{|l+m|}\right)\\
&\quad \times\left\|\mathbf{H}\left(\cdot,\Xi_n(\cdot),\mathbb{P}\Xi_n(\cdot),\mathbb{B}\Xi_n(\cdot)\right)-\mathbf{H}(\cdot,\Xi(\cdot),\mathbb{P}\Xi(\cdot),\mathbb{B}\Xi(\cdot))\right\|_{\infty}+\|\Xi_n-\Xi\|\,\rho.
\end{aligned}$$

Claim 2: In Φ, the operator $\mathcal{P}$ maps bounded sets into bounded sets. In fact, it is sufficient to prove that there exists a positive constant ζ for each $\Xi\in\mathbf{D}_q=\{\Xi\in\Phi:\|\Xi\|_{\infty}\leqslant q\}$, where $q>0$ such that $\|\mathcal{P}(\Xi)\|_{\infty}\leqslant\zeta$. By (H3), we have

$$\begin{aligned}
|\mathcal{P}(\Xi)(\vartheta)| &\leqslant \frac{1}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}|\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))|ds\\
&\quad +\frac{|m|}{|l+m|}\frac{1}{\mu(\beta)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\beta-1}|\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))|ds\\
&\quad +\frac{|n|}{|l+m|}+\sum_{0<\vartheta_j<\vartheta}\left\|\mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right)\right\|\\
&\leqslant \frac{\mathbf{K}}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}ds+\frac{|m|}{|l+m|}\frac{\mathbf{K}}{\mu(\beta)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds+\frac{|n|}{|l+m|}+\rho_1\\
&\leqslant \frac{\mathbf{K}}{\mu(\beta+1)}(\psi(t))^{\beta}+\frac{\mathbf{K}|m|}{|\beta+1|\mu(l+m)}(\psi(t))^{\beta}+\frac{|n|}{|l+m|}+\rho_1
\end{aligned}$$

for every $\vartheta\in\mathbb{F}$. Therefore we find

$$\|\mathcal{P}(\Xi)\|_{\infty}\leqslant\frac{\mathbf{K}}{\mu(\beta+1)}(\psi(t))^{\beta}+\frac{\mathbf{K}|m|}{|l+m|\mu(\beta+1)}(\psi(t))^{\beta}+\frac{|n|}{|l+m|}+\rho_1:=\zeta.$$

Claim 3: The operator $\mathcal{P}$ maps bounded sets into equicontinuous sets of Φ. Let $\vartheta_1,\vartheta_2\in\mathbb{F}$ with $\vartheta_1<\vartheta_2$ and $\mathbf{D}_q$ be the bounded set with $\Xi\in\mathbf{D}_q$ in Φ as in Claim 2. Then

$$\begin{aligned}
&|\mathcal{P}(\Xi)\left(\vartheta_2\right)-\mathcal{P}(\Xi)\left(\vartheta_1\right)|\\
&\quad\leqslant\Big|\frac{1}{\mu(\beta)}\int_0^{\vartheta_1}\psi'(s)\left[\left(\psi\left(\vartheta_2\right)-\psi(s)\right)^{\beta-1}-\left(\psi\left(\vartheta_1\right)-\psi(s)\right)^{\beta-1}\right]\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds\\
&\qquad+\frac{1}{\mu(\beta)}\int_{\vartheta_1}^{\vartheta_2}\psi'(s)\left(\psi\left(\vartheta_2\right)-\psi(s)\right)^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds\Big|+\sum_{0<\vartheta_j<\vartheta}\left\|\mathbb{I}_j\left(\Xi\left(\vartheta_2^-\right)\right)-\mathbb{I}_j\left(\Xi\left(\vartheta_1^-\right)\right)\right\|\\
&\quad\leqslant\frac{\mathbf{K}}{\mu(\beta)}\int_0^{\vartheta_1}\psi'(s)\left[\left(\psi\left(\vartheta_2\right)-\psi(s)\right)^{\beta-1}-\left(\psi\left(\vartheta_1\right)-\psi(s)\right)^{\beta-1}\right]ds\\
&\qquad+\frac{\mathbf{K}}{\mu(\beta)}\int_{\vartheta_1}^{\vartheta_2}\psi'(s)\left(\psi\left(\vartheta_2\right)-\psi(s)\right)^{\beta-1}ds+\left\|\Xi\left(\vartheta_2\right)-\Xi\left(\vartheta_1\right)\right\|\rho
\end{aligned}$$

$$\leqslant \frac{\mathbf{K}}{\mu(\beta+1)}\left(\psi\left(\vartheta_2\right)-\psi\left(\vartheta_1\right)\right)^{\beta}+\frac{\mathbf{K}}{\mu(\beta+1)}\left(\left(\psi\left(\vartheta_1\right)\right)^{\beta}-\left(\psi\left(\vartheta_2\right)\right)^{\beta}\right)\left\|\Xi\left(\vartheta_2\right)-\Xi\left(\vartheta_1\right)\right\|\rho.$$

Since $\vartheta_1 \to \vartheta_2$, the above inequality's right-hand side tends to zero. We will able to finish that $\mathbf{H}:\Phi\to\Phi$ is continuous because of a result of Stages 1 to 3 together with the Arzela-Ascoli theorem.

Claim 4: A priori bounds. Now it remains to prove that

$$\kappa=\{\Xi\in\Phi:\Xi=\mu\mathcal{P}(\Xi) \ \text{ for any } \ \mu\in(0,1)\}$$

is bounded. If $\Xi\in\kappa$, then $\mu\mathcal{P}(x)$ for any $\mu\in(0,1)$. Hence, for each $\vartheta\in\mathbb{F}$, we get the following:

$$\begin{aligned}\Xi(\vartheta)=\mu\Big[&\frac{1}{\mu(\beta)}\int_0^{\vartheta}\psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds-\frac{1}{l+m}\\&\times\left[\frac{m}{\mu(\beta)}\int_0^{t}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}\mathbf{H}(s,\Xi(s),\mathbb{P}\Xi(s),\mathbb{B}\Xi(s))ds-n\right]\Big]+\sum_{0<\vartheta_j<\vartheta}\mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right).\end{aligned}$$

We complete this stage by considering the estimation in Claim 2. As a result of Schaefer's fixed point theorem, we finish the proof that $\mathcal{P}$ has fixed point which is the solution of the problem (1.1). □

4. Ulam-Hyers-Rassias stability

In this part, we study the Ulam stability of BVP for ψ-fractional differential system (1.1).

$$^{c}\mathbf{D}^{\beta;\psi}\Xi(\vartheta)=\mathbf{H}(\vartheta,\Xi(\vartheta),\mathbb{P}\Xi(\vartheta),\mathbb{B}\Xi(\vartheta)),\quad \vartheta\in\mathbb{F}:=[0,t]. \tag{4.1}$$

Let $\mathbf{H}:\mathbb{F}\times\mathbb{E}\times\mathbb{E}\times\mathbb{E}\to\mathbb{E}$, $\mathfrak{p},\mathfrak{B}:\Delta\times\mathbb{E}\to\mathbb{E}$ be continuous functions. For the simplicity, we make use of

$$\mathbb{P}\Xi(\vartheta)=\int_0^{\vartheta}\mathfrak{p}(\vartheta,s,\Xi(s))ds,\quad \mathbb{B}\Xi(\vartheta)=\int_0^{\vartheta}\mathfrak{B}(\vartheta,s,\Xi(s))ds.$$

We pay attention to the topic of a novel operator with respect to another function, as it covers many fractional systems that are special cases for various values. More precisely, the existence and the Ulam-Hyers-Rassias stability of solutions to the system (4.1) are obtained in weighted spaces by using standard Schaefer's fixed point theorem and the following inequalities:

$$\left|^{c}\mathbf{D}^{\beta;\psi}\Upsilon(\vartheta)-\mathbf{H}(\vartheta,\Upsilon(\vartheta),\mathbb{P}\Upsilon(\vartheta),\mathbb{B}\Upsilon(\vartheta))\right|\leqslant\epsilon,\quad\vartheta\in\mathbb{F}, \tag{4.2}$$

$$\left|^{c}\mathbf{D}^{\beta;\psi}\Upsilon(\vartheta)-\mathbf{H}(\vartheta,\Upsilon(\vartheta),\mathbb{P}\Upsilon(\vartheta),\mathbb{B}\Upsilon(\vartheta))\right|\leqslant\epsilon\varphi(\vartheta),\quad\vartheta\in\mathbb{F}, \tag{4.3}$$

$$\left|^{c}\mathbf{D}^{\beta;\psi}\Upsilon(\vartheta)-\mathbf{H}(\vartheta,\Upsilon(\vartheta),\mathbb{P}\Upsilon(\vartheta),\mathbb{B}\Upsilon(\vartheta))\right|\leqslant\varphi(\vartheta),\quad\vartheta\in\mathbb{F}. \tag{4.4}$$

Definition 4.1 ([23, 29])**.** System (4.1) is Ulam-Hyers stable if there exists $C_f>0$ which is a real number such that for each $\epsilon>0$ and for every solution $\Upsilon\in\Phi$ of inequality (4.2) there exists a solution $\Xi\in\Phi$ of system (4.1) along with

$$|\Upsilon(\vartheta)-\Xi(\vartheta)|\leqslant C_{f^{\epsilon}},\quad\vartheta\in J.$$

Definition 4.2 ([30])**.** System (4.1) is generalized Ulam-Hyers stable if there exists $\psi_f\in C([0,\infty),[0,\infty))$, $\psi_f(0)=0$ such that for one solution $\Upsilon\in\Phi$ of inequality (4.2) there exists a solution $\Xi\in\Phi$ of system (4.1) with

$$|\Upsilon(\vartheta)-\Xi(\vartheta)|\leqslant\psi_{f^{\epsilon}},\quad\vartheta\in J.$$

Definition 4.3 ([29])**.** System (4.1) is Ulam-Hyers-Rassias stable with respect to $\varphi\in\Phi$ if there exists a real number $C_f>0$ such that for each $\epsilon>0$ and for any solution $\Upsilon\in\Phi$ of inequality (4.1) there exists a solution $\Xi\in\Phi$ of system (4.3) with

$$|\Upsilon(\vartheta)-\Xi(\vartheta)|\leqslant C_{f^{\epsilon}}\varphi(\vartheta),\quad\vartheta\in J.$$

Definition 4.4 ([30])**.** System (4.1) is generalized Ulam-Hyers-Rassias stable with respect to $\varphi \in \Phi$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\Upsilon \in \Phi$ of inequality (4.4) there exists a solution $\Xi \in \Phi$ of system (4.1) with

$$|\Upsilon(\vartheta) - \Xi(\vartheta)| \leqslant C_{f,\varphi}\varphi(\vartheta), \quad \vartheta \in J.$$

Remark 4.5. A function $\Upsilon \in \Phi$ is a solution of (4.2) if and only if there exists a function $g \in \Phi$ (which depend on Υ) such that

$$(1)\ |g(\vartheta)| \leqslant \epsilon, \vartheta \in \mathbb{F}; \qquad (2)\ {}^{c}\mathbf{D}^{\beta;\psi}\Upsilon(\vartheta) = \mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta)) + g(\vartheta), \vartheta \in \mathbb{F}.$$

Remark 4.6. Let $\beta \in (0, 1)$. If $\Upsilon \in \Phi$ is a solution of the inequality (4.2), then the inequality has a solution of Υ,

$$\left|\Upsilon(\vartheta) - \mathfrak{A}_{\Upsilon} - \frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta))ds\right| \leqslant \epsilon\frac{(\psi(t))^{\beta}}{\lambda(\beta+1)}\left(1 + \frac{|m|}{|l+m|}\right).$$

In fact, by Remark 4.5, we have

$$\begin{aligned}
{}^{c}\mathbf{D}^{\beta;\psi}\Upsilon(\vartheta) &= \mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta)) + g(\vartheta), \text{ where } \vartheta \in \mathbb{F}, \\
\Upsilon(\vartheta) &= \mathfrak{A}_{\Upsilon} + \frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta))ds \\
&\quad + \frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}g(s)ds \\
&\quad - \left(\frac{m}{l+m}\right)\frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}g(s)ds, \quad \vartheta \in \mathbb{F},
\end{aligned}$$

with

$$\mathfrak{A}_{\Upsilon} = \frac{1}{l+m}\left[n - \frac{m}{\lambda(\beta)}\int_0^{t} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta))ds\right].$$

From this, it follows that

$$\begin{aligned}
&|\ \Upsilon(\vartheta) - \mathfrak{A}_{\Upsilon} - \frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta))ds\ | \\
&\quad = |\ \frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}g(s)ds - \left(\frac{m}{l+m}\right)\frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}g(s)ds\ | \\
&\quad \leqslant \frac{1}{\lambda(\beta)}\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}|g(s)|ds - \left(\frac{m}{l+m}\right)\int_0^{\vartheta} \psi'(s)(\psi(\vartheta) - \psi(s))^{\beta-1}|g(s)|ds \\
&\quad \leqslant \epsilon\frac{(\psi(t))^{\beta}}{\lambda(\beta+1)}\left(1 + \frac{|m|}{|l+m|}\right).
\end{aligned}$$

Remark 4.7. It is obvious that Definition 4.1 $\Rightarrow$ Definition 4.2 and Definition 4.3 $\Rightarrow$ Definition 4.4.

Remark 4.8. A solution of the ψ-fractional differential equations with boundary condition inequality (4.2) is said to be an ϵ-solution of problem (4.1).

Theorem 4.9. *Suppose* (H1), (H2), (H4), (H5) *and*

$$\frac{\mathbf{K}\left(1 + \mathbb{P}_1 + \mathbb{B}_1\right)(\psi(t))^{\beta}}{\mu(\beta+1)}\left(1 + \frac{|m|}{|l+m|}\right) + \rho\|\Xi - \mathfrak{v}\|_{\infty} < 1, \tag{4.5}$$

hold. Then, the problem (1.1) *is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0, \Upsilon \in \Phi$ be a function satisfying inequality (4.2) and $\Xi \in \Phi$ be the unique solution of the following problem,

$$\begin{aligned} {}^{c}\mathbf{D}^{\beta;\psi}\Xi(\vartheta) &= \mathbf{H}(\vartheta, \Xi(\vartheta), \mathbb{P}\Xi(\vartheta), \mathbb{B}\Xi(\vartheta)), \quad \vartheta \in \mathbb{F}, \quad \beta \in (0,1), \\ \Delta\Xi|_{\vartheta=\vartheta_j} &= \mathbb{I}_j\left(\Xi\left(\vartheta_k^-\right)\right), \quad \vartheta = \vartheta_j, \quad j = 1, 2, \ldots, n, \\ \Xi(0) &= \Upsilon(0),\ \Xi(t) = \Upsilon(t). \end{aligned}$$

Using Lemma 3.3, we obtain

$$\Xi(\vartheta) = \mathfrak{A}_\Xi + \frac{1}{\lambda(\beta)}\int_0^\vartheta \psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Xi(\vartheta), \mathbb{P}\Xi(\vartheta), \mathbb{B}\Xi(\vartheta))ds + \sum_{0<\vartheta_j<\vartheta} \mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right).$$

Alternatively, if $\Xi(0) = \Upsilon(0), \Xi(t) = \Upsilon(t)$, then $\mathfrak{A}_\Xi = \mathfrak{A}_\Upsilon$. In fact,

$$\begin{aligned} |\mathfrak{A}_\Xi - \mathfrak{A}_\Upsilon| &\leqslant \frac{|m|}{|l+m|\lambda(\beta)}\int_0^t \psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1} \mid \mathbf{H}(\vartheta, \Xi(\vartheta), \mathbb{P}\Xi(\vartheta), \mathbb{B}\Xi(\vartheta)) \\ &\quad \left. -\mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta)) \mid ds + \sum_{0<\vartheta_j<\vartheta} \|\mathbb{I}_j\Xi\left(\vartheta_j^-\right) - \mathbb{I}_j\Upsilon\left(\vartheta_j^-\right)\| \right) \\ &\leqslant \frac{\mathbf{K}\left(1+\mathbb{P}_1+\mathbb{B}_1\right)|m|}{|l+m|} I^{\beta;\psi}|\Xi(t)-\Upsilon(t)| + \rho\|\Xi-\Upsilon\| = 0. \end{aligned}$$

Therefore we obtain $\mathfrak{A}_\Xi = \mathfrak{A}_\Upsilon$ and so

$$\Xi(\vartheta) = \mathfrak{A}_\Xi + \frac{1}{\lambda(\beta)}\int_0^\vartheta \psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Xi(\vartheta), \mathbb{P}\Xi(\vartheta), \mathbb{B}\Xi(\vartheta))ds + \sum_{0<\vartheta_j<\vartheta} \mathbb{I}_j\left(\Xi\left(\vartheta_j^-\right)\right).$$

By integration of inequality (4.2) and using Remark 4.6, we conclude that

$$\begin{aligned} &\left|\Upsilon(\vartheta) - \mathfrak{A}_\Upsilon - \frac{1}{\lambda(\beta)}\int_0^\vartheta \psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta))ds\right| + \sum_{0<\vartheta_j<\vartheta} \mathbb{I}_j\left(\Upsilon\left(\vartheta_j^-\right)\right) \\ &\leqslant \epsilon\frac{(\psi(t))^\beta}{\lambda(\beta+1)}\left(1+\frac{|m|}{|l+m|}\right) + \rho. \end{aligned}$$

For any $\vartheta \in \mathbb{F}$, we find

$$\begin{aligned} |\Upsilon(\vartheta) - \Xi(\vartheta)| &\leqslant \left|\Upsilon(\vartheta) - \mathfrak{A}_\Upsilon - \frac{1}{\lambda(\beta)}\int_0^\vartheta \psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}\mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta))ds\right| \\ &\quad + \frac{1}{\lambda(\beta)}\int_0^\vartheta \psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1} \mid \mathbf{H}(\vartheta, \Upsilon(\vartheta), \mathbb{P}\Upsilon(\vartheta), \mathbb{B}\Upsilon(\vartheta)) \\ &\quad \left. -\mathbf{H}(\vartheta, \Xi(\vartheta), \mathbb{P}\Xi(\vartheta), \mathbb{B}\Xi(\vartheta)) \mid ds + \sum_{0<\vartheta_j<\vartheta} \|\mathbb{I}_j\Upsilon\left(\vartheta_j^-\right) - \mathbb{I}_j\Xi\left(\vartheta_j^-\right)\| \right) \\ &\leqslant \epsilon\frac{(\psi(t))^\beta}{\lambda(\beta+1)}\left(1+\frac{|m|}{|l+m|}\right) \\ &\quad + \frac{\mathbf{K}\left(1+\mathbb{P}_1+\mathbb{B}_1\right)}{\lambda(\beta)}\int_0^\vartheta \psi'(s)(\psi(\vartheta)-\psi(s))^{\beta-1}|\Upsilon(s)-\Xi(s)|ds + \rho\|\Upsilon-\Xi\|. \end{aligned}$$

Using the Gronwall inequality, Lemma 2.3, and Remark 2.4, we obtain

$$|\Upsilon(\vartheta)-\Xi(\vartheta)| \leqslant \left(1+\frac{|m|}{|l+m|}\right)\frac{\epsilon(\psi(t))^{\beta}}{\lambda(\beta+1)}E_{\beta;\psi}\left(\mathbf{K}(1)+\mathbb{P}_1(\psi(t))^{\beta}+\mathbb{B}_1(\psi(t))^{\beta}\right)+\rho\|\Upsilon-\Xi\|.$$

Thus, problem (1.1) is Ulam-Hyers stable. □

Theorem 4.10. *Suppose* (H1), (H2), (H4), *and inequality* (4.5) *hold. If there exists an increasing function* $\varphi \in \Phi$ *and* $\mu_\varphi > 0$, *then the system* (1.1) *is Ulam-Hyers-Rassias stable.*

Proof. By using Theorem 4.9, we get

$$I^{\beta;\psi}\varphi(\vartheta) \leqslant \mu_\varphi \varphi(\vartheta)$$

for each $\vartheta \in \mathbb{F}$. Thus, in view of Definition 4.4, then the system (1.1) is Ulam-Hyers stable. □

Remark 4.11. Under the assumptions of Theorem 4.9, we consider problem (1.1) and inequality (4.4). One can repeat the same process to verify that problem (1.1) is Ulam-Hyers-Rassias stable.

5. Concluding remarks

In this work, we establish some new conditions for the existence of solutions for a class of nonlinear ψ-Caputo fractional Volterra-Fredholm integro-differential systems with boundary conditions. In addition, the Ulam-Hyers-Rassias stability for solutions of the given problem are also discussed. The desired results are proved by using fixed point theorems due to Banach and Schaefer's in Banach spaces. Our results in the given configuration are novel and substantially contribute to the literature on this new field of study. We feel that there are multiple potential study avenues such as coupled systems, problems with infinite delays, and many more due to the limited number of publications on fractional Volterra-Fredholm integro-differential equations, particularly with non-instantaneous impulses. We hope that this work will serve as a starting point for such an undertaking.

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