Available online at www.isr-publications.com/jmcs
J. Math. Computer Sci. 16 (2016), 298–307
Research Article

Online: ISSN 2008-949x

Journal of Mathematics and Computer Science

Journal Homepage: www.tjmcs.com - www.isr-publications.com/jmcs

Exponential extinction of discrete Nicholson's blowflies systems with patch structure and mortality terms

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Abstract

Discrete Nicholson's blowflies systems with patch structure and mortality terms are considered in this paper. Based on certain discrete inequalities, we prove the boundedness of the systems. Using this result, sufficient conditions are then established to guarantee the exponential extinction for the systems. We provide numerical examples verified by illustrative figures to demonstrate the validity of the proposed results.

Keywords: Extinction, discrete Nicholson's blowflies model, patch structure, mortality terms.
2010 MSC: 39A11, 92B05.

1. Introduction and preliminaries

One of the most celebrated population models which was introduced by Gurney et al. in [7] to describe the laboratory population of the Australian sheep–blowfly and to agree with Nicholson's experimental data obtained in [18] is the so called Nicholson's blowflies model

$$x'(t) = -\alpha x(t) + \gamma x(t-\tau)e^{-\delta x(t-\tau)}. \tag{1.1}$$

In biomedical terms, x is the density of the population at time t, α is the *pair capita* daily adult death rate, γ is the maximum *per capita* daily egg production, $1/\delta$ is the size at which the blowfly population reproduces at its maximum rate and τ is the generation time or the time taken from birth to maturity. The dynamical behavior of solutions for various modifications of this model has been extensively studied by many authors during the last decades. We suggest the reader to consult some relevant papers [1, 11–13, 19, 22].

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Received 2016-03-07

It is well known that the optimal management of renewable resources has direct relationship to the sustainable development of exploitation of population. The recent studies has also indicated that the consideration of population models with density–dependent mortality will be more accurate at low densities. One way to handle these two phenomena is to study population models with harvesting and mortality terms. In his remarkable paper [5], Berezansky et al. has put forward an open problem about the dynamical behavior of Nicholson's blowflies models of the forms

$$x'(t) = -H(x) - \alpha x(t) + \gamma x(t-\tau)e^{-\delta x(t-\tau)} \tag{1.2}$$

and

$$x'(t) = -M(x) + \gamma x(t-\tau)e^{-\delta x(t-\tau)}, \tag{1.3}$$

where H and M denote the harvesting and mortality terms, respectively. The papers [4, 6, 8, 9, 14–17, 21, 26] have given an affirmative answer to the problem raised by Berezansky et al. for equations (1.2) and (1.3). In comparison to the extensive investigation of these two equations in the literature, the study of the discrete Nicholson's model is considered to be seldom [10, 20, 24, 25]. Indeed, the discrete counterpart of equation (1.2) has been lately attacked by the current authors in [2, 3, 23]. To the best of authors' observation, however, the discrete analogue of equation (1.3) has not been considered yet.

Motivated by the reality that the difference equations provide more adequate and accurate descriptions for population models, we study the discrete analogue of equation (1.3) which can be formulated as

$$\Delta x(n) = -M(x) + \gamma x(n-\tau)e^{-\delta x(n-\tau)}, \tag{1.4}$$

where Δx denotes the difference $x(n+1) - x(n)$. The mortality term M might take the forms

$$M(x) = \alpha x/\beta + x \text{ or } M(x) = \alpha - \beta e^{-x},\ \alpha, \beta > 0. \tag{1.5}$$

Taking into account, moreover, the migration rates of populations from one patch to another one, it would be more realistic to carry out our investigations for equation (1.4) under certain patch structure. We shall consider the following generalized Nicholson's blowflies systems with patch structure and mortality terms of the forms

$$\begin{aligned}\Delta x_i(n) = &-\frac{\alpha_{ii}(n)x_i(n)}{\beta_{ii}(n)+x_i(n)} + \sum_{j=1, j\neq i}^{l} \frac{\alpha_{ij}(n)x_j(n)}{\beta_{ij}(n)+x_j(n)} \\ &+ \sum_{j=1}^{m} \gamma_{ij}(n)x_i(n-\tau_{ij}(n))e^{-\delta_{ij}(n)x_i(n-\tau_{ij}(n))}\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}\Delta x_i(n) = &-\alpha_{ii}(n) + \beta_{ii}(n)e^{-x_i(n)} + \sum_{j=1, j\neq i}^{l} \alpha_{ij}(n) - \beta_{ij}(n)e^{-x_j(n)} \\ &+ \sum_{j=1}^{m} \gamma_{ij}(n)x_i(n-\tau_{ij}(n))e^{-\delta_{ij}(n)x_i(n-\tau_{ij}(n))},\end{aligned} \tag{1.7}$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ik}, \delta_{ik}$, and $\tau_{ik} \geq 0$ are all nonnegative bounded sequences, $i, j = 1, 2, \ldots, l$, $k = 1, 2, \ldots, m$.

It is known that the ecosystems in real life are often facing unpredictable circumstances which might result in changing their biological behavior. This has urged many authors to investigate the models that describe these systems. Indeed, they studied the dynamical behavior of their solutions such as the existence and uniqueness, existence of periodic and almost periodic solutions, oscillation and asymptotic stability of solutions. Although it is considered as an important topic for the study of mathematical biology, it is noticed that not much research has been achieved in the direction of the extinction dynamic. A primary purpose of this paper is to establish sufficient conditions for the boundedness and the exponential extinction of systems (1.6) and (1.7). Numerical examples corresponding to (1.6) and (1.7) are provided at the end of each section to support the main results.

2. Exponential extinction

Let $\mathbb{Z}$ and $\mathbb{Z}^+$ denote the sets of integer and nonnegative integer numbers, respectively. For a bounded sequence g defined on $\mathbb{Z}$, we define g^+ and g^- as follows

$$g^+ = \sup_{n\in\mathbb{Z}} g(n) \quad \text{and} \quad g^- = \inf_{n\in\mathbb{Z}} g(n).$$

Denote by $|x|$, the absolute value vector and let $\tau_i = \max_{1\le j\le m}\{\sup_{n\in\mathbb{Z}} \tau_{ij}(n)\}$. We consider systems (1.6) and (1.7) together with the following initial conditions:

$$x_i(s) = \varphi_i(s) \ge 0, \quad \varphi_i(0) > 0,\ s \in I = \{-\tau_i, -\tau_i+1, \ldots, 0\},\ i = 1, 2, \ldots, l. \tag{2.1}$$

One can easily show that the solutions of systems (1.6) or (1.7) with the initial conditions (2.1) are defined and remain positive for $n \in \mathbb{Z}^+$.

Definition 2.1. The system (1.6) (or (1.7)) with the initial conditions (2.1) is said to be *exponentially extinct* if there are constants L and κ such that

$$|x_i(n, n_0, \varphi_i)| \le Le^{-\kappa(n-n_0)}, \quad i = 1, 2, \ldots, l.$$

Throughout this section, we assume that there exists a positive constant Γ_1 such that

(I) $\frac{\alpha_{ii}^-}{\beta_{ii}^+ + \Gamma_1} > \sum_{j=1, j\ne i}^l \frac{\alpha_{ij}^+}{\beta_{ij}^-} + \sum_{j=1}^m \frac{\gamma_{ij}^+}{\delta_{ij}^- e\Gamma_1},\ i = 1, 2, \ldots, l;$

(II) $\frac{\alpha_{ii}^-}{\beta_{ii}^+ + \Gamma_1} - 1 > \sum_{j=1, j\ne i}^l \frac{\alpha_{ij}^+}{\beta_{ij}^-} + \sum_{j=1}^m \gamma_{ij}^+,\ i = 1, 2, \ldots, l.$

Define

$$C^0 := \{\varphi_i :\ \varphi_i(0) > 0 \ \text{and}\ 0 \le \varphi_i(n) < \Gamma_1\ \forall\, n \in I\},\ i = 1, 2, \ldots, l.$$

Lemma 2.2. *Let (I) hold. Then, the solution $x_i(n, n_0, \varphi_i)$ of (1.6) with $\varphi_i \in C^0$ satisfies*

$$0 \le x_i(n, n_0, \varphi_i) < \Gamma_1 \ \ \forall\ \ n \in [n_0, \infty),\ i = 1, 2, \ldots, l. \tag{2.2}$$

Proof. Set $x_i(n) = x_i(n, n_0, \varphi_i)$ for all $n \in [n_0, \infty)$. Let $[n_0, T) \subseteq [n_0, \infty)$ be an interval such that

$$x_i(n) > 0 \ \ \forall\, n \in [n_0, T). \tag{2.3}$$

We claim that

$$0 \le x_i(n) < \Gamma_1 \ \ \forall\, n \in [n_0, T). \tag{2.4}$$

For the sake of contradiction, we assume that (2.4) is not true. Then, one can find $n_1 \in (n_0, T)$ such that

$$x_i(n_1) = \Gamma_1 \quad \text{and} \quad 0 \le x_j(n) < \Gamma_1, \ \ j = 1, 2, \ldots, l \tag{2.5}$$

for all $n \in [n_0 - \tau_j^+, n_1)$. In view of (1.6), (I), (2.5) and the facts that $\frac{\alpha_{ii}(n)x}{\beta_{ii}(n)+x} \le \frac{\alpha_{ii}(n)x}{\beta_{ii}(n)}$ and $\sup_{u\ge 0} ue^{-u} = \frac{1}{e}$, we have

$$\begin{aligned} 0 \le & \Delta x_i(n_1) \\ = & -\frac{\alpha_{ii}(n_1)x_i(n_1)}{\beta_{ii}(n_1) + x_i(n_1)} + \sum_{j=1, j\ne i}^{l} \frac{\alpha_{ij}(n_1)x_j(n_1)}{\beta_{ij}(n_1) + x_j(n_1)} \\ & + \sum_{j=1}^{m} \gamma_{ij}(n_1)x_i(n_1 - \tau_{ij}(n_1))e^{-\delta_{ij}(n_1)x_i(n_1 - \tau_{ij}(n_1))} \end{aligned}$$

$$\le -\frac{\alpha_{ii}(n_1)x_i(n_1)}{\beta_{ii}(n_1)+x_i(n_1)}+\sum_{j=1,j\neq i}^{l}\frac{\alpha_{ij}(n_1)x_j(n_1)}{\beta_{ij}(n_1)}+\sum_{j=1}^{m}\frac{\gamma_{ij}(n_1)}{\delta_{ij}(n_1)}\frac{1}{e}$$
$$\le\Big[-\frac{\alpha_{ii}^-}{\beta_{ii}^+ +\Gamma_1}+\sum_{j=1,j\neq i}^{l}\frac{\alpha_{ij}^+}{\beta_{ij}^-}+\sum_{j=1}^{m}\frac{\gamma_{ij}^+}{\delta_{ij}^- e\Gamma_1}\Big]\Gamma_1<0,$$

which is a contradiction. This implies that (2.4) holds. Based on the validity of (2.4), it follows that relation (2.2) is true. The proof is complete. □

Theorem 2.3. *Let* (I) *and* (II) *hold. Then, the solution* $x_i(n,n_0,\varphi_i)$ *of system* (1.6) *with* $\varphi_i\in C^0$ *is exponentially extinct as* n *tends to* ∞.

Proof. Define the functions $\Phi_i(u)$ by setting

$$\Phi_i(u)=e^u-\frac{\alpha_{ii}^- e^u}{\beta_{ii}^+ +\Gamma_1}+\sum_{j=1,j\neq i}^{l}\frac{\alpha_{ij}^+ e^u}{\beta_{ij}^-}+\sum_{j=1}^{m}\gamma_{ij}^+ e^{u(\tau_i^+ +1)} \tag{2.6}$$

for $u\in[0,1],\ i=1,2,\ldots,l$. It is clear that Φ_i are continuous on $[0,1]$. Then, by (II) we have

$$\Phi_i(0)=1-\frac{\alpha_{ii}^-}{\beta_{ii}^+ +\Gamma_1}+\sum_{j=1,j\neq i}^{l}\frac{\alpha_{ij}^+}{\beta_{ij}^-}+\sum_{j=1}^{m}\gamma_{ij}^+<0,\ \ i=1,2,\ldots,l,$$

which implies that there exists a constant $\lambda>0$ such that

$$\Phi_i(\lambda)=e^\lambda-\frac{\alpha_{ii}^- e^\lambda}{\beta_{ii}^+ +\Gamma_1}+\sum_{j=1,j\neq i}^{l}\frac{\alpha_{ij}^+ e^\lambda}{\beta_{ij}^-}+\sum_{j=1}^{m}\gamma_{ij}^+ e^{\lambda(\tau_i^+ +1)}<0,\ \ i=1,2,\ldots,l. \tag{2.7}$$

We consider the functions

$$y_i(n)=x_i(n)e^{\lambda(n-n_0)},\ \ i=1,2,\ldots,l. \tag{2.8}$$

Calculating the difference of $y_i(n)$ along the solution $x_i(n)$ of (1.6), we have

$$\begin{aligned}\Delta y_i(n)=&\Delta\big(x_i(n)e^{\lambda(n-n_0)}\big)\\ =&\Delta x_i(n)e^{\lambda(n+1-n_0)}+x_i(n)\Delta e^{\lambda(n-n_0)}\\ =&\Big(-\frac{\alpha_{ii}(n)x_i(n)}{\beta_{ii}(n)+x_i(n)}+\sum_{j=1,j\neq i}^{l}\frac{\alpha_{ij}(n)x_j(n)}{\beta_{ij}(n)+x_j(n)}\Big)e^{\lambda(n+1-n_0)}+x_i(n)\big(e^{\lambda(n+1-n_0)}-e^{\lambda(n-n_0)}\big)\\ \le& x_i(n)e^{\lambda(n+1-n_0)}-\frac{\alpha_{ii}(n)x_i(n)}{\beta_{ii}(n)+x_i(n)}e^{\lambda(n+1-n_0)}+\sum_{j=1,j\neq i}^{l}\frac{\alpha_{ij}(n)x_j(n)}{\beta_{ij}(n)+x_j(n)}e^{\lambda(n+1-n_0)}\\ &+\sum_{j=1}^{m}\gamma_{ij}(n)x_i(n-\tau_{ij}(n))e^{-\delta_{ij}(n)x_i(n-\tau_{ij}(n))}e^{\lambda(n+1-n_0)},\ \ i=1,2,\ldots,l\end{aligned} \tag{2.9}$$

for all $n\ge n_0$.

Let L_1 be a positive number such that $L_1>y_i(n)\ \forall\ n\in[n_0-\tau_i^+,n_0],\ i=1,2,\ldots,l$. We claim that

$$y_i(n)=x_i(n)e^{\lambda(n-n_0)}<L_1\ \ \forall\ \ n\ge n_0,\ i=1,2,\ldots,l. \tag{2.10}$$

Assume, on the contrarily, that there exists $n_2>n_0$ such that

$$y_i(n_2)=L_1\ \text{ and }\ y_j(n)<L_1\ \ \forall\ \ n\in[n_0-\tau_i^+,n_2),\ j=1,2,\ldots,l. \tag{2.11}$$

Then, from (2.2) and (2.9), we obtain

$$\begin{aligned}
0 \le & \Delta y_i(n_2)\\
\le & x_i(n_2)e^{\lambda(n_2+1-n_0)} - \frac{\alpha_{ii}(n_2)x_i(n_2)}{\beta_{ii}(n_2)+x_i(n_2)}e^{\lambda(n_2+1-n_0)}\\
& + \sum_{j=1,j\neq i}^{l} \frac{\alpha_{ij}(n_2)x_j(n_2)}{\beta_{ij}(n_2)+x_j(n_2)}e^{\lambda(n_2+1-n_0)}\\
& + \sum_{j=1}^{m} \gamma_{ij}(n_2)x_i(n_2-\tau_{ij}(n_2))e^{-\delta_{ij}(n_2)x_i(n_2-\tau_{ij}(n_2))}e^{\lambda(n_2+1-n_0)}\\
\le & y_i(n_2)e^{\lambda} - \frac{\alpha_{ii}^-}{\beta_{ii}^+ + \Gamma_1}y_i(n_2)e^{\lambda} + \sum_{j=1,j\neq j}^{l} \frac{\alpha_{ij}(n_2)}{\beta_{ij}(n_2)}y_j(n_2)e^{\lambda}\\
& + \sum_{j=1}^{m} \gamma_{ij}^+ y_i(n_2-\tau_{ij}(n_2))e^{-\delta_{ij}(n_2)x_i(n_2-\tau_{ij}(n_2))}e^{\lambda(n_2+1-n_0)}e^{-\lambda(n_2-\tau_i^+-n_0)}\\
\le & \Big(e^{\lambda} - \frac{\alpha_{ii}^- e^{\lambda}}{\beta_{ii}^+ + \Gamma_1} + \sum_{j=1,j\neq i}^{l} \frac{\alpha_{ij}^+ e^{\lambda}}{\beta_{ij}^-} + \sum_{j=1}^{m} \gamma_{ij}^+ e^{\lambda(\tau_i^+ + 1)}\Big)L_1.
\end{aligned}$$

Thus,

$$e^{\lambda} - \frac{\alpha_{ii}^- e^{\lambda}}{\beta_{ii}^+ + \Gamma_1} + \sum_{j=1,j\neq i}^{l} \frac{\alpha_{ij}^+ e^{\lambda}}{\beta_{ij}^-} + \sum_{j=1}^{m} \gamma_{ij}^+ e^{\lambda(\tau_i^+ + 1)} \ge 0,$$

which contradicts by (2.7). Hence, (2.10) holds. It follows that $x_i(n) < L_1 e^{-\lambda(n-n_0)}$ for all $n \ge n_0$, $i = 1, 2, \dots, l$. The proof is complete.

□

Example 2.4. Consider the system corresponding to (1.6)

$$\begin{aligned}
\Delta x_1(n) =& -\frac{(15+|\cos n|)x_1(n)}{(2+|\sin n|)+x_1(n)} + \frac{(1+|\sin n|)x_2(n)}{(6+|\cos n|)+x_2(n)} + \frac{(1+|\cos n|)x_3(n)}{(6+|\sin n|)+x_3(n)}\\
&+ \frac{1}{4}(1+\cos^2 n)x_1(n-2|\sin n|)e^{-x_1(n-2|\sin n|)}\\
&+ \frac{1}{4}(1+\sin^2 n)x_1(n-2|\cos n|)e^{-x_1(n-2|\cos n|)},\\
\Delta x_2(n) =& -\frac{(15+|\sin n|)x_2(n)}{(2+|\cos n|)+x_2(n)} + \frac{(1+|\cos n|)x_1(n)}{(6+|\sin n|)+x_1(n)} + \frac{(1+|\sin n|)x_3(n)}{(6+|\cos n|)+x_3(n)}\\
&+ \frac{1}{4}(1+\sin^2 n)x_2(n-2|\cos n|)e^{-x_2(n-2|\cos n|)}\\
&+ \frac{1}{4}(1+\cos^2 n)x_2(n-2|\sin n|)e^{-x_2(n-2|\sin n|)},\\
\Delta x_3(n) =& -\frac{(15+|\sin n|)x_3(n)}{(2+|\cos 6n|)+x_3(n)} + \frac{(1+|\cos n|)x_1(n)}{(6+|\sin n|)+x_1(n)} + \frac{(1+|\sin n|)x_2(n)}{(6+|\cos n|)+x_2(n)}\\
&+ \frac{1}{4}(1+\sin^2 n)x_3(n-2|\cos n|)e^{-x_3(n-2|\cos n|)}\\
&+ \frac{1}{4}(1+\cos^2 n)x_3(n-2|\sin n|)e^{-x_3(n-2|\sin 2n|)}.
\end{aligned} \tag{2.12}$$

By simple calculations, one can figure out that $\alpha_{ii}^- = 15$, $\beta_{ii}^+ = 3$ $(i = 1, 2, 3)$, $\alpha_{ij}^+ = 2$, $\beta_{ij}^- = 6$, $(i, j =$

$1,2,3$ and $j \neq i$), $\gamma_{ij}^{+} = \frac{1}{2}$ and $\delta_{ij}^{-} = 1$, ($i = 1,2,3$ and $j = 1,2$). Choosing $\Gamma_1 = 2$, one can easily see that

$$\frac{15}{3+2} > \frac{2}{6} + \frac{2}{6} + \frac{1}{4e} + \frac{1}{4e} \quad \text{and} \quad \frac{15}{3+2} - 1 > \frac{2}{6} + \frac{2}{6} + \frac{1}{2} + \frac{1}{2},$$

which means that conditions (I) and (II) are satisfied, respectively. This tells that the solution of system (2.12) with $0 \leq x_i(n) < 2$ is exponentially extinct as n tends to ∞. The solution behavior of system (2.12) is illustrated in Figure 1.

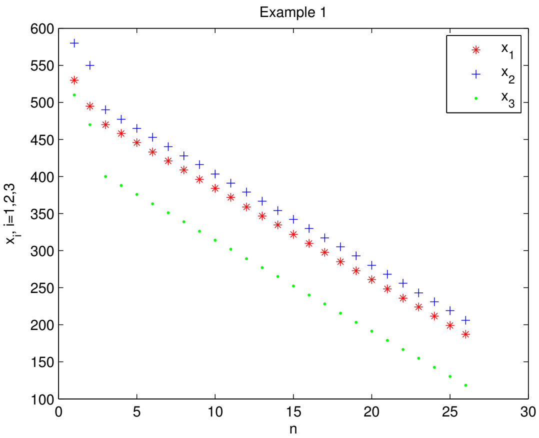

Figure 1: The solution behavior of equation (2.12) with the initial conditions $\phi_1(n) = 580, \phi_2(n) = 530, \phi_3(n) = 510$.

The remaining part of the paper is dedicated to the study system (1.7). To preserve the meaning of the the mortality terms in system (1.7), we stress that the condition $\alpha_{ij}(n) > \beta_{ij}(n)$ holds for $n \in \mathbb{Z}$ and $i,j = 1,2,\ldots,l$. We also assume that there exist positive constants Γ_2, L_2 and ν such that

(III) $\alpha_{ii}^{-} > \sum_{j=1,j\neq i}^{l} \alpha_{ij}^{+} + \left[\beta_{ii}^{+} - \sum_{j=1,j\neq i}^{l} \beta_{ij}^{-}\right] e^{-\Gamma_2} + \sum_{j=1}^{m} \frac{\gamma_{ij}^{+}}{\delta_{ij}^{-} e}$;

(IV) $\beta_{ii}^{-} > 2 + \frac{\Gamma_2}{2} \beta_{ii}^{+} + \sum_{j=1,j\neq i}^{l} \beta_{ij}^{+} + \sum_{j=1}^{m} \gamma_{ij}^{+}$;

(V) $-\alpha_{ii}(n) + \beta_{ii}(n) + \sum_{j=1,j\neq i}^{l} \left[\alpha_{ij}(n) - \beta_{ij}(n)\right] \leq L_2 e^{-\nu n}$.

Define

$$C_0 := \{\varphi_i : \ \varphi_i(0) > 0 \ \text{ and } \ 0 \leq \varphi_i(n) < \Gamma_2 \ \ \forall\, n \in I\}, \ i = 1,2,\ldots,l.$$

Lemma 2.5. *Let* (III) *hold. Then, the solution $x_i(n, n_0, \varphi_i)$ of* (1.7) *with $\varphi_i \in C_0$ satisfies*

$$0 \leq x_i(n, n_0, \varphi_i) < \Gamma_2 \ \ \forall \ \ n \in [n_0, \infty). \tag{2.13}$$

Proof. Set $x_i(n) = x_i(n, n_0, \varphi_i)$ for all $n \in [n_0, \infty)$. Let $[n_0, T) \subseteq [n_0, \infty)$ be an interval such that

$$x_i(n) > 0 \ \ \forall\, n \in [n_0, T). \tag{2.14}$$

We claim that

$$0 \leq x_i(n) < \Gamma_2 \ \ \forall\, n \in [n_0, T). \tag{2.15}$$

For the sake of contradiction, we assume that (2.15) is not true. Then, one can find $n_3 \in (n_0, T)$ such that

$$x_i(n_3) = \Gamma_2 \quad \text{and} \quad 0 \leq x_j(n) < \Gamma_2, \ j = 1,2,\ldots,l \tag{2.16}$$

for all $n \in [n_0 - \tau_j^{+}, n_3)$. In view of (1.7), (III), (2.16), and the fact that $\sup_{u\geq 0} ue^{-u} = \frac{1}{e}$, we have

$$\begin{aligned}
0 \leq & \Delta x_i(n_3) \\
= & -\alpha_{ii}(n_3) + \beta_{ii}(n_3)e^{-x_i(n_3)} + \sum_{j=1,j\neq i}^{l} \alpha_{ij}(n_3) - \beta_{ij}(n_3)e^{-x_j(n_3)} \\
& + \sum_{j=1}^{m} \gamma_{ij}(n_3)x_j(n_3 - \tau_{ij}(n_3))e^{-\delta_{ij}(n_3)x_j(n_3-\tau_{ij}(n_3))} \\
\leq & -\alpha_{ii}^{-} + \sum_{j=1,j\neq i}^{l} \alpha_{ij}^{+} + \Big[\beta_{ii}^{+} - \sum_{j=1,j\neq i}^{l} \beta_{ij}^{-}\Big]e^{-\Gamma_2} + \sum_{j=1}^{m} \frac{\gamma_{ij}^{+}}{\delta_{ij}^{-}e} < 0,
\end{aligned}$$

which is a contradiction and this implies that (2.15) holds. In view of (2.15), it follows that relation (2.13) is true. Hence the proof is complete. □

Theorem 2.6. *Let* (III)–(V) *hold. Then, the solution* $x_i(n, n_0, \varphi_i)$ *of system* (1.7) *with* $\varphi_i \in C_0$ *is exponentially extinct as* n *tends to* ∞.

Proof. Define the functions Φ_i by setting

$$\Phi_i(u) = \Big[2 - \beta_{ii}^{-} + \frac{\Gamma_2}{2}\beta_{ii}^{+} + \sum_{j=1,j\neq i}^{l} \beta_{ij}^{+}\Big]e^{u} + \sum_{j=1}^{m} \gamma_{ij}^{+}e^{u(\tau_i^{+}+1)} \tag{2.17}$$

for $u \in [0, 1]$, $i = 1, 2, \ldots, l$. It is clear that Φ_i are continuous on $[0, 1]$. Then, by (IV) we have

$$\Phi_i(0) = 2 - \beta_{ii}^{-} + \frac{\Gamma_2}{2}\beta_{ii}^{+} + \sum_{j=1,j\neq i}^{l} \beta_{ij}^{+} + \sum_{j=1}^{m} \gamma_{ij}^{+} < 0, \tag{2.18}$$

which implies that there exists a constant λ with $0 < \lambda < \nu$ such that

$$\Phi_i(\lambda) = \Big[2 - \beta_{ii}^{-} + \frac{\Gamma_2}{2}\beta_{ii}^{+} + \sum_{j=1,j\neq i}^{l} \beta_{ij}^{+}\Big]e^{\lambda} + \sum_{j=1}^{m} \gamma_{ij}^{+}e^{\lambda(\tau_i^{+}+1)} < 0. \tag{2.19}$$

Let

$$y_i(n) = x_i(n)e^{\lambda(n-n_0)}, \ i = 1, 2, \ldots, l. \tag{2.20}$$

Calculating the difference of $y_i(n)$ along the solution $x_i(n)$ of (1.7), we have

$$\begin{aligned}
\Delta y_i(n) = & \Delta\big(x_i(n)e^{\lambda(n-n_0)}\big) = \Delta x_i(n)e^{\lambda(n+1-n_0)} + x_i(n)\Delta e^{\lambda(n-n_0)} \\
\leq & y_i(n)e^{\lambda} + \Big[-\alpha_{ii}(n) + \beta_{ii}(n)\big(1 - x_i(n) + \frac{1}{2}x_i^2(n)\big)\Big]e^{\lambda(n+1-n_0)} \\
& + \sum_{j=1,j\neq i}^{l} \Big[\alpha_{ij}(n) - \beta_{ij}(n)\big(1 - x_j(n)\big)\Big]e^{\lambda(n+1-n_0)} \\
& + \Big[\sum_{j=1}^{m} \gamma_{ij}(n)x_i(n - \tau_{ij}(n))e^{-\delta_{ij}(n)x_i(n-\tau_{ij}(n))}\Big]e^{\lambda(n+1-n_0)} \\
\leq & y_i(n)e^{\lambda} + \big[-\alpha_{ii}(n) + \beta_{ii}(n) + \sum_{j=1,j\neq i}^{l} (\alpha_{ij}(n) - \beta_{ij}(n))\big]e^{\lambda(n+1-n_0)} \\
& - \beta_{ii}(n)y_i(n)e^{\lambda} + \frac{1}{2}\beta_{ii}(n)y_i(n)x_i(n)e^{\lambda} + \sum_{j=1,j\neq i}^{l} \beta_{ij}(n)y_j(n)e^{\lambda} \\
& + \sum_{j=1}^{m} \gamma_{ij}^{+}e^{\lambda(\tau_i^{+}+1)}y_i(n - \tau_{ij}(n))
\end{aligned}$$

for all $n \geq n_0$. In view of (V) and (2.13), we obtain

$$\begin{aligned}\Delta y_i(n) \leq & \Big[y_i(n) + L_2 e^{(\lambda-\nu)n} - \beta_{ii}^- y_i(n) + \frac{1}{2}\Gamma_2 \beta_{ii}^+ y_i(n)\Big] e^{\lambda} \\ & + \sum_{j=1, j\neq i}^{l} \beta_{ij}(n) y_j(n) e^{\lambda} + \sum_{j=1}^{m} \gamma_{ij}^+ e^{\lambda(\tau_i^+ + 1)} y_i(n-\tau_{ij}(n)).\end{aligned} \tag{2.21}$$

Let L_3 be a positive number such that

$$L_3 = \max\{y_i(n), L_2\} \ \ \forall \ \ n \in [n_0 - \tau_i^+, n_0].$$

We claim that

$$y_i(n) = x_i(n) e^{\lambda(n-n_0)} < L_3 \ \ \forall \ \ n \geq n_0. \tag{2.22}$$

If this is not true, then there exists $n_4 > n_0$ such that

$$y_i(n_4) = L_3 \ \text{ and } \ y_j(n) < L_3 \ \ \forall \ \ n \in [n_0 - \tau_i^+, n_4), \ j = 1, 2, \ldots, l. \tag{2.23}$$

Then from (2.19) and (2.21), we obtain

$$\begin{aligned}0 \leq \Delta y_i(n_4) \leq & \Big[y_i(n_4) + L_2 e^{(\lambda-\nu)n_4} - \beta_{ii}^- y_i(n_4) + \frac{1}{2}\Gamma_2 \beta_{ii}^+ y_i(n_4)\Big] e^{\lambda} \\ & + \sum_{j=1, j\neq i}^{l} \beta_{ij}(n) y_j(n_4) e^{\lambda} + \sum_{j=1}^{m} \gamma_{ij}^+ e^{\lambda(\tau_i^+ + 1)} y_i(n_4-\tau_{ij}(n_4)) \\ \leq & \Big(2e^{\lambda} - \beta_{ii}^- e^{\lambda} + \frac{1}{2}\Gamma_2 \beta_{ii}^+ e^{\lambda} + \sum_{j=1, j\neq i}^{l} \beta_{ij}^+ e^{\lambda} + \sum_{j=1}^{m} \gamma_{ij}^+ e^{\lambda(\tau_i^+ + 1)}\Big) L_3.\end{aligned}$$

Thus,

$$2e^{\lambda} - \beta_{ii}^- e^{\lambda} + \frac{1}{2}\Gamma_2 \beta_{ii}^+ e^{\lambda} + \sum_{j=1, j\neq i}^{l} \beta_{ij}^+ e^{\lambda} + \sum_{j=1}^{m} \gamma_{ij}^+ e^{\lambda(\tau_i^+ + 1)} \geq 0,$$

which contradicts by (2.19). Hence (2.22) holds. It follows that $x_i(n) < L_3 e^{-\lambda(n-n_0)}$ for all $n \geq n_0$. The proof is complete. □

Example 2.7. Consider the system corresponding to (1.7)

$$\begin{aligned}\Delta x_1(n) = & -(15 + |\sin 3n|) + (14 + |\cos 3n|) e^{-x_1(n)} + \big(\frac{3}{2} + \frac{1}{2}|\sin 3n|\big) - \big(1 + \frac{1}{2}|\cos 3n|\big) e^{-x_2(n)} \\ & + \big(1 + \frac{1}{2}|\sin 3n|\big) - \big(\frac{1}{2} + \frac{1}{2}|\cos 3n|\big) e^{-x_3(n)} \\ & + \frac{1}{4}(\cos^2 n) x_1(n - 2|\sin n|) e^{-x_1(n-2|\sin n|)} \\ & + \frac{1}{4}(\sin^2 n) x_1(n - 2|\cos n|) e^{-x_1(n-2|\cos n|)}, \\ \Delta x_2(n) = & -(15 + |\cos n|) + (14 + |\sin n|) e^{-x_2(n)} + \big(\frac{3}{2} + \frac{1}{2}|\cos n|\big) - \big(1 + \frac{1}{2}|\sin n|\big) e^{-x_1(n)} \\ & + \big(1 + \frac{1}{2}|\cos n|\big) - \big(\frac{1}{2} + \frac{1}{2}|\sin n|\big) e^{-x_3(n)} + \frac{1}{4}(\sin^2 n) x_2(n - 2|\cos n|) e^{-x_2(n-2|\cos n|)} \\ & + \frac{1}{4}(\cos^2 n) x_2(n - 2|\sin n|) e^{-x_2(n-2|\sin n|)}, \\ \Delta x_3(n) = & -(15 + |\sin n|) + (14 + |\cos n|) e^{-x_3(n)} + \big(\frac{3}{2} + \frac{1}{2}|\sin n|\big) - \big(1 + \frac{1}{2}|\cos n|\big) e^{-x_1(n)} \\ & + \big(1 + \frac{1}{2}|\sin n|\big) - \big(\frac{1}{2} + \frac{1}{2}|\cos n|\big) e^{-x_2(n)} \\ & + \frac{1}{4}(\cos^2 n) x_3(n - 2|\sin n|) e^{-x_3(n-2|\sin n|)} \\ & + \frac{1}{4}(\sin^2 n) x_3(n - 2|\cos n|) e^{-x_3(n-2|\cos n|)}.\end{aligned} \tag{2.24}$$

By simple calculations, we find that $\alpha_{ii}^- = 15$, $\beta_{ii}^- = 14$, $\beta_{ii}^+ = 15$ $(i = 1, 2, 3)$, $\alpha_{ij}^+ = 2$, $\beta_{ij}^- = 1$, $\beta_{ij}^+ = \frac{3}{2}$ $(i, j = 1, 2, 3$ and $j \neq i)$, $\gamma_{ij}^+ = 0.25$ and $\delta_{ij}^- = 1$, $(i = 1, 2, 3$ and $j = 1, 2)$. Choosing $\Gamma_2 = 1$, one can easily see that

$$15 > 2 + 2 + \frac{15 - 1 - 1}{e} + \frac{1}{4e} + \frac{1}{4e}, \quad 14 > 2 + \frac{15}{2} + \frac{3}{2} + \frac{3}{2} + 0.25 + 0.25$$

and

$$-\alpha_{ii}(n) + \beta_{ii}(n) + \sum_{j=1, j\neq i}^{3} \left[\alpha_{ij}(n) - \beta_{ij}(n)\right] = 0, \quad i = 1, \ldots, 3,$$

which means that conditions (III), (IV) and (V) are satisfied, respectively. This tells that the solution of equation (2.24) with $0 \leq x_i(n) < 1$ is exponentially extinct as n tends to ∞. The solution of system (2.24) behaves as illustrated in Figure 2.

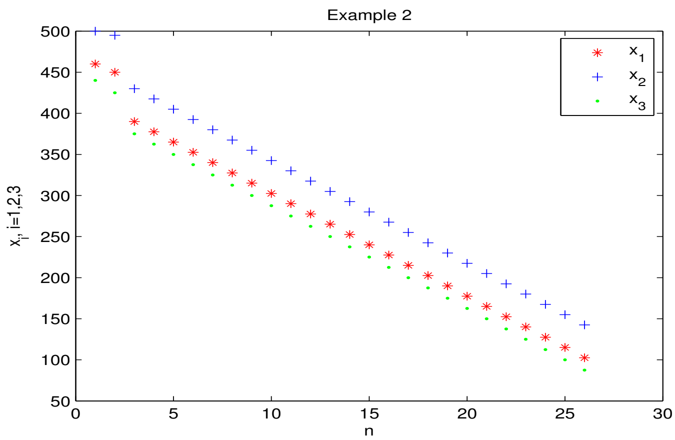

Figure 2: The solution behavior of equation (2.24) with the initial conditions $\phi_1(n) = 500, \phi_2(n) = 460, \phi_3(n) = 440$.

3. Concluding remark

This paper investigated a generalized discrete Nicholson's blowflies systems with patchy structure and mortality terms which are expressed in exponential and fractional forms. According to authors' knowledge, no systems in the addressed forms have been under consideration in the literature. The authors provide essentially new sufficient conditions for the boundedness as well as the exponential extinction of the solutions of the systems. The results obtained in this paper are of great significance for the audience working on mathematical biology as they could be used in laboratory experiments to avoid certain unwanted situations.

Based on the results of this paper, further study could be conducted. Indeed, for systems (1.6) and (1.7),

1. Certain fixed point theorems could be used to prove the existence of almost periodic positive solutions;
2. Certain Lyapunov's functionals could be employed to prove the exponential convergence of the solutions;
3. The oscillatory and the global attractivity could be considered;
4. The stability and the existence of Hopf bifurcations could be discussed.

We leave these research topics for future consideration.

Acknowledgment

The first two authors would like to express their sincere thanks to King Abdulaziz City for Science and Technology (KACST) for supporting this project through the grant LGP 35-117.

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