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On the elliptical solutions of models connected to the short pulse equation

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Abstract

In the present paper, we consider a special hierarchy of equations comprising the short pulse equation, the sine-Gordon integrable hierarchy and the elastic beam equation. These equations are highly non-linear and rely on transformations to arrive at solutions. Previously, recursion operators and hodograph mappings were successful in reducing these equations. However, we show that via the conservation laws or the one-parameter Lie group, the special hierarchy may be integrated and will admit the exact solutions that feature elliptical functions.

Keywords: Lie symmetries, elastic beam equations, sine-Gordon.

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1. Introduction

In the world of nonlinear partial differential equations, hierarchies of equations are of superior interest [12, 23]. While there are many reasons for this, the primary reason that hierarchical equations attract so much interest is due to its admission of a recursion operator. We consider one such equation, viz. the well known short pulse equation (SPE)

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}, \tag{1.1}$$

where $u = u(t, x)$ represents the magnitude of an electric field, that appeared in [27], associated with ultra-short pulses in a nonlinear medium. A recursion operator may be constructed to build a hierarchy containing Eq. (1.1) (see [5]), and also, a second related equation

$$u_{xt} = \frac{1}{4}\left(\frac{u_{xx}}{(1+{u_x}^2)}^{\frac{3}{2}}\right)_{xx}, \tag{1.2}$$

known as the elastic beam equation (EBE) [9]. The EBE is a significant model in the theory of nonlinear waves and describes the evolution of nonlinear transverse oscillations of elastic beams under tension. The EBE can also be embedded in the Wadati-Konno-Ichikawa system [29].

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Moreover, it was shown in [25], that Eq. (1.1) is related, through a hodograph transformation [8], to the sine-Gordon (SG) equation. We consider the full hierarchy that contains the sine-Gordon equation, viz.

$$u_{tt} - u_{xx} = \sum_{i=1}^{N} \beta_i \sin(iu). \tag{1.3}$$

The hodograph transformation connecting (1.1) and (1.3) was further explored [17] to construct solitonic and quasi-periodic solutions. The SG equation offers a variety of physical applications and is structural similarity to the linear relativistic Klein-Gordon equation. SG equations have wide applications and admit many interesting computational results [10, 11]. The double SG equation (DSG) equation, with $N = 2$ supports the study of charge density waves and ferromagnetic materials (refer to [14] and references therein), while the triple SG equation (TSG) equation ($N = 3$) is frequently studied in the propagation of optical pulses [7]. Since the short pulse equation (1.1) and the SG equation both admit the bi-Hamiltonian feature [6, 16, 26], each of them is associated with the corresponding infinite bi-Hamiltonian integrable hierarchy due to Magri's theory [15].

In this paper, we study the corresponding solutions between the two equations with deep connections to the short-pulse integrable hierarchy, namely, the classical SG integrable hierarchy and the EBE equation. More precisely, in view of the correspondence between the SPE and the SG equation, it is anticipated that the respective solutions may be related in a certain manner.

Lie symmetries and conservation laws are important tools for analysis [2–4], and many recent studies have relied on these methods [18–22].

The structure of the paper is as follows. In Section 2, some preliminary notation and theory is defined. Section 3 contains the exact solutions and conservation laws of all equations under study, whereby we search for commonality between the equations, their symmetries, conserved values and derived solutions. Lastly, Section 4 concludes the paper.

2. Preliminary discussion

Let

$$G_\alpha\left(x, u^{(k)}\right) = 0, \tag{2.1}$$

be a system of nonlinear differential equations, where $u^{(k)}$ represents the k^{th} derivative of u with respect to x, where $x = (x^1, \ldots, x^p)$. We consider the symmetry vector field

$$X = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}, \tag{2.2}$$

where the infinitesimal criterion for invariance is given by [24]

$$X\left[G_\alpha\left(x, u^{(k)}\right)\right] = 0, \quad \text{when} \quad G_\alpha\left(x, u^{(k)}\right) = 0. \tag{2.3}$$

The operator in Eq. (2.2) can be used to define the Lagrange system

$$\frac{dx^i}{\xi^i} = \frac{du^\alpha}{\eta^\alpha},$$

whose solution provides the zero-order invariants that reduce the equation,

$$W^{[0]}(x^i, u^\alpha).$$

A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0, \tag{2.4}$$

along the solutions of (2.1). If X and T satisfy

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \tag{2.5}$$

then X is said to be associated with T [13].

3. Exact solutions and conservation laws

In this section, we derive the solutions and conservation laws of the SPE and its two associated models, the SG hierarchy and EBE model.

3.1. The short pulse equation

Suppose we consider

$$X = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \eta^1(x,t,u)\partial_u,$$

to be the symmetry generator for equation (1.1). The Lie symmetry for condition from equation (2.3), solves to provide the following individual symmetries

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = -x\partial_x + t\partial_t - u\partial_u, \tag{3.1}$$

which form a 3-dimensional algebra and have the commutator relations as Table 1.

Table 1: Lie commutator table of (3.1).

$[,]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	$-X_2$
X_3	$-X_1$	X_2	0

As defined above, for us to calculate the conserved vector we use the multiplier approach. Therefore, we find that there exists nontrivial differential functions $\Lambda(x,t,u)$, called multipliers, such that

$$\Lambda(1.1) = D_t T^t + D_x T^x, \tag{3.2}$$

and

$$\frac{\delta}{\delta u}(\text{LHS of } (3.2)) = 0,$$

where $\frac{\delta}{\delta u}$ is the standard Euler operator which annihilates divergence expressions.

In each case, we have the conserved vector $T = (T^t, T^x)$, where T^t is the conserved density and T^x is the conserved flux.

The conservation laws of the SPE, are calculated to be

$$T_1{}^t = \frac{1}{2}\frac{u_{xx}{}^2}{(u_x{}^2+1)^{\frac{5}{2}}}, \quad T_1{}^x = \frac{1}{4}\frac{u^2u_{xx}{}^2 + 4uu_x{}^2u_{xx} + 4u_x{}^4 + 4uu_{xx} - 4u_{xt}u_{xx} + 8u_x{}^2 + 4}{(u_x{}^2+1)^{\frac{5}{2}}},$$

given by

$$\Lambda(x,t,u) = \frac{1}{2}\frac{2u_x{}^2u_{xxx} - 5u_xu_{xx}^2 + 2u_{xxx}}{(u_x{}^2+1)^{\frac{7}{2}}},$$

and

$$T_2{}^t = -\sqrt{u_x{}^2+1}, \quad T_2{}^x = \frac{u^2}{2}\sqrt{u_x{}^2+1},$$

given by

$$\Lambda(x,t,u) = \frac{u_x}{\sqrt{u_x{}^2+1}}.$$

In testing for association between the symmetries and the conserved vector T_2, we find that X_1 and X_2 meet the association condition (2.5). Now, considering a linear combination $X = X_1 + cX_2$ (c is an arbitrary constant), yielding the transformations

$$s = x, \quad r = cx - t.$$

Hence we proceed to find the reduced conservation law T^r, which is given by:

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_x(s)D_t(r) - D_t(s)D_x(r)}, \tag{3.3}$$

and using T^r, along with the transformations obtained, we can derive the analytical solutions of the equations understudy [28].

Now transforming the component of the conserved vector using (3.3) gives that

$$T^r = -\sqrt{c^2 {u_r}^2 + 1}\left(1 + \frac{cu^2}{2}\right).$$

Since $D_r T^r = 0$, which implies that $T^r = k, k \in \mathbb{R}$, we get the ODE

$$k + \sqrt{c^2 u'^2 + 1}\left(1 + \frac{cu^2}{2}\right) = 0. \tag{3.4}$$

Solving Eq. (3.4) gives the implicit solution

$$r \pm \int^{u(r)} \frac{\left(a^2 c + 2\right) c}{\sqrt{-a^4 c^2 - 4\, a^2 c + 4\, k^2 - 4}} da - C_1 = 0,$$

where C_1 is an arbitrary constant. The integral may be evaluated, and we have the solution

$$\begin{aligned} r \pm c\Bigg(&(k+1)\,\mathrm{EllipticE}\left(\frac{u(r)\sqrt{2}}{2}\sqrt{\frac{c}{k-1}}, \sqrt{-\frac{k-1}{k+1}}\right) - \mathrm{EllipticF}\left(\frac{u(r)\sqrt{2}}{2}\sqrt{\frac{c}{k-1}}, \sqrt{-\frac{k-1}{k+1}}\right) k\Bigg) \\ &\times \sqrt{\frac{c\,(u(r))^2 + 2\,k + 2}{k+1}}\sqrt{\frac{-2\,c\,(u(r))^2 + 4\,k - 4}{c}}\frac{1}{\sqrt{-(u(r))^4 c^2 - 4\,c\,(u(r))^2 + 4\,k^2 - 4}} - C_1 = 0, \end{aligned}$$

where EllipticF is the Elliptic integral of the first kind and EllipticE is the Elliptic integral of the second kind [1]. We leave the solution in implicit form, and when inverting the transformation, for the SPE solution we have

$$\begin{aligned} cx - t \pm c\Bigg(&(k+1)\,\mathrm{EllipticE}\left(\frac{u(x,t)\sqrt{2}}{2}\sqrt{\frac{c}{k-1}}, \sqrt{-\frac{k-1}{k+1}}\right) - \mathrm{EllipticF}\left(\frac{u(x,t)\sqrt{2}}{2}\sqrt{\frac{c}{k-1}}, \sqrt{-\frac{k-1}{k+1}}\right) k\Bigg) \\ &\times \sqrt{\frac{c\,(u(x,t))^2 + 2\,k + 2}{k+1}}\sqrt{\frac{-2\,c\,(u(x,t))^2 + 4\,k - 4}{c}}\frac{1}{\sqrt{-(u(x,t))^4 c^2 - 4\,c\,(u(x,t))^2 + 4\,k^2 - 4}} - C_1 = 0. \end{aligned}$$

3.2. *The SG equations*

Considering the vector field (2.2) for Eq. (1.3), the Lie symmetry for condition from equation (2.3), solves to provide the symmetries

$$X_1,\ X_2,\ X_4 = t\partial_x + x\partial_t,$$

which form a 3-dimensional Lie algebra with commutator relations as in Table 2.

Table 2: Lie commutator table.

$[,]$	X_1	X_2	X_4
X_1	0	0	$-X_2$
X_2	0	0	$-X_1$
X_4	X_2	X_1	0

The components that satisfy the conservation law (2.4) of (1.3), are calculated to be

$$T^t = u_t u_x, \quad \forall\, N,$$

for all conserved densities, and the first few corresponding fluxes are

$$\begin{aligned}
N = 1 : T^x &= \beta_1 \cos u - \frac{1}{2}(u_t^2 - u_x^2),\\
N = 2 : T^x &= \beta_2 \cos^2 u + \beta_1 \cos u - \frac{1}{2}{u_t}^2 - \frac{1}{2}{u_x}^2 - \beta_2,\\
N = 3 : T^x &= \frac{4}{3}\beta_3 \cos^3 u + \beta_2 \cos^2 u - \beta_3 \cos u + \beta_1 \cos u - \frac{1}{2}{u_t}^2 - \frac{1}{2}{u_x}^2 - \beta_2.
\end{aligned}$$

Similar to the previous case, X_1 and X_2 meet the association condition, and via the same transformations, gives that for $N = 1$,

$$T^r = \left(\frac{c^3}{2} - \frac{c}{2}\right) u_r^2 - c\beta_1 \cos u,$$

hence we get the ODE in r

$$k - \left(\frac{c^3}{2} - \frac{c}{2}\right) u'^2 - c\beta_1 \cos u = 0. \tag{3.5}$$

For $k, c, \beta_1 \in \mathbb{R}$, equation (3.5) has the solution

$$\begin{aligned}
u(r) = 2\mathrm{am}\Bigg[\frac{1}{4}\Bigg(& 2C_1\sqrt{k + c\beta_1} + \frac{r\sqrt{2}\sqrt{c}\sqrt{(c-1)(c+1)}\sqrt{k + c\beta_1}}{1 - c}\\
&+ \frac{2r\sqrt{2}\sqrt{(c-1)(c+1)}\sqrt{k + c\beta_1}}{\sqrt{c}} - \frac{\sqrt{2}\sqrt{c}\sqrt{(c-1)(c+1)}r\sqrt{k + c\beta_1}}{c+1}\Bigg)\Bigg|\frac{2c\beta_1}{k + c\beta_1}\Bigg].
\end{aligned} \tag{3.6}$$

The function $\mathrm{am}[\mu|\kappa]$ is the Jacobi amplitude. The above solution maps back to

$$\begin{aligned}
u(x,t) = 2\mathrm{am}\Bigg[\frac{1}{4}\Bigg(& 2C_1\sqrt{k + c\beta_1} + \frac{(cx - t)\sqrt{2}\sqrt{c}\sqrt{(c-1)(c+1)}\sqrt{k + c\beta_1}}{1 - c}\\
&+ \frac{2(cx - t)\sqrt{2}\sqrt{(c-1)(c+1)}\sqrt{k + c\beta_1}}{\sqrt{c}} - \frac{\sqrt{2}\sqrt{c}\sqrt{(c-1)(c+1)}(cx - t)\sqrt{k + c\beta_1}}{c+1}\Bigg)\Bigg|\frac{2c\beta_1}{k + c\beta_1}\Bigg],
\end{aligned} \tag{3.7}$$

for a solution of (1.3).

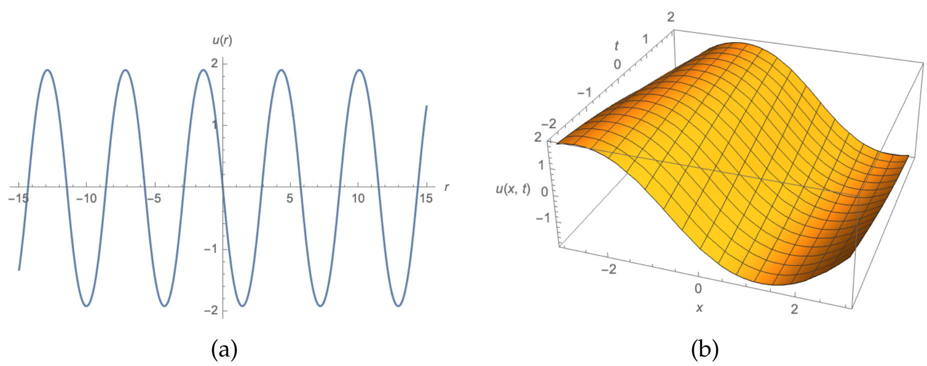
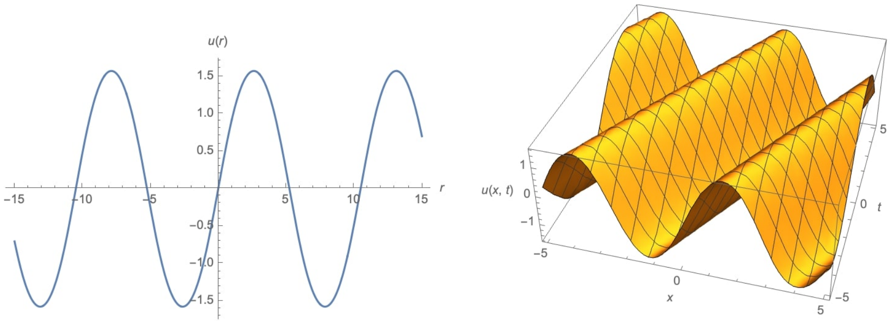


Figure 1: Graphical illustration of the analytical solutions are depicted, we select the parameter values $C_1 = 0, c = 3, \beta_1 = 1, k = 1$: (a) 2D Plot of Eq. (3.6); and (b) 3D Plot of Eq. (3.7).



(a) (b)

Figure 2: Graphical illustration of the analytical solutions are depicted, we select the parameter values $k = 0$, $c = 2$, $\beta_1 = \frac{3}{2}$, $C_1 = 0$: (a) 2D Plot of Eq. (3.6); and (b) 3D Plot of Eq. (3.7).

For $N = 2$, we have the reduced conservation law

$$T^r = c\beta_2 + \frac{1}{2}\left(c^2 - c\right)u_r^2 - c\beta_2\cos^2 u - c\beta_1\cos u,$$

so that the ODE in r is

$$\beta_2 + \frac{(c-1)}{2}u'^2 - \beta_2\cos^2 u - \beta_1\cos u - \frac{k}{c} = 0. \tag{3.8}$$

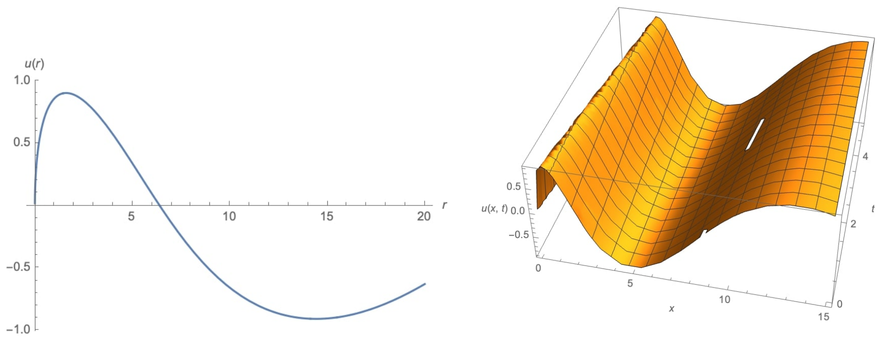
For $k, c, \beta_1, \beta_2 \in \mathbb{R}$, Eq. (3.8) has a constant solution. However, in solving Eq. (3.8), some special cases exist for particular values for β_1, β_2, c, and k.

For $\beta_1 = 1$, $k = 0$, $c = 3$, and $\beta_2 = 1$, we have the solution

$$u(r) = -2i\tanh^{-1}\left(\sqrt{\sqrt{5}+2}\,\mathrm{sn}\left[i\sqrt{\tfrac{1}{2}\left(\sqrt{5}-2\right)}\left(C_1 \pm \tfrac{r}{\sqrt{2}}\right)\middle| -4\sqrt{5}-9\right]\right), \tag{3.9}$$

where the function $\mathrm{sn}[\mu|\kappa]$ is used in Jacobi elliptic functions. Reverting to the original variables, the solution for Eq. (1.3), the DSG is

$$u(x,t) = -2i\tanh^{-1}\left(\sqrt{\sqrt{5}+2}\,\mathrm{sn}\left[i\sqrt{\tfrac{1}{2}\left(\sqrt{5}-2\right)}\left(C_1 \pm \tfrac{(3x-t)}{\sqrt{2}}\right)\middle| -4\sqrt{5}-9\right]\right). \tag{3.10}$$



(a) (b)

Figure 3: Graphical illustration of the analytical solutions are depicted, we select the parameter value $C_1 = 0$: (a) 2D Plot of Eq. (3.9); and (b) 3D Plot of Eq. (3.10).

As for $N = 3$, the conservation law is

$$T^r = c\left(\beta_2 + \frac{1}{2}\left(c^2 - 1\right)u_r^2 - \frac{4}{3}\beta_3\cos^3 u - \beta_2\cos^2 u + \beta_3\cos u - \beta_1\cos u\right).$$

Setting this $T^r = 0$, and solving in general with β_1, β_2, β_3, c, and k as arbitrary yields the implicit solutions

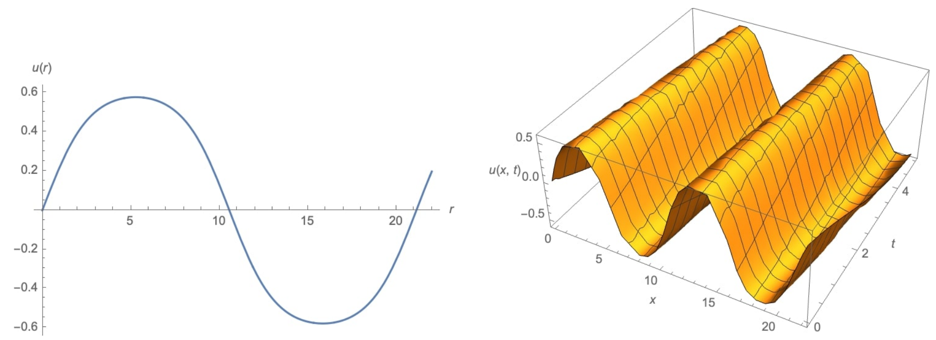
$$0 = r \mp \int^{u(r)} \frac{\sqrt{6}\rho\, da}{2\sqrt{\rho\left(4\,\beta_3\,(\cos(a))^3\,c + 3\,\beta_2\,(\cos(a))^2\,c - 3\,\beta_3\,\cos(a)\,c + 3\,\beta_1\,\cos(a)\,c - 3\,\beta\,c + 3\,k\right)}} - C_1,$$

where $\rho = c\,(c-1)\,(c+1)$. Solving for particular values for β_1, β_2, β_3, c, and k yields some interesting solutions. For example, where $C_1 = 0, \beta_2 = 0, c = 2, \beta_3 = 1, \beta_1 = 0, k = 1/3$, the solution becomes

$$u(r) = \frac{2}{3}\mathrm{sn}\left[\frac{\sqrt{2}(r - C_2)}{4}\middle|\frac{4}{3}\right], \tag{3.11}$$

in original variables, the solution for the TSG is

$$u(x,t) = \frac{2}{3}\mathrm{sn}\left[\frac{\sqrt{2}(2x - t - C_2)}{4}\middle|\frac{4}{3}\right]. \tag{3.12}$$



(a) (b)

Figure 4: Graphical illustration of the analytical solutions are depicted, we select the parameter value $C_2 = 0$: (a) 2D Plot of Eq. (3.11); and (b) 3D Plot of Eq. (3.12).

3.3. *The elastic beam equation*

Next, we turn our attention to the EBE (1.2). The individual symmetries of the EBE (1.2) are

$$X_1,\ X_2,\ X_5 = x\partial_x + 3t\partial_t + u\partial_u,$$

which form a 3-dimensional algebra and have the commutator relations as in Table 3.

Table 3: Lie commutator table.

[,]	X_1	X_2	X_5
X_1	0	0	$\frac{-1}{3}X_1$
X_2	0	0	$-X_2$
X_5	$\frac{1}{3}X_1$	X_2	0

Conservation laws are challenging to find and not very useful in obtaining solutions to the EBE, therefore we use the traditional Lie symmetry reduction method. A reduction of (1.2) by $cX_2 - X_1$ leads to the reduced equation

$$(3c^6 - 12c^8(F'(z))^2)(F''(z))^3 + c(1 + (cF'(z))^2)(F''(z))(4\sqrt{1 + (cF'(z))^2} + 8(cF'(z))^2\sqrt{1 + (cF'(z))^2}$$

$$+\ 4(cF'(z))^4\sqrt{1+(cF'(z))^2}+9c^5(F'(z))F'''(z))-\ c^4(1+(cF'(z))^2)^2F''''(z)=0,$$

where $u(x,t)=F(z)$, $z=t+cx$.

The solution here is $F(z)=\int G(z)dz+C_4$, where $G(z)$ is given by the expression

$$\int^{G(z)}\frac{\pm\sqrt{2}c^3}{\sqrt{-c\left(-C_1\,f^2c^7+C_2\,\sqrt{f^2c^2+1}c^5f-C_1\,c^5+4\,\sqrt{f^2c^2+1}\right)(f^2c^2+1)}}df+2\,C_3+2\,z=0.$$

A simpler result is found if we set $c=-1, C_1=C_2=0$ in the above. The integral equals

$$-\frac{G(z)}{\sqrt[4]{G(z)^2+1}}+\frac{1}{2}\,G(z)\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{3}{2};-G(z)^2\right),$$

where ${}_2F_1$ is a Hypergeometric function. Hence $G(z)$ satisfies

$$\pm\sqrt{2}\left(-\frac{G(z)}{\sqrt[4]{G(z)^2+1}}+\frac{1}{2}\,G(z)\,{}_2F_1\left(\frac{1}{4},\frac{1}{2};\frac{3}{2};-G(z)^2\right)\right)+2\,C_3+2\,z=0.$$

The solutions of Eq. (1.2) are dependent on the Hypergeometric function and influenced by the function $G(z)$. Note that the Hypergeometric function is expressible in terms of an elliptical integral.

4. Conclusion

It is indeed very surprising that the SPE shares strong mathematical links with the EBE and the SGEs. Whilst the SPE describes the evolution of very short pulses in nonlinear media, the EBE describes nonlinear transverse oscillations of elastic beams under tension and the SGE has applications to waves and pulses. This study explored the conservation law properties of the above-mentioned equations, except the EBE which admits complicated expressions. In searching for solutions through conservation laws, we exploited the association between the conservation laws and the space and time translation symmetries. Most of the equations admit multipliers that generate conserved components, that is, the densities and fluxes. The SPE possesses two conservation laws, the class of SGEs all admit one conservation law each, all with the same conserved density but different fluxes as expected.

The SPE conservation law was reduced from second-order to first-order, and the resulting ODE when solved produced an implicit solution with elliptical functions. The family of SGEs were analyzed using first-order ODEs; this family had varying solution types. In the case of $N=1$, the solutions were found explicitly. In the case of $N=2$ and $N=3$, some solutions were in the implicit form expressed as integrals, while other solutions involved special functions such as the elliptical integral of the first kind or a sinusoidal function. The EBE was reduced from fourth-order PDE to a fourth-order ODE via an invariant function related to a travelling wave transformation. It is found that the resulting ODE, is extremely non-linear and difficult to solve. However, it has an integral solution that consists of a hypergeometric function.

In the literature, the recursion operator that connects the SPE to the EBE and the hodograph transformation in the case of the SGE, are used to find solutions of these above mentioned equations. We have shown that solutions may be achieved either via symmetries or conservation laws, and interestingly, all solutions are elliptical in nature.

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