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Generalized neutrosophic ideal convergent sequence spaces

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Abstract

Kostyrko et al. initiated the concept of ideal convergence in [P. Kostyrko, T. Šalát, W. Wilczyński, Real Anal. Exchange, **26** (2000), 669–686]. The purpose of this paper is to introduce and define spaces of the neutrosophic convergent sequence via ideal, namely ${}^{I}\mathcal{S}_{\mathcal{M}}$ and ${}^{I}\mathcal{S}_{\mathcal{M}_0}$. We prove that new spaces are linear and Hausdorff topological spaces. Further, we examine the relation between I-Cauchy and I-convergent sequences and show that every separable space ${}^{I}\mathcal{S}_{\mathcal{M}}$ is second countable. Moreover, we prove that the space ${}^{I}\mathcal{S}_{\mathcal{M}}$ is complete.

Keywords: Ideal, filter, I-convergence, I-Cauchy, t-norm, t-conorm, neutrosophic normed space.

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1. Introduction

The theory named as fuzzy theory initiated by Zadeh [22] in 1965 since then, fuzzification of various classical theories have been made. In 1986, Atanassov [1] generalized fuzzy sets and introduced concepts of intuitionistic fuzzy (IF) sets. Park [17] analyzed the intuitionistic fuzzy topological spaces. Saadati and Park [18] further examined the intuitionistic fuzzy normed linear spaces. Recently, Smarandache [20] extended the intuitionistic fuzzy sets and introduced the notion known as neutrosophic sets (NS). This notion is more flexible and effective because it handles, besides independent components, also partially independent and partially dependent components, while intuitionistic fuzzy sets cannot deal with these. Moreover, Bera and Mahapatra [2] studied neutrosophic soft linear space and examined convexity and Cauchy sequences. Kirişci [11] introduced neutrosophic normed spaces and studied statistical convergence on it. Khan et al. [8, 10] analyzed the continuous, bounded linear operator and Fibonacci matrix in neutrosophic normed spaces.

Fast [5] and Steinhaus [21] independently introduced statistical convergence of sequences of real numbers. Later on, Kostyrko et al. [12] generalized this concept and introduced I-convergence. Esi [3] analyzed strongly summable double sequence spaces in n-normed spaces defined by ideal convergence and

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an Orlicz function. Hazarika and Esi [6] further examined theLacunary ideal summability and its applications to approximation theorems. Mursaleen and Mohiuddine [13, 14] studied ideal and statistical convergence of sequences in IF-normed space. Several researchers examined these theories in different areas (see, [4, 7, 9, 15, 16, 19]).

Now, Recalling some basic notion, definitions and remarks.

Definition 1.1 ([12]). Suppose W be a non-void set. A family of subsets I of the power set 2^W is known as an ideal if $\emptyset \in I$, for each $I_1, I_2 \in I$ implies $I_1 \cup I_2 \in I$ and for each $I_1 \in I, I_2 \subset I_1$ implies $I_2 \in I$.

Definition 1.2 ([12]). Suppose W be a non-void set. Then a family of subsets $\mathcal{F}$ of 2^W is said to be a filter on W if and only if $\emptyset \notin \mathcal{F}$, for $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{F}$ we have $\mathcal{F}_1 \cap \mathcal{F}_2 \in \mathcal{F}$ and for each $\mathcal{F}_1 \in \mathcal{F}$ and $\mathcal{F}_2 \supset \mathcal{F}_1$ implies $\mathcal{F}_2 \in \mathcal{F}$.

Remark 1.3 ([12]). For evry ideal I there is a filter corresponding to I such that

$$\mathcal{F}(I) = \{T \subset \mathbb{N} : T^c \in I\}.$$

Definition 1.4 ([12]). A sequence $w = (w_k)$ is called I-convergent to a number L if, for each $\epsilon > 0$, there exists a positive integer $\mathbb{N}$ such that, the set

$$\{k \in \mathbb{N} :| w_k - L |\geqslant \epsilon\} \in I.$$

Definition 1.5 ([17]). A binary operation $\star$ on $[0, 1]$ is said to be a continuous t-norm if $\star$ is associative and commutative and it satisfies the following conditions

(i) $d_1 \star 1 = d_1$ for all $d_1 \in [0, 1]$;
(ii) $d_1 \star d_2 \leqslant d_3 \star d_4$, whenever $d_1 \leqslant d_3$ and $d_2 \leqslant d_4$, for each $d_1, d_2, d_3, d_4 \in [0, 1]$.

Definition 1.6 ([17]). A binary operation $\diamond$ on $[0, 1]$ is said to be a continuous t-conorm if $\diamond$ is associative and commutative and it satisfies the following conditions

(i) $d_1 \diamond 0 = d_1$ for all $d_1 \in [0, 1]$;
(ii) $d_1 \diamond d_2 \leqslant d_3 \diamond d_4$, whenever $d_1 \leqslant d_3$ and $d_2 \leqslant d_4$ for each $d_1, d_2, d_3, d_4 \in [0, 1]$.

Remark 1.7 ([17]). The relation between t-norm and t-conorm which are as follows, if one select $0 < d_1, d_2 < 1$ for $d_1 > d_2$, then there exist $0 < d_3, d_4 < 1$ in such a way that $d_1 \star d_3 \geqslant d_2, d_1 \geqslant d_4 \diamond d_2$. Further, if one select $d_5 \in (0, 1)$, then there exist $d_6, d_7 \in (0, 1)$ in such a way that $d_6 * d_6 \geqslant d_5$ and $d_7 \diamond d_7 \leqslant d_5$.

Definition 1.8 ([20]). Suppose $\mathcal{S}$ be a subset of a universe of discourse W. Then,

$$\mathcal{S} = \{< w, \mathcal{P}(w), \mathcal{Q}(w), \mathcal{R}(w) >: w \in W\},$$

where, neutrosophic component $\mathcal{P}(w)$ denotes the degree of truth-membership, $\mathcal{Q}(w)$ denotes degree of indeterminacy-membership, and $\mathcal{R}(w)$ denotes the degree of false-nonmembership, respectively, such that $0 \leqslant \mathcal{P}(w) + \mathcal{Q}(w) + \mathcal{P}(w) \leqslant 3$.

Definition 1.9 ([11]). Take W as a vector space and $\mathcal{M} : W \times (0, \infty) \to [0, 1]$ be a norm. Assume $\star$ and $\diamond$ be the continuous t-norm and t-conorm, respectively, then, the four-tuple $(W, \mathcal{M}, \star, \diamond)$ is known as neutrosophic normed space (NNS) if the subsequent conditions are hold; for all $w, z \in W$ and $t, s > 0$:

(i) $0 \leqslant \mathcal{P}(w, t),\ \ \mathcal{Q}(w, t),\ \ \ \mathcal{R}(w, t) \leqslant 1$;
(ii) $0 \leqslant \mathcal{P}(w, t) + \mathcal{Q}(w, t) + \mathcal{R}(w, t) \leqslant 3$;
(iii) $\mathcal{P}(w, t) = 0$ for $t \leqslant 0$;
(vi) $\mathcal{P}(w, t) = 1$ for $t > 0$ iff $w = 0$;
(v) $\mathcal{P}(cw, t) = \mathcal{P}(x, \frac{t}{|c|})$, $\forall\ c \neq 0,\ \ t > 0$;

(vi) $\mathcal{P}(w,t) \star \mathcal{P}(z,s) \leqslant \mathcal{P}(w+z,t+s)$;
(vii) $\mathcal{P}(w,\star)$ is non-decreasing continuous function for $t > 0$, $\lim_{t\to\infty} \mathcal{P}(w,t) = 1$;
(viii) $\mathcal{Q}(w,t) = 1$ for $t \leqslant 0$;
(ix) $\mathcal{Q}(w,t) = 0$ for $t > 0$ iff $w = 0$;
(x) $\mathcal{Q}(\alpha w,t) = \mathcal{Q}(w, \frac{t}{|c|})$, $c \neq 0$;
(xi) $\mathcal{Q}(w,t) \diamond \mathcal{Q}(z,s) \geqslant \mathcal{Q}(w+z,t+s)$;
(xii) $\mathcal{Q}(w,\diamond)$ is non-increasing continuous function, $\lim_{t\to\infty} \mathcal{Q}(w,t) = 0$;
(xiii) $\mathcal{R}(w,t) = 1$ for $t \leqslant 0$;
(xiv) $\mathcal{R}(w,t) = 0$ for $t > 0$ iff $w = 0$;
(xv) $\mathcal{R}(cw,t) = \mathcal{R}(w, \frac{t}{|c|})$, $\forall\ c \neq 0$;
(xvi) $\mathcal{R}(w,t) \diamond \mathcal{R}(z,s) \geqslant \mathcal{R}(w+z,t+s)$;
(xvii) $\mathcal{R}(w,\diamond)$ is non-increasing continuous function, $\lim_{t\to\infty} \mathcal{R}(w,t) = 0$.

In this case $\mathcal{M} = (\mathcal{P},\mathcal{Q},\mathcal{R})$ is called a neutrosophic norm.

Example 1.10 ([11]). Suppose $(W, \|\ .\ \|)$ be a normed space. Give the operations as $w \star z = w + z - wz$ and $w \diamond z = \min(w,z)$. For $t > \|w\|$,

$$\mathcal{P}(w,t) = \frac{t}{t + \|w\|}, \quad \mathcal{Q}(w,t) = \frac{\|w\|}{t + \|w\|}, \quad \mathcal{R}(w,t) = \frac{\|w\|}{t}$$

for all $w, z \in W$ and $0 < t$. If one take $t \leqslant \|w\|$, then $\mathcal{P}(w,t) = 0$, $\mathcal{Q}(w,t) = 1$ and $\mathcal{R}(w,t) = 1$. Therefore, $(W, \mathcal{M}, \star, \diamond)$ is NNS.

Definition 1.11 ([11]). Let $(W, \mathcal{M}, \star, \diamond)$ be NNS. Then a sequence $w = (w_k)$ is called convergent to $L \in W$ with respect to the $\mathcal{M}$ if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mathcal{P}(w_k - L, t) > 1 - \epsilon$, $\mathcal{Q}(w_k - L, t) < \epsilon$ and $\mathcal{R}(w_k - L, t) < \epsilon$ for all $k \geqslant k_0$.

Definition 1.12 ([11]). Let $(W, \mathcal{M}, \star, \diamond)$ be a NNS. Then a sequence $w = (w_k)$ is said to be a Cauchy with respect to the $\mathcal{M}$ if, for every $\epsilon > 0$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mathcal{P}(w_k - w_l, t) > \epsilon$, $\mathcal{Q}(w_k - w_l, t) < \epsilon$ and $\mathcal{R}(w_k - w_l, t) < \epsilon$ for all $k, l \geqslant k_0$.

2. Main results

In this paper, we introduce and define new sequence spaces with the help of neutrosophic normed space and ideal which are as follows

$$^{I}\mathcal{S}_{\mathcal{M}} = \Big\{(w_k) \in \ell_\infty : \{k \in \mathbb{N} : \mathcal{P}(w_k - L, t) \leqslant 1 - \epsilon \text{ or } \mathcal{Q}(w_k - L, t) \geqslant \epsilon, \mathcal{R}(w_k - L, t) \geqslant \epsilon\}\Big\} \in I;$$
$$^{I}\mathcal{S}_{\mathcal{M}_0} = \Big\{(w_k) \in \ell_\infty : \{k \in \mathbb{N} : \mathcal{P}(w_k, t) \leqslant 1 - \epsilon \text{ or } \mathcal{Q}(w_k, t) \geqslant \epsilon, \mathcal{R}(w_k, t) \geqslant \epsilon\}\Big\} \in I.$$

The open ball with radius γ and center w with respect to t is defined as follows:

$$\mathcal{O}_w(\gamma, t) = \Big\{(z_k) \in \ell_\infty : \{k \in \mathbb{N} : \mathcal{P}(w_k - z_k, t) \leqslant 1 - \epsilon \text{ or } \mathcal{Q}(w_k - z_k, t) \geqslant \epsilon, \mathcal{R}(w_k - z_k, t) \geqslant \epsilon\}\Big\}.$$

Theorem 2.1. *The spaces* $^{I}\mathcal{S}_{\mathcal{M}}$ *and* $^{I}\mathcal{S}_{\mathcal{M}_0}$ *are linear over* $\mathbb{R}$.

Proof. Assume $w = (w_k), z = (z_k) \in\ ^{I}\mathcal{S}_{\mathcal{M}}$ and α, β be scalars. Then for a given $\epsilon > 0$, one obtains

$$A_1 = \Big\{k \in \mathbb{N} : \mathcal{P}\Big(w_k - L_1, \tfrac{t}{2|\alpha|}\Big) \leqslant 1 - \epsilon \text{ or } \mathcal{Q}\Big(w_k - L_1, \tfrac{t}{2|\alpha|}\Big) \geqslant \epsilon,\ \ \mathcal{R}\Big(w_k - L_1, \tfrac{t}{2|\alpha|}\Big) \geqslant \epsilon\Big\},$$

$$A_2 = \Big\{k \in \mathbb{N} : \mathcal{P}\Big(z_k - L_2, \tfrac{t}{2|\beta|}\Big) \leqslant 1-\epsilon \text{ or } \mathcal{Q}\Big(z_k - L_2, \tfrac{t}{2|\beta|}\Big) \geqslant \epsilon,\ \mathcal{R}\Big(z_k - L_2, \tfrac{t}{2|\beta|}\Big) \geqslant \epsilon\Big\}.$$

Therefore

$$A_1^c = \Big\{k \in \mathbb{N} : \mathcal{P}\Big(w_k - L_1, \tfrac{t}{2|\alpha|}\Big) > 1-\epsilon \text{ or } \mathcal{Q}\Big(w_k - L_1, \tfrac{t}{2|\alpha|}\Big) < \epsilon,\ \mathcal{R}\Big(w_k - L_1, \tfrac{t}{2|\alpha|}\Big) < \epsilon\Big\}$$
$$A_2^c = \Big\{k \in \mathbb{N} : \mathcal{P}\Big(z_k - L_2, \tfrac{t}{2|\beta|}\Big) > 1-\epsilon \text{ or } \mathcal{Q}\Big(z_k - L_2, \tfrac{t}{2|\beta|}\Big) < \epsilon,\ \mathcal{R}\Big(z_k - L_2, \tfrac{t}{2|\beta|}\Big) < \epsilon\Big\}.$$

Determine set $A_3 = A_1 \cup A_2$, therefore it implies that the set A_3^c is a non-void set . One shall prove that for each $(w_k), (z_k) \in \ {}^I\mathcal{S}_\mathcal{M}$,

$$A_3^c \subset \Big\{k \in \mathbb{N} : \mathcal{P}((\alpha w_k + \beta z_k) - (\alpha L_1 + \beta L_2), t) > 1-\epsilon \text{ or } \mathcal{Q}((\alpha w_k + \beta z_k) - (\alpha L_1 + \beta L_2), t) < \epsilon \text{ or } \mathcal{R}((\alpha w_k + \beta z_k) - (\alpha L_1 + \beta L_2), t) < \epsilon\Big\}.$$

Let $m \in A_3^c$. In this case

$$\mathcal{P}\Big(w_m - L_1, \frac{t}{2\mid\alpha\mid}\Big) > 1-\epsilon \text{ or } \mathcal{Q}\Big(w_m - L_1, \frac{t}{2\mid\alpha\mid}\Big) < \epsilon \text{ or } \mathcal{R}\Big(w_m - L_1, \frac{t}{2\mid\alpha\mid}\Big) < \epsilon$$

and

$$\mathcal{P}\Big(z_m - L_2, \frac{t}{2\mid\beta\mid}\Big) > 1-\epsilon \text{ or } \mathcal{Q}\Big(z_m - L_2, \frac{t}{2\mid\beta\mid}\Big) < \epsilon \text{ or } \mathcal{R}\Big(w_m - L_1, \frac{t}{2\mid\alpha\mid}\Big) < \epsilon.$$

We have

$$\begin{aligned}
\mathcal{P}\Big(\alpha w_m + \beta z_m - \alpha L_1 + \beta L_2, t\Big) &\geqslant \mathcal{P}\Big(alpha w_m - \alpha L_1, \frac{t}{2}\Big) \star \mathcal{P}\Big(\beta w_m - \beta L_2, \frac{t}{2}\Big)\\
&= \mathcal{P}\Big(w_m - L_1, \frac{t}{2\mid\alpha\mid}\Big) \star \mathcal{P}\Big(w_m - L_2, \frac{t}{2\mid\beta\mid}\Big)\\
&> (1-\epsilon)\star(1-\epsilon) > 1-\epsilon,\\
\mathcal{Q}\Big(\alpha w_m + \beta z_m - \alpha L_1 + \beta L_2, t\Big) &\leqslant \mathcal{Q}\Big(\alpha w_m - \alpha L_1, \frac{t}{2}\Big) \diamond \mathcal{Q}\Big(\beta w_m - \beta L_2, \frac{t}{2}\Big)\\
&= \mathcal{Q}\Big(w_m - L_1, \frac{t}{2\mid\alpha\mid}\Big) \diamond \mathcal{Q}\Big(w_m - L_2, \frac{t}{2\mid\beta\mid}\Big) < \epsilon \diamond \epsilon = \epsilon,\\
\mathcal{R}\Big(\alpha w_m + \beta z_m - \alpha L_1 + \beta L_2, t\Big) &\leqslant \mathcal{R}\Big(\alpha w_m - \alpha L_1, \frac{t}{2}\Big) \diamond \mathcal{R}\Big(\beta w_m - \beta L_2, \frac{t}{2}\Big)\\
&= \mathcal{R}\Big(w_m - L_1, \frac{t}{2\mid\alpha\mid}\Big) \diamond \mathcal{R}\Big(w_m - L_2, \frac{t}{2\mid\beta\mid}\Big) < \epsilon \diamond \epsilon = \epsilon.
\end{aligned}$$

This implies that

$$A_3^c \subset \Big\{k \in \mathbb{N} : \mathcal{P}((\alpha w_k + \beta z_k) - (\alpha L_1 + \beta L_2), t) > 1-\epsilon \text{ or } \mathcal{Q}((\alpha w_k + \beta z_k) - (\alpha L_1 + \beta L_2), t) < \epsilon, \mathcal{R}((\alpha w_k + \beta z_k) - (\alpha L_1 + \beta L_2), t) < \epsilon\Big\}.$$

Therefore, the space ${}^I\mathcal{S}_\mathcal{M}$ is linear over $\mathbb{R}$.

On a similar manner, one can prove that the space ${}^I\mathcal{S}_{\mathcal{M}_0}$ is linear over $\mathbb{R}$. □

Theorem 2.2. *In the space ${}^I\mathcal{S}_\mathcal{M}$, every open ball $\mathcal{O}_w(\gamma, t)$ is an open set.*

Proof. Given that $\mathcal{O}_w(\gamma, t)$ be an open ball with center w, therefore

$$\mathcal{O}_w(\gamma, t) = \Big\{z_k \in \ell_\infty : \{k \in \mathbb{N} : \mathcal{P}(w_k - z_k, t) \leqslant 1-\epsilon \text{ or } \mathcal{Q}(w_k - z_k, t) \geqslant \epsilon, \mathcal{R}(w_k - z_k, t) \geqslant \epsilon\}\Big\}.$$

Assume that $z \in \mathcal{O}^c_w(\gamma, t)$. Then $\mathcal{P}(w_k - z_k, t) > 1-\gamma$, $\mathcal{Q}(w_k - z_k, t) < \gamma$, and $\mathcal{R}(w_k - z_k, t) < \gamma$. Since $\mathcal{P}(w_k - z_k, t) > 1-\gamma$, there exists $t_0 \in (0, t)$ such that $\mathcal{P}(w_k - z_k, t_0) > 1-\gamma$ and $\mathcal{Q}(w_k - z_k, t_0) < \gamma$. Putting $\gamma_0 = \mathcal{P}((w_k) - (z_k), t_0)$, one obtains $\gamma_0 > 1-\gamma$, there exists $s \in (0, 1)$ in such a way that $\gamma_0 > 1-s > 1-\gamma$. For $\gamma_0 > 1-s$, one gets $\gamma_1, \gamma_2 \in (0, 1)$ so that $\gamma_0 \star \gamma_1 > 1-s$ and $(1-\gamma_0) \diamond (1-\gamma_0) \leqslant s$. Put $\gamma_3 = \max\{\gamma_1, \gamma_2\}$. Consider the ball $\mathcal{O}^c_z(1-\gamma_3, t-t_0)$. One can show that

$$\mathcal{O}^c_z(1-\gamma_3, t-t_0) \subset \mathcal{O}^c_w(\gamma, t).$$

Let $x = (x_k) \in \mathcal{O}^c_z(1-\gamma_3, t-t_0)$, then $\mathcal{P}(w_k - x_k, t-t_0) > \gamma_3$, $\mathcal{Q}(w_k - x_k, t-t_0) < 1-\gamma_3$, and $\mathcal{R}(w_k - x_k, t-t_0) < 1-\gamma_3$. Therefore

$$\begin{aligned}
\mathcal{P}((w_k) - (z_k), t) &\geqslant \mathcal{P}((w_k) - (x_k), t_0) \star \mathcal{P}((z_k) - (x_k), t-t_0) \geqslant (\gamma_0 \star \gamma_3) \geqslant (\gamma_0 \star \gamma_1) \geqslant (1-s) \geqslant (1-\gamma),\\
\mathcal{Q}(w_k - z_k, t) &\leqslant \mathcal{Q}(w_k - x_k, t_0) \diamond \mathcal{Q}(z_k - x_k, t-t_0) \leqslant (1-\gamma_0) \diamond (1-\gamma_3) \leqslant (1-\gamma_0) \diamond (1-\gamma_2) \leqslant s \leqslant \gamma,\\
\mathcal{R}(w_k - z_k, t) &\leqslant \mathcal{R}(w_k - x_k, t_0) \diamond \mathcal{R}(z_k - x_k, t-t_0) \leqslant (1-\gamma_0) \diamond (1-\gamma_3) \leqslant (1-\gamma_0) \diamond (1-\gamma_2) \leqslant s \leqslant \gamma.
\end{aligned}$$

Thus $x \in \mathcal{O}^c_w(\gamma, t)$ and hence

$$\mathcal{O}^c_z(1-\gamma_3, t-t_0) \subset \mathcal{O}^c_w(\gamma, t).$$

□

Remark 2.3. Let ${}^I\mathcal{S}_\mathcal{M}$ be a neutrosophic normed space. Define $\tau_\mathcal{M} = \big\{D \subset {}^I\mathcal{S}_\mathcal{M} : \text{for each } w \in D$ there exists $t > 0$ and $\gamma \in (0, 1)$ such that $\mathcal{O}_w(\gamma, t) \subset D\big\}$. Then $\tau_\mathcal{M}$ is a topology on ${}^I\mathcal{S}_\mathcal{M}$.

Remark 2.4.

(i) From Theorem 2.2 and Remark 2.3, every neutrosophic norm $\mathcal{M}$ on ${}^I\mathcal{S}_\mathcal{M}$ generates a topology $\tau_\mathcal{M}$ on ${}^I\mathcal{S}_\mathcal{M}$ which has a base as the family of open sets of the form $\big\{\mathcal{O}_w(\gamma, t) : w \in {}^I\mathcal{S}_\mathcal{M}, \gamma \in (0, 1), t > 0\big\}$.

(ii) Since $\big\{\mathcal{O}_w(\frac{1}{k}, \frac{1}{k}) : k = 1, 2, 3, \dots\big\}$ is a local base at w, the topology $\tau_\mathcal{M}$ on ${}^I\mathcal{S}_\mathcal{M}$ is first countable.

Theorem 2.5. *The spaces* ${}^I\mathcal{S}_\mathcal{M}$ *and* ${}^I\mathcal{S}_{\mathcal{M}_0}$ *are Hausdorff.*

Proof. We prove the result for ${}^I\mathcal{S}_\mathcal{M}$. Similarly the proof follows for ${}^I\mathcal{S}_{\mathcal{M}_0}$. Let $w, z \in {}^I\mathcal{S}_\mathcal{M}$ such that $w \neq z$. Then $0 < \mathcal{P}(w-z, t) < 1$, $0 < \mathcal{Q}(w-z, t) < 1$ and $0 < \mathcal{R}(w-z, t) < 1$. Putting $\gamma_1 = \mathcal{P}(w-z, t)$, $\gamma_2 = \mathcal{Q}(w-z, t)$ and $\gamma_3 = \mathcal{R}(w-z, t)$ and $\gamma = \max\{\gamma_1, 1-\gamma_2, 1-\gamma_3\}$. For each $\gamma_0 \in (\gamma, 1)$, there exist γ_4 and γ_5 such that $\gamma_4 \star \gamma_5 \geqslant \gamma_0$ and $(1-\gamma_5) \diamond (1-\gamma_6) \leqslant (1-\gamma_0)$. Putting $\gamma_6 \max\{\gamma_4, 1-\gamma_5\}$ and considering the open balls $\mathcal{O}_w(1-\gamma_6, \frac{t}{2})$ and $\mathcal{O}_z(1-\gamma_5, \frac{t}{2})$, then clearly $\mathcal{O}^c_w(1-\gamma_6, \frac{t}{2}) \cap \mathcal{O}^c_z(1-\gamma_6, \frac{t}{2}) = \emptyset$. For if there exists $x \in \mathcal{O}^c_w(1-\gamma_6, \frac{t}{2}) \cap \mathcal{O}^c_z(1-\gamma_6, \frac{t}{2})$, then

$$\begin{aligned}
\gamma_1 &= \mathcal{P}(w-z, t) \geqslant \mathcal{P}(w-z, \frac{t}{2}) \star \mathcal{P}(x-z, \frac{t}{2}) \geqslant \gamma_5 \star \gamma_5 \geqslant \gamma_3 \star \gamma_3 \geqslant \gamma_0 > \gamma_1,\\
\gamma_2 &= \mathcal{Q}(w-z, t) \leqslant \mathcal{Q}(w-x, \frac{t}{2}) \diamond \mathcal{Q}(x-z, \frac{t}{2}) \leqslant (1-\gamma_5) \diamond (1-\gamma_5) \leqslant (1-\gamma_4) \diamond (1-\gamma_4) \leqslant (1-\gamma_0) < \gamma_2,\\
\gamma_2 &= \mathcal{R}(w-z, t) \leqslant \mathcal{R}(w-x, \frac{t}{2}) \diamond \mathcal{R}(x-z, \frac{t}{2}) \leqslant (1-\gamma_5) \diamond (1-\gamma_5) \leqslant (1-\gamma_4) \diamond (1-\gamma_4) \leqslant (1-\gamma_0) < \gamma_2,
\end{aligned}$$

which is a contradiction. Hence, the space ${}^I\mathcal{S}_\mathcal{M}$ is Hausdorff. □

Theorem 2.6. *Every separable space* ${}^I\mathcal{S}_\mathcal{M}$ *is second countable.*

Proof. Given that the neutrosophic space ${}^I\mathcal{S}_\mathcal{M}$ separable. Therefore $\mathcal{K} = \{w_k : k \in \mathbb{N}\}$ is a countable dense subset of W. Let $\mathcal{G} = \{g(w_i, \frac{1}{p}, \frac{1}{k}) : i, p \in \mathbb{N}\}$. Therefore $\mathcal{G}$ is countable. One claims that $\mathcal{G}$ is a base for the family of all open sets in W. Suppose V is an open set in W and let $w \in V$. Then there exist $t > 0$ and $\gamma \in (0, 1)$ such that $g(w, \gamma, t) \subset V$. Since $\gamma \in (0, 1)$, one can select $s \in (0, 1)$ such a way that $\gamma > s \diamond s$ and $1-\gamma < (1-s) \star (1-s)$. Assume $m \in \mathbb{N}$ in such way that $\frac{1}{m} < \min\{s, \frac{t}{2}\}$. Since $\mathcal{K}$ is dense in W, there exists $w_i \in \mathcal{K}$ in such way that $w_i \in g(w, \frac{1}{m}, \frac{1}{m})$. Now if $z_i \in g(w_i, \frac{1}{m}, \frac{1}{m})$, then

$$\begin{aligned}\mathcal{P}\Big(w-z,t\Big) &\geqslant \mathcal{P}\Big(w-w_i,\frac{t}{2}\Big)\star\mathcal{P}\Big(z-w_i,\frac{t}{2}\Big)\\ &\geqslant \mathcal{P}\Big(w-w_i,\frac{1}{m}\Big)\star\mathcal{P}\Big(z-w_i,\frac{1}{m}\Big)\geqslant\Big(1-\frac{1}{m}\Big)\star\Big(1-\frac{1}{m}\Big)\geqslant(1-s)\star(1-s)>1-\gamma,\\ \mathcal{Q}\Big(w-z,t\Big)&\leqslant\mathcal{Q}\Big(w-w_i,\frac{t}{2}\Big)\diamond\mathcal{Q}\Big(z-w_i,\frac{t}{2}\Big)\leqslant\mathcal{Q}\Big(w-w_i,\frac{1}{m}\Big)\diamond\mathcal{Q}\Big(z-w_i,\frac{1}{m}\Big)\leqslant\frac{1}{m}\diamond\frac{1}{m}\leqslant s\diamond s<\gamma,\\ \mathcal{R}\Big(w-z,t\Big)&\leqslant\mathcal{R}\Big(w-w_i,\frac{t}{2}\Big)\diamond\mathcal{R}\Big(z-w_i,\frac{t}{2}\Big)\leqslant\mathcal{R}\Big(w-w_i,\frac{1}{m}\Big)\diamond\mathcal{R}\Big(z-w_i,\frac{1}{m}\Big)\leqslant\frac{1}{m}\diamond\frac{1}{m}\leqslant s\diamond s<\gamma.\end{aligned}$$

Thus, $z \in g(w, \gamma, t) \subset V$ and hence $\mathcal{G}$ is a base. □

Theorem 2.7. *A sequence $w = (w_k)$ in ${}^{I}\mathcal{S}_{\mathcal{M}}$ is* I*-convergent with respect to the neutrosophic norm $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ if and only if it is* I*-Cauchy with respect to same norm.*

Proof.

Necessity: Let $w = (w_k)$ in ${}^{I}\mathcal{S}_{\mathcal{M}}$ is I-convergent. Assume $s > 0$ in such a way that $(1-s)\star(1-s) > 1-\epsilon$ and $\epsilon > s \diamond s$. Then, for each $t > 0$, one has

$$B = \{k \in \mathbb{N} : \mathcal{P}(w_k - L, t) \leqslant 1 - s \text{ or } \mathcal{Q}(w_k - L, t) \geqslant s, \mathcal{R}(w_k - L, t) \geqslant s\} \in I.$$

Therefore,

$$B^c = \{k \in \mathbb{N} : \mathcal{P}(w_k - L, t) > 1 - s \text{ or } \mathcal{Q}(w_k - L, t) < s, \mathcal{R}(w_k - L, t) < s\} \in \mathcal{F}(I).$$

Suppose $m \in B^c$. Then we obtain

$$\mathcal{P}(w_m - L, t) > 1 - s, \quad \mathcal{Q}(w_m - L, t) < s, \text{ and } \mathcal{R}(w_m - L, t) < s.$$

Let,

$$C = \{k \in \mathbb{N} : \mathcal{P}(w_k - w_m, t) \leqslant 1 - \epsilon \text{ or } \mathcal{Q}(w_k - w_m, t) \geqslant \epsilon, \mathcal{R}(w_k - w_m, t) \geqslant \epsilon\}.$$

One have to prove that $C \subset B$. Take $k \in Q$. We obtain

$$\mathcal{P}(w_k - w_m, \frac{t}{2}) \leqslant 1 - \epsilon \text{ or } \mathcal{Q}(w_k - w_m, \frac{t}{2}) \geqslant \epsilon, \mathcal{Q}(w_k - w_m, \frac{t}{2}) \geqslant \epsilon.$$

There are two possible cases, firstly consider that $\mathcal{P}(w_k - w_m, t) \leqslant 1 - \epsilon$. Therefore, one obtains $\mathcal{P}(w_k - L, \frac{t}{2}) \leqslant 1 - s$, then $k \in B$. On the contrary, if $\mathcal{P}(w_k - L, \frac{t}{2}) > 1 - s$ then

$$1 - \epsilon \geqslant \mathcal{P}(w_k - w_m, t) \geqslant \mathcal{P}(w_k - L, \frac{t}{2}) \star \mathcal{P}(w_m - L, \frac{t}{2}) > (1 - s) \star (1 - s) > 1 - \epsilon,$$

which is not possible. Therefore, $C \subset B$.

Similarly, let $\mathcal{Q}(w_k - w_m, t) \geqslant \epsilon$ and $\mathcal{R}(w_k - w_m, t) \geqslant \epsilon$. Then we have $\mathcal{Q}(w_k - L, \frac{t}{2}) \geqslant s$ and $\mathcal{Q}(w_k - L, \frac{t}{2}) \geqslant s$, hence $k \in B$. Otherwise, if $\mathcal{Q}(w_k - L, \frac{t}{2}) < s$ and $\mathcal{R}(w_k - L, \frac{t}{2}) < s$, then

$$\epsilon \leqslant \mathcal{Q}(w_k - w_m, t) \leqslant \mathcal{Q}(w_k - L, \frac{t}{2}) \diamond \mathcal{Q}(w_m - L, \frac{t}{2}) < s \diamond s < \epsilon$$

and

$$\epsilon \leqslant \mathcal{R}(w_k - w_m, t) \leqslant \mathcal{R}(w_k - L, \frac{t}{2}) \diamond \mathcal{R}(w_m - L, \frac{t}{2}) < s \diamond s < \epsilon,$$

which is impossible. Therefore, $C \subset B$. Hence, in both cases one concludes that $C \subset B$. Therefore $C \in I$. Therefore, w is a Cauchy with respect to neutrosophic norm $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$.

Sufficiency: assume that $w = (w_k)$ is I Cauchy sequence with respect to neutrosophic norm $(\mathcal{P}, \mathcal{Q}, \mathcal{R})$ but not I-convergent. Then there exists p in such way that

$$B(\epsilon, t) = \{k \in \mathbb{N} : \mathcal{P}(w_k - w_p, t) \leqslant 1 - \epsilon \text{ or } \mathcal{Q}(w_k - w_p, t) \geqslant \epsilon, \mathcal{Q}(w_k - w_p, t) \geqslant \epsilon\} \in I$$

and

$$C(\epsilon, t) = \left\{k \in \mathbb{N} : \mathcal{P}(w_k - L, \frac{t}{2}) > 1 - \epsilon \text{ and } \mathcal{Q}(w_k - L, \frac{t}{2}) < \epsilon, \mathcal{Q}(w_k - L, t) \geqslant \epsilon\right\} \in I.$$

Equivalently, $C^c(\epsilon, t) \in \mathcal{F}(I)$. Since

$$\mathcal{P}(w_k - w_m, t) \geqslant 2\mathcal{P}(w_k - L, \frac{t}{2}) > 1 - \epsilon,$$
$$\mathcal{Q}(w_k - w_p, t) \leqslant 2\mathcal{Q}(w_k - L, \frac{t}{2}) < \epsilon,$$
$$\mathcal{R}(w_k - w_p, t) \leqslant 2\mathcal{Q}(w_k - L, \frac{t}{2}) < \epsilon,$$

if $\mathcal{P}(w_k - L, \frac{t}{2}) > \frac{1-\epsilon}{2}$, $\mathcal{Q}(w_k - L, \frac{t}{2}) < \frac{\epsilon}{2}$, and $\mathcal{R}(w_k - L, \frac{t}{2}) < \frac{\epsilon}{2}$, respectively, one obtains $B^c(\epsilon, t) \in I$ and so $B(\epsilon, t) \in \mathcal{F}(I)$, which contradicts our assumption. □

Theorem 2.8. *Let ${}^I\mathcal{S}_\mathcal{M}$ be a neutrosophic normed space such that every Cauchy sequence in ${}^I\mathcal{S}_\mathcal{M}$ has a convergent subsequence. Then the space ${}^I\mathcal{S}_\mathcal{M}$ is complete.*

Proof. Let (w_{k_q}) be a subsequence of Cauchy sequence (w_k) that converges to w. We show that $(w_k) \to w$ as $k \to \infty$. Let $t > 0$ and $\epsilon \in (0, 1)$. Select $s \in (0, 1)$ in a such way that $(1 - s) \star (1 - s) \geqslant 1 - \epsilon$ and $s \diamond s \leqslant \epsilon$. Since (w_k) is a Cauchy sequence, $\exists$ $N_0 \in \mathbb{N}$ such that $\mathcal{P}(w_p - w_k, \frac{t}{2}) > 1 - s$, $\mathcal{P}(w_p - w_k, \frac{t}{2}) < s$ and $\mathcal{R}(w_p - w_k, \frac{t}{2}) < s$ for all $p, k \in N_0$. Since $(w_{k_q}) \to w$, there is a positive integer $q_m > N_0$ such that $\mathcal{P}(w_{q_m} - w, \frac{t}{2}) > 1 - s$, $\mathcal{Q}(w_{q_m} - w, \frac{t}{2}) < s$, and $\mathcal{R}(w_{q_m} - w, \frac{t}{2}) < s$. Then, if $k \geqslant N_0$,

$$\mathcal{P}(w_k - w, t) \geqslant \mathcal{P}(w_k - w_{q_m}, \frac{t}{2}) \star \mathcal{P}(w_{q_m j} - w, \frac{t}{2}) \geqslant (1 - s) \star (1 - s) \geqslant 1 - \epsilon,$$
$$\mathcal{Q}(w_k - w, t) \leqslant \mathcal{Q}(w_k - w_{q_m j}, \frac{t}{2}) \diamond \mathcal{Q}(w_{q_m} - w, \frac{t}{2}) \leqslant s \diamond s \leqslant \epsilon,$$
$$\mathcal{R}(w_k - w, t) \leqslant \mathcal{R}(w_k - w_{q_m j}, \frac{t}{2}) \diamond \mathcal{R}(w_{q_m} - w, \frac{t}{2}) \leqslant s \diamond s \leqslant \epsilon.$$

Therefore $w_k \to w$ as $k \to \infty$. Hence the space ${}^I\mathcal{S}_\mathcal{M}$ is complete. □

3. Conclusion

In this paper, we provide researchers the spaces of all convergent sequences and null sequences via ideal in a more general setting known as neutrosophic normed space. We proved that spaces are linear and Hausdorff topological spaces. These spaces and results help in many branches of science and engineering.

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