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Existence and Hyers-Ulam stability of solutions to the implicit second-order differential equation via fractional integral boundary conditions

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Abstract

In this paper, the existence and Ulam–Hyers stability of solutions for implicit second order fractional differential equations are investigated via fractional-orders integral boundary conditions. Our results are based on Krasnoselskii's fixed point Theorem and Banach contraction principle. We provide examples at the end to clarify our acquired outcomes..

Keywords: ϕ-Caputo fractional order, existence results, Green's function, boundary value problems, Ulam-Hyers stability.

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1. Introduction

The topic of Fractional calculus is part of a classical analysis that includes integration and differentiation of fractional-order. There has been a huge advancement in fractional derivatives with Riemann-Liouville, Caputo, Hilfer, Hadamard, and others. Ongoing outcomes on fractional calculus and fractional differential equations can be found in [7, 17–20, 29]. The Caputo fractional derivative has recently been applied in different areas of the physical interpretation of initial conditionsand boundary conditions. For additional details on their different applications, see the monographs of Abbas et al. [2], Baleanu et al. [16], Kilbas et al. [24], Lakshmikantham et al. [26], Agarwal et al. [4, 5], Babakhani and Daftardar-Gejji [10], Belmekki and Benchohra [11], Benchohra et al. [12–14], Kilbas and Marzan [23], and the references therein.

The principle of impulsive fractional differential equations has received a lot of attention, and several researchers have documented some very useful results in this field [21, 27]. It has been discovered that impulsive fractional differential equations can provide ideal models to describe real processes that deviate from their states at random times and cannot be interpreted using conventional fractional differential equations. Several mathematicians have recently looked into boundary value problems for fractional differential equations with fractional derivatives. More specifically Chasreechai and Tariboon in

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[15] demonstrated existence theorems for the positive solutions of the following types of problems with integral boundary condition

$$\mu''(\mathfrak{r}) + \chi g(\mathfrak{r})\ h(\mu(\mathfrak{r})) = 0, \quad \mathfrak{r} \in (0,1), \quad \mu(0) = \alpha_1 \int_0^{\eta} \mu(\sigma)\ d\sigma, \quad \mu(1) = \alpha_2 \int_0^{\eta} \mu(\sigma)\ d\sigma.$$

Hu and Wang [21] proved the existence of solution for the nonlinear fractional differential equation with integral boundary condition

$$D^{\zeta}\mu(\mathfrak{r}) = g\big(\mathfrak{r}, \mu(\mathfrak{r}), D^{\delta}\mu(\mathfrak{r})\big), \quad \mathfrak{r} \in (0,1), 1 < \zeta \leqslant 2,\ 0 < \delta \leqslant 1, \quad \mu(0) = \mu_{\circ}, \quad \mu(1) = \int_0^1 h(\sigma)\mu(\sigma))\ d\sigma,$$

where D^{α} is the Riemann-Liouville fractional derivative. Murad and Hadid et al. [27] studied the boundary value problem of the fractional differential equations given by

$$D^{\zeta}\mu(\mathfrak{r}) = g\big(\mathfrak{r}, \mu(\mathfrak{r}), D^{\delta}\mu(\mathfrak{r})\big), \quad \mathfrak{r} \in (0,1), 1 < \zeta \leqslant 2,\ 0 < \delta \leqslant 1, \quad \mu(0) = 0, \quad \mu(1) = I_0^{\gamma}\mu(\mathfrak{r}).$$

where D^{ζ} is the Riemann-Liouville fractional derivative.

On the other hand, the stability analysis of integral and differential equations is critical in many applications, a great interest has been shown in the study of Ulam-Hyers stability of non-linear functional differential equations with different type of boundary conditions, we mean by Ulam-Hyers stability that functional differential equation has an exact solution that is very close to its approximative solution and that the error can be determined. Ulam introduced a novel concept of stability analysis on the topic of stability theory in 1940, and Hyers advanced it in 1941. Later, Rassias in 1997 expanded on the aforementioned concept, for essential results and new additions on Ulam-Hyers stability of f integro-differential equations, we suggest the references [1, 22, 28].

Fractional derivatives of a function with respect to function ϕ have been considered in the classical articles as a generalization of Riemann-Liouville derivative. This fractional derivative is called the ϕ-fractional derivative and it differs from the classical one as the kernel appears in terms of ϕ. Recently, this derivative has been reassessed by Almeida in [8], where the Caputo-type regularization of the existing definition and some interesting properties are provided. Several properties of this operator could be found in [3, 25, 30].

Motivated by the results in the shown above articles, here we develop existence and stability theorem for implicit fractional order differential problem (ISDP).

$$\frac{d^2}{d\mathfrak{r}^2} y(\mathfrak{r}) = f\Big(\mathfrak{r}, y(\mathfrak{r}), {}^{c}D^{\beta;\phi} y(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\, {}^{c}D^{\alpha;\phi} y(\sigma) d\sigma\Big), \tag{1.1}$$

$$y(0) = \frac{1}{\Gamma(\gamma)} \int_0^1 \phi'(\sigma)(\phi(1) - \phi(\sigma))^{\gamma-1}\ h_1(\sigma, y(\sigma))\ d\sigma, \tag{1.2}$$

$$y'(1) = \frac{1}{\Gamma(\gamma)} \int_0^1 \phi'(\sigma)(\phi(1) - \phi(\sigma))^{\gamma-1}\ h_2(\sigma, y(\sigma))\ d\sigma, \tag{1.3}$$

where $1 < \beta < \alpha \leqslant 2$ and $0 < \gamma < 1$, here $\phi\,(\mathfrak{r})$ is non decreasing function with $\phi'\,(\mathfrak{r}) \neq 0,\ \forall\, \mathfrak{r} \in \mathbb{I} = [0,1]$, and ${}^{c}D^{\alpha,\phi}$ is the ϕ-Caputo fractional derivative.

The paper contents are separated into four sections. Section 2 contains the main results and is divided into two parts. The equivalence between ISDP (1.1)-(1.3) and the integral equation was discussed in part one. Part two includes two existence results of the problem (1.1)-(1.3), one of which is proven using the principle of Banach contraction and the other uses the fixed point theorem of Krasnosel'skii. We will also discuss Ulam-Hyers Russian stability of our problem in Section 3. Also, in Section 4 we present some particular cases and examples that demonstrate our findings.

Now, we state the definition of fractional integral and fractional derivative used in developing our results throughout this paper.

Definition 1.1 ([8]). For any real number $\alpha > 0$, the ϕ-Riemann-Liouville fractional integral of order α for an integrable function $u : \mathbb{I} \to \mathbb{R}$ with respect to another function $\phi : \mathbb{I} \to \mathbb{R}$, which is an increasing differentiable function such that $\phi'(\mathfrak{r}) \neq 0$ for all $\mathfrak{r} \in \mathbb{I}$ is defined by:

$$I^{\alpha,\phi} u(\mathfrak{r}) = \frac{1}{\Gamma(\alpha)} \int_0^{\mathfrak{r}} \phi'(\sigma)(\phi(\mathfrak{r}) - \phi(\sigma))^{\alpha-1} u(\sigma)\, d\sigma,$$

where Γ is the classical Euler Gamma function.

Definition 1.2 ([8]). If $n \in N$ and $\phi, u \in C^n(\mathbb{I}, \mathbb{R})$ are two functions such that ϕ is increasing and $\phi'(\mathfrak{r}) \neq 0$ for all $\mathfrak{r} \in \mathbb{I}$, then the ϕ-Capato fractional derivative of a function u of order α is defined by:

$${}^cD^{\alpha,\psi} u(\mathfrak{r}) = I^{n-\alpha,\phi} \left(\frac{1}{\phi'(\mathfrak{r})} \frac{d}{d\mathfrak{r}} \right)^n u(\mathfrak{r}) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\mathfrak{r}} \psi'(\sigma)(\psi(\mathfrak{r}) - \psi(\sigma))^{n-\alpha-1} u_{\psi}^{[n]}(\sigma)\, d\sigma,$$

where $u_{\phi}^{[n]}(\mathfrak{r}) = \left(\frac{1}{\phi'(\mathfrak{r})} \frac{d}{d\mathfrak{r}} \right)^n u(\mathfrak{r})$ and $n = [\alpha] + 1$ for $\alpha \notin N$, and $n = \alpha$ for $\alpha \in N$.

Remark 1.3. Let $\alpha > 0$, then the differential equation $({}^cD_{a+}^{\alpha,\phi} h)(\mathfrak{r}) = 0$ has solution

$$h(\mathfrak{r}) = c_0 + c_1 (\phi(\mathfrak{r}) - \phi(0)) + c_2 (\phi(\mathfrak{r}) - \phi(0))^2 + \cdots + c_{n-1} (\phi(\mathfrak{r}) - \phi(0))^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

2. Existence and uniqueness

We show the existence of solution for ISDP (1.1)-(1.3), by assuming the following conditions.

(H_1) $\mathcal{F}_i : \mathbb{I} \times \mathbb{R} \to \mathbb{R}, i = 1, 2$ are continuous and there exist constants $k_i \in [0, 1)$ such that

$$|\mathcal{F}_i(\mathfrak{r}, u) - \mathcal{F}_i(\mathfrak{r}, v)| \leqslant k_i |u - v|.$$

(H_2) $\theta : \mathbb{I} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and there exists $\psi \in C(\mathbb{I}, \mathbb{R}_+)$, with norm $\|\psi\|$, such that:

$$|\theta(\mathfrak{r}, u_1, u_2, u_3) - \theta(\mathfrak{r}, v_1, v_2, v_3)| \leqslant \psi(\mathfrak{r}) \big(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| \big),$$

$\forall\, \mathfrak{r} \in \mathbb{I},\ u_i, v_i \in \mathbb{R},\ (i = 1, 2, 3)$.

(H_3) $k(\mathfrak{r}, \sigma)$ is continuous for all $(\mathfrak{r}, \sigma) \in \mathbb{I} \times \mathbb{I}$, and there is a positive constant K such that

$$\max_{\mathfrak{r}, \sigma \in \mathbb{I}} |k(\mathfrak{r}, \sigma)| = K.$$

Remark 2.1. From assumptions (H_1) and (H_2), we have

$$|\mathcal{F}_i(\mathfrak{r}, u)| \leqslant H_i + k_i |u|, \quad \text{where} \quad F_i = \sup_{\mathfrak{r} \in \mathbb{I}} |h_i(\mathfrak{r}, 0)|, \ i = 1, 2$$

and

$$|\theta(\mathfrak{r}, u_1, u_2, u_3)| \leqslant \|\psi\| (|u_1| + |u_2| + |u_3|) + \Theta, \quad \text{where} \quad \Theta = \sup_{\mathfrak{r} \in \mathbb{I}} |f(\mathfrak{r}, 0, 0, 0)|.$$

Lemma 2.2. *The mild solution of ISDP* (1.1)-(1.3) *is the solution of the integral equation*

$$y(\mathfrak{r}) = \mathcal{F}(\mathfrak{r}, y(\mathfrak{r})) + \int_0^1 G(\mathfrak{r}, \sigma) u(\sigma) d\sigma,$$

here u *is the solution of integral equation*

$$u(\mathfrak{r}) = \theta\big(\mathfrak{r}, \digamma(\mathfrak{r}) + \int_0^1 G(\mathfrak{r},\sigma)u(\sigma)d\sigma, I^{2-\beta;\phi}u(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)I^{2-\alpha;\phi}u(\sigma)d\sigma\big),$$

$G(\mathfrak{r},\sigma)$ *is the Green function described by*

$$G(\mathfrak{r},\sigma) = \begin{cases} \mathfrak{r} - (1-\sigma), & 0 \leqslant \sigma \leqslant \mathfrak{r} \leqslant 1, \\ \mathfrak{r}, & 0 \leqslant \mathfrak{r} \leqslant \sigma \leqslant 1, \end{cases} \tag{2.1}$$

$G_\circ := \max\{|G(\mathfrak{r},\sigma)|, (\mathfrak{r},\sigma) \in I \times I\}$, *and*

$$\digamma(\mathfrak{r}, y(\mathfrak{r})) = \int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \digamma_1(\sigma, y(\sigma))d\sigma + \mathfrak{r}\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \digamma_2(\sigma, y(\sigma))\, d\sigma. \tag{2.2}$$

Proof. We know that ${}^cD^{\alpha;\phi}y(\mathfrak{r}) = I^{2-\alpha;\phi}\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r})$ and ${}^cD^{\beta;\phi}y(\mathfrak{r}) = I^{2-\beta;\phi}\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r})$ for $\mathfrak{r} \in \mathbb{I}$. Consequently, if y is a solution of ISDP (1.1), then

$$u(\mathfrak{r}) = \theta(\mathfrak{r}, y(\mathfrak{r}), I^{2-\beta;\phi}u(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)I^{2-\alpha;\phi}u(\sigma)d\sigma)$$

and

$$y(\mathfrak{r}) = a_\circ + a_1\,\mathfrak{r} + \int_0^{\mathfrak{r}} (1-\sigma)u(\sigma)d\sigma. \tag{2.3}$$

We can get the following from (1.2) and (1.3),

$$a_\circ = \int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \digamma_1(\sigma, y(\sigma))d\sigma.$$

Differentiating (2.3) we receive

$$y'(\mathfrak{r}) = a_1 + \int_0^{\mathfrak{r}} u(\sigma)d\sigma.$$

So,

$$a_1 = \frac{1}{\Gamma(\gamma)}\int_0^1 \phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}\digamma_2(\sigma, y(\sigma))\, d\sigma - \int_0^1 u(\sigma)d\sigma.$$

Then the solution of (1.1)-(1.3) is as follows:

$$y(\mathfrak{r}) = \int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \digamma_1(\sigma, y(\sigma))d\sigma + \mathfrak{r}\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \digamma_2(\sigma, y(\sigma))\, d\sigma$$
$$- \mathfrak{r}\int_0^1 u(\sigma)d\sigma + \int_0^{\mathfrak{r}} (1-\sigma)u(\sigma)d\sigma.$$

□

Lemma 2.3. $\digamma : \mathbb{I} \times \mathbb{R} \to \mathbb{R}$, *is Lipschitzian function with a Lipschitz constant* c,

$$\|\digamma(\mathfrak{r}, x) - \digamma(\mathfrak{r}, y)\| \leqslant c\,\|x - y\|,$$

where $c = \frac{(\phi(1)-\phi(0))^\gamma[k_1 + k_2]}{\Gamma(\gamma+1)}$.

Proof. For all $u, v \in X$ and $\mathfrak{r} \in \mathbb{I}$, we obtain

$$\big|\digamma(\mathfrak{r}, x(\mathfrak{r})) - \digamma(\mathfrak{r}, y(\mathfrak{r}))\big|$$
$$\leqslant \Big|\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \digamma_1(\sigma, x(\sigma))d\sigma + \mathfrak{r}\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \digamma_2(\sigma, x(\sigma))\, d\sigma$$

$$
\begin{aligned}
&-\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \mathcal{F}_1(\sigma,y(\sigma))d\sigma - \mathfrak{r}\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)} \mathcal{F}_2(\sigma,y(\sigma))\, d\sigma\Big| \\
&\leqslant \int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)}\big|\big(\mathcal{F}_1(\sigma,x(\sigma)) - \mathcal{F}_1(\sigma,y(\sigma))\big| d\sigma \\
&\quad + \mathfrak{r}\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)}\big|\big(\mathcal{F}_2(\sigma,x) - \mathcal{F}_2(\sigma,y)\big)\big| d\sigma \\
&\leqslant \frac{k_1(\phi(1)-\phi(0))^{\gamma}}{\Gamma(\gamma+1)}\|x-y\| + \frac{k_2\,(\phi(1)-\phi(0))^{\gamma}}{\Gamma(\gamma+1)}\|x-y\| \\
&\leqslant \frac{(\phi(1)-\phi(0))^{\gamma}[k_1\ + k_2]}{\Gamma(\gamma+1)}\|x-y\|.
\end{aligned}
$$

Then

$$\|\mathcal{F}(\mathfrak{r},x) - \mathcal{F}(\mathfrak{r},y)\| \leqslant c\ \|x-y\|.$$

$\square$

2.1. Main results

The first existence result for ISDP (1.1)-(1.3) is depending on the fixed point Theorem of Krasnoselskii [9].

Theorem 2.4. *Let assumptions* (H_1)-(H_3) *hold. If*

$$\frac{(\phi(1)-\phi(0))^{\gamma}k_1 + \mathfrak{r}(\phi(1)-\phi(0))^{\gamma+1}k_2}{\Gamma(\gamma)} + \frac{G_\circ\ \|\psi\|}{1-\mathfrak{M}} < 1,$$

where

$$\mathfrak{M} = \frac{\|\psi\|(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{\|\psi\|(\phi(1)-\phi(0))^{2-\alpha}\ K}{\Gamma(3-\alpha)},$$

then the ISDP (1.1)-(1.3) *has at least one mild solution on* $\mathbb{I}$.

Proof. Set the operator $\mathcal{A} : C(\mathbb{I},\mathbb{R}) \to C(\mathbb{I},\mathbb{R})$ by:

$$\mathcal{A}y(\mathfrak{r}) = \mathcal{F}(\mathfrak{r},y(\mathfrak{r})) + \int_0^1 G(\mathfrak{r},\sigma)\nu(\sigma)d\sigma, \tag{2.4}$$

with $\mathcal{F}$ is the functions defined by (2.2),

$$\nu(\mathfrak{r}) = \theta\Big(\mathfrak{r},y(\mathfrak{r}),I^{2-\beta;\phi}\nu(\mathfrak{r}),\int_0^{\mathfrak{r}} k(t,\sigma)\ I^{2-\alpha;\phi}\nu(\sigma)d\sigma\Big).$$

Define the set

$$B_\rho = \{y \in C(\mathbb{I},\mathbb{R}) : \|y\| \leqslant \rho\},$$

with

$$\rho \geqslant \frac{\frac{(\phi(1)-\phi(0))^{\gamma}F_1 + (\phi(1)-\phi(0))^{\gamma}F_2}{\Gamma(\gamma+1)} + \frac{G_\circ\ \Theta}{1-\mathfrak{M}}}{1-\Big(\frac{(\phi(1)-\phi(0))^{\gamma}k_1 + \mathfrak{r}(\phi(1)-\phi(0))^{\gamma}k_2}{\Gamma(\gamma+1)} + \frac{G_\circ\ \|\psi\|}{1-\mathfrak{M}}\Big)}.$$

Moreover, we characterize the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on B_ρ by

$$\mathcal{A}_1y(\mathfrak{r}) = \mathcal{F}(\mathfrak{r},y(\mathfrak{r})), \quad \mathcal{A}_2y(t) = \int_0^1 G(\mathfrak{r},\sigma)\nu(\sigma)d\sigma.$$

Consider that $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined on B_ρ, and for any $y \in C(\mathbb{I},\mathbb{R})$,

$$\mathcal{A}y(\mathfrak{r}) = \mathcal{A}_1y(\mathfrak{r}) + \mathcal{A}_2y(\mathfrak{r}), \quad \mathfrak{r} \in \mathbb{I}.$$

The proof will be decomposed into 3 steps.

Step 1: Let $y_1, y_2 \in B_\rho$ and $\mathfrak{t} \in \mathbb{I}$, we have

$$|A_1y_1(\mathfrak{t}) + A_2y_2(\mathfrak{t})| \leqslant |A_1y_1(\mathfrak{t})| + |A_2y_2(\mathfrak{t})| \leqslant |\mathcal{F}(\mathfrak{t}, y_1(\mathfrak{t}))| + \int_0^1 |G(\mathfrak{t},\sigma)||\nu(\sigma)|d\sigma, \tag{2.5}$$

where $\nu(\mathfrak{t}) = \theta\big(\mathfrak{t}, y_2(\mathfrak{t}), I^{2-\beta;\phi}\nu(\mathfrak{t}), \int_0^{\mathfrak{t}} k(\mathfrak{t},\sigma)\ I^{2-\alpha;\phi}\nu(\sigma)d\sigma\big)$,

$$\begin{aligned}
|\nu(\mathfrak{t})| &= \Big|\theta\big(\mathfrak{t}, y_2(\mathfrak{t}), I^{2-\beta;\phi}\nu(\mathfrak{t}), \int_0^{\mathfrak{t}} k(\mathfrak{t},\sigma)\ I^{2-\alpha;\phi}\nu(\sigma)d\sigma\big)\Big| \\
&\leqslant\ \theta + \psi(\mathfrak{t})|y_2(\mathfrak{t})| + \psi(\mathfrak{t})\ \int_0^{\mathfrak{t}} \frac{\phi'(\sigma)(\phi(\mathfrak{t})-\phi(\sigma))^{2-\beta-1}}{\Gamma(2-\beta)}|\nu(\sigma)|\ d\sigma \\
&\quad + \psi(\mathfrak{t})\int_0^{\mathfrak{t}} |k(\mathfrak{t},\sigma)| \int_0^{\sigma} \frac{\phi'(\eta)(\phi(\sigma)-\phi(\eta))^{2-\alpha-1}}{\Gamma(2-\alpha)}|\nu(\eta)|\ d\eta\ d\sigma \\
&\leqslant \theta + \|\psi\|\ \|y_2\| + \|\psi\|\frac{(\phi(\mathfrak{t})-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}\|\nu\| + \|\psi\|K\frac{(\phi(\mathfrak{t})-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\|\nu\|\mathfrak{t}, \\
\|\nu\| &\leqslant \theta + \|\psi\|\ \|y_2\| + \|\psi\|\frac{(\phi(\mathfrak{t})-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}\|\nu\| + \|\psi\|K\frac{(\phi(\mathfrak{t})-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\|\nu\|.
\end{aligned}$$

Then

$$\|\nu\| \leqslant\ \frac{\theta + \|\psi\|\ \rho}{1-\big(\frac{\|\psi\|(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{\|\psi\|(\phi(1)-\phi(0))^{2-\alpha}\ K}{\Gamma(3-\alpha)}\big)},$$

and

$$\begin{aligned}
|\mathcal{F}(\mathfrak{t}, y_1(\mathfrak{t}))| &\leqslant \int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)}|\mathcal{F}_1(\sigma, y_1(\sigma))|d\sigma \\
&\quad + \mathfrak{t}\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)}|\mathcal{F}_2(\sigma, y_1(\sigma))|d\sigma \\
&\leqslant \int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)}|[F_1 + k_1|y_1(\sigma)|]d\sigma \\
&\quad + \mathfrak{t}\int_0^1 \frac{\phi'(\sigma)(\phi(1)-\phi(\sigma))^{\gamma-1}}{\Gamma(\gamma)}|[F_2 + k_2|y_2(\sigma)|]d\sigma \\
&\leqslant \frac{(\phi(1)-\phi(0))^{\gamma}[F_1 + k_1\|y_1\|]}{\Gamma(\gamma+1)} + \frac{(\phi(1)-\phi(0))^{\gamma}\ [F_2 + k_2\|y_2\|]}{\Gamma(\gamma+1)}.
\end{aligned}$$

Hence (2.5) implies that, for each $\mathfrak{t} \in \mathbb{I}$,

$$|A_1y_1(\mathfrak{t}) + A_2y_2(\mathfrak{t})| \leqslant \frac{(\phi(1)-\phi(0))^{\gamma}[F_1 + k_1\|y_1\|]}{\Gamma(\gamma+1)} + \frac{(\phi(1)-\phi(0))^{\gamma}\ [F_2 + k_2\|y_2\|]}{\Gamma(\gamma+1)} + \frac{G_\circ\ (\theta + \|\psi\|\ \rho)}{1-\ \mathfrak{M}} \leqslant\ \rho.$$

For $\mathfrak{t} \in \mathbb{I}$, we have

$$\|A_1y_1 + A_2y_2\| \leqslant\ \rho.$$

This proves that $A_1y_1 + A_2y_2 \in B_\rho$ for every $y_1, y_2 \in B_\rho$.

Step 2: The operator A_1 is a contraction mapping on B_ρ. It is clear that from Lemma 2.3, A_1 is a contration mapping for $c < 1$.

Step 3: The operator A_2 is completely continuous (compact and continuous) on B_ρ. First, we prove that the operator A_2 is continuous. Let $\{y_n\}_{n\in N}$ be a sequence such that $y_n \to y$ as $n \to \infty$ in $C(\mathbb{I}, \mathbb{R})$. Then for each $\mathfrak{r} \in \mathbb{I}$, we have

$$|A_2 y_n - A_2 y| \leqslant \int_0^1 |G(\mathfrak{r},\sigma)||\nu_n(\sigma) - \nu(\sigma)|d\sigma, \tag{2.6}$$

where $\nu_n, \nu \in C(\mathbb{I}, \mathbb{R})$, such that

$$\nu_n(\mathfrak{r}) = \theta\left(\mathfrak{r}, y_n(\mathfrak{r}), I^{2-\beta;\phi}\nu_n(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\ I^{2-\alpha;\phi}\nu_n(\sigma)d\sigma\right),$$
$$\nu(\mathfrak{r}) = \theta\left(\mathfrak{r}, y_2(\mathfrak{r}), I^{2-\beta;\phi}\nu(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\ I^{2-\alpha;\phi}\nu(\sigma)d\sigma\right),$$

and by (H_2), we have

$$\begin{aligned}|\nu_n(\mathfrak{r}) - \nu(\mathfrak{r})| &= \Big|\theta\big(\mathfrak{r}, y_n(\mathfrak{r}), I^{2-\beta;\phi}\nu_n(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\nu_n(\sigma)d\sigma\big) - \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\beta;\phi}\nu(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\nu(\sigma)d\sigma\big)\Big| \\ &\leqslant \psi(\mathfrak{r})\left(|y_n(\mathfrak{r}) - y(\mathfrak{r})| + \int_0^{\mathfrak{r}} \frac{\phi'(\sigma)(\phi(\mathfrak{r}) - \phi(\sigma))^{2-\beta-1}}{\Gamma(2-\beta)}|\nu_n(\sigma) - \nu(\sigma)|d\sigma\right. \\ &\quad \left. + \int_0^{\mathfrak{r}} |k(\mathfrak{r},\sigma)| \int_0^{\sigma} \frac{\phi'(\eta)(\phi(\sigma) - \phi(\eta))^{2-\alpha-1}}{\Gamma(2-\alpha)}|\nu_n(\eta) - \nu(\eta)|\ d\eta\ d\sigma\right) \\ &\leqslant \|\psi\|\big(\|y_n - y\| + \frac{\|\nu_n - \nu\|(\phi(1) - \phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{K\ \|\nu_n - \nu\|(\phi(1) - \phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\big).\end{aligned}$$

Thus

$$\|\nu_n - \nu\| \leqslant \frac{\|\psi\|}{1 - \|\psi\|\big(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{K\ (\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\big)}\|y_n - y\|.$$

Since $y_n \to y$, then we get $\nu_n(\mathfrak{r}) \to \nu(\mathfrak{r})$ as $n \to \infty$ for each $\mathfrak{r} \in \mathbb{I}$. And let $\varepsilon > 0$ be such that, for each $\mathfrak{r} \in \mathbb{I}$, we have $|\nu_n(\mathfrak{r})| \leqslant \varepsilon$, and $|\nu(\mathfrak{r})| \leqslant \varepsilon$. Then, we have

$$|G(\mathfrak{r},\sigma)||\nu_n(\sigma) - \nu(\sigma)| \leqslant |G(\mathfrak{r},\sigma)|[|\nu_n(\sigma)| + |\nu(\sigma)|] \leqslant 2\varepsilon|G(\mathfrak{r},\sigma)|.$$

For each $\mathfrak{r} \in \mathbb{I}$, the function $\sigma \to 2\varepsilon|G(\mathfrak{r},\sigma)|$ is integrable on $\mathbb{I}$. Then applying Lebesgue dominated convergence theorem, and (2.6) imply that

$$\|A_2 y_n - A_2 y\| \to 0 \ \text{ as } \ n \to \infty.$$

Consequently, A_2 is continuous.

Next, it is easy to verify that

$$\|A_2 y\| \leqslant G_0\ \big(\frac{\Theta + \|\psi\|\ \rho}{1 - \mathfrak{M}}\big)\ \leqslant \rho,$$

due to definitions of $\mathfrak{M}$ and ρ. This proves that A_2 is uniformly bounded on B_ρ.

Finally, we show that A_2 maps bounded sets into equicontinuous sets of $C(\mathbb{I}, \mathbb{R})$, i.e., B_ρ is equicontinuous. Now, $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ and $\mathfrak{r}_1, \mathfrak{r}_2 \in \mathbb{I}$, $\mathfrak{r}_1 < \mathfrak{r}_2$, $|\mathfrak{r}_2 - \mathfrak{r}_1| < \delta$, then we have

$$\begin{aligned}|A_2 y(\mathfrak{r}_2) - A_2 y(\mathfrak{r}_1)| &\leqslant \int_0^1 |G(\mathfrak{r}_2,\sigma) - G(\mathfrak{r}_1,\sigma)|\ |\nu(\sigma)|d\sigma \\ &\leqslant \|\nu\| \int_0^1 |G(\mathfrak{r}_2,\sigma) - G(\mathfrak{r}_1,\sigma)|\ d\sigma \leqslant \frac{\Theta + \|\psi\|\ \rho}{1 - \mathfrak{M}} \int_0^1 |G(\mathfrak{r}_2,\sigma) - G(\mathfrak{r}_1,\sigma)|\ d\sigma.\end{aligned}$$

As $\mathfrak{r}_1 \to \mathfrak{r}_2$, the right-hand side of the above inequality is not dependent on y and tends to zero. Consequently,

$$|A_2y(\mathfrak{r}_2) - A_2y(\mathfrak{r}_1)| \to 0, \quad \forall\, |\mathfrak{r}_2 - \mathfrak{r}_1| \to 0.$$

Thus, $\{Ay\}$ is equi-continuous on B_ρ and A is compact operator by the Arzela-Ascoli Theorem [6], we conclude that $A : C(\mathbb{I}, \mathbb{R}) \to C(\mathbb{I}, \mathbb{R})$ is continuous and compact. Hence, hypotheses of Krasnoselskii's fixed point theorem are hold and shows that $A_1 + A_2$ has a fixed point on B_ρ. Therefore, the ISDP (1.1)-(1.3) has a mild solution, and the proof is complete. □

Our second finding establishes the existence of a unique solution to the ISDP (1.1)-(1.3) using Banach's fixed point theorem.

Theorem 2.5. *Assume that assumptions of Theorem 2.4 hold. If*

$$c + \frac{G_\circ \,\|\psi\|}{1 - \|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{K\,(\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)} < 1, \tag{2.7}$$

then the ISDP (1.1)-(1.3) *has a unique solution on* $\mathbb{I}$.

Proof. It follows, from Theorem 2.4, that ISDP (1.1)-(1.3) has at least one solution. Therefore, we only need to show that the operator A described in (2.4) is a contraction.

Now, we take $x, y \in C(\mathbb{I}, \mathbb{R})$. Then for $\mathfrak{r} \in \mathbb{I}$, we have

$$Ax(\mathfrak{r}) - Ay(\mathfrak{r}) = \mathcal{F}(\mathfrak{r}, x(\mathfrak{r})) + \int_0^1 G(\mathfrak{r},\sigma)u(\sigma)d\sigma - h(\mathfrak{r}, y(\mathfrak{r})) - \int_0^1 G(\mathfrak{r},\sigma)v(\sigma)d\sigma,$$

where $u, v \in C(\mathbb{I}, \mathbb{R})$ be such that

$$u(\mathfrak{r}) = \theta\big(\mathfrak{r}, x(\mathfrak{r}), I^{2-\beta;\phi}u(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\; I^{2-\alpha;\phi}\; u(\sigma)d\sigma\big),$$
$$v(\mathfrak{r}) = \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\beta;\phi}v(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\; I^{2-\alpha;\phi}\; v(\sigma)d\sigma\big).$$

Then, for $\mathfrak{r} \in \mathbb{I}$

$$|Ax(\mathfrak{r}) - Ay(\mathfrak{r})| \leqslant \; |\mathcal{F}(\mathfrak{r}, x(\mathfrak{r})) - \mathcal{F}(\mathfrak{r}, y(\mathfrak{r}))| + \int_0^1 G(\mathfrak{r},\sigma)\; |u(\sigma) - v(\sigma)|d\sigma, \tag{2.8}$$

but by condition (H_2), we have

$$\begin{aligned}
&|u(\mathfrak{r}) - v(\mathfrak{r})| \\
&= |\theta\big(\mathfrak{r}, x(\mathfrak{r}), I^{2-\beta;\phi}u(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\; I^{2-\alpha;\phi}\; u(\sigma)d\sigma\big) - \theta\big(\mathfrak{r}, y(\mathfrak{r}), I^{2-\beta;\phi}\; v(\mathfrak{r}), \int_0^{\mathfrak{r}} k(\mathfrak{r},\sigma)\; I^{2-\alpha;\phi}\; v(\sigma)d\sigma\big)| \\
&\leqslant \psi(\mathfrak{r}) \left(|x(\mathfrak{r}) - y(\mathfrak{r})| + \int_0^{\mathfrak{r}} \frac{\phi'(\sigma)(\phi(\mathfrak{r}) - \phi(\sigma))^{2-\beta-1}}{\Gamma(2-\beta)} |u(\sigma) - v(\sigma)|\; d\sigma \right. \\
&\quad \left. + \int_0^{\mathfrak{r}} |k(\mathfrak{r},\sigma)| \int_0^{\sigma} \frac{\phi'(\eta)(\phi(\sigma) - \phi(\eta))^{2-\alpha-1}}{\Gamma(2-\alpha)} |u(\eta) - v(\eta)|\; d\eta\; d\sigma \right) \\
&\leqslant \|\psi\|\big(\|x - y\| + \frac{(\phi(1) - \phi(0))^{2-\beta}\|u - v\|}{\Gamma(2-\beta+1)} + \frac{K\;\|u - v\|(\phi(1) - \phi(0))^{2-\alpha}}{\Gamma(2-\alpha+1)}\big),
\end{aligned}$$

for $\mathfrak{r} \in \mathbb{I}$, we have

$$\|u - v\| \leqslant \; \|\psi\|\big(\|x - y\| + \frac{(\phi(1) - \phi(0))^{2-\beta}\|u - v\|}{\Gamma(3-\beta)} + \frac{K\;\|u - v\|(\phi(1) - \phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\big).$$

Thus
$$\|u-v\| \leqslant \frac{\|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\,(\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\|x-y\|.$$
Return to (2.8) and by Lemma 2.3, we have
$$\begin{aligned}\|Ax-Ay\| &\leqslant c\|x-y\|+\frac{G_\circ\,\|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\,(\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\|x-y\| \\ &\leqslant \left(c+\frac{G_\circ\,\|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\,(\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\right)\|x-y\|.\end{aligned}$$
By $\left(c+\frac{G_\circ\,\|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\,(\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\right)<1$, then this proves contraction of operator A. Hence, by Banach's contraction principle, A has a unique fixed point which is a mild solution of the ISDP (1.1)-(1.3) on $\mathbb{I}$. □

3. Ulam-Hyers stability

Now, we examine the Ulam stability for ISDP (1.1)-(1.3). Let $\epsilon>0$ and $\Phi:\mathbb{I}\to\mathbb{R}_+$ be a continuous function. Inequalities such as these are taken into account:
$$|\frac{d^2}{dt^2}y(t)-\theta\big(t,y(t),{}^cD^{\beta;\phi}y(t),\int_0^t k(t,\sigma)\,{}^cD^{\alpha;\phi}y(\sigma)d\sigma\big)|\leqslant\epsilon,\quad t\in\mathbb{I}, \tag{3.1}$$
$$|\frac{d^2}{dt^2}y(t)-\theta\big(t,y(t),{}^cD^{\beta;\phi}y(t),\int_0^t k(t,\sigma)\,{}^cD^{\alpha;\phi}y(\sigma)d\sigma\big)|\leqslant\Phi(t),\ t\in\mathbb{I}, \tag{3.2}$$
$$|\frac{d^2}{dt^2}y(t)-\theta\big(t,y(t),{}^cD^{\beta;\phi}y(t),\int_0^t k(t,\sigma)\,{}^cD^{\alpha;\phi}y(\sigma)d\sigma\big)|\leqslant\epsilon\ \Phi(t),\quad t\in I. \tag{3.3}$$

Definition 3.1 ([28])**.** The ISDP (1.1)-(1.3) is Ulam-Hyers stable if there exists a real number $c_f>0$ such that for each $\epsilon>0$ and for each solution $y\in C(\mathbb{I},\mathbb{R})$ of the inequality (3.1) there exists a solution $z\in C(\mathbb{I},\mathbb{R})$ of (1.1)-(1.3) with
$$|y(t)-z(t)|\leqslant\epsilon\,c_f,\quad t\in\mathbb{I}.$$

Definition 3.2 ([28])**.** The problem ISDP (1.1)-(1.3) is generalized Ulam-Hyers stable if there exists $c_f\in C(\mathbb{R}_+,\mathbb{R}_+)$ with $c_f(0)=0$ such that for each $\epsilon>0$ and for each solution $y\in C(\mathbb{I},\mathbb{R})$ of the inequality (3.1) there exists a solution $z\in C(\mathbb{I},\mathbb{R})$ of (1.1)-(1.3) with
$$|y(t)-z(t)|\leqslant c_f(\epsilon),\quad t\in\mathbb{I}.$$

Definition 3.3 ([28])**.** The problem ISDP (1.1)-(1.3) is Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi}>0$ such that for each $\epsilon>0$ and for each solution $y\in C(\mathbb{I},\mathbb{R})$ of the inequality (3.2) there exists a solution $z\in C(\mathbb{I},\mathbb{R})$ of (1.1)-(1.3) with
$$|y(t)-z(t)|\leqslant\epsilon\,c_{f,\Phi}\Phi(t),\quad t\in\mathbb{I}.$$

Definition 3.4 ([28])**.** The problem ISDP (1.1)-(1.3) is generalized Ulam-Hyers-Rassias stable with respect to Φ if there exists a real number $c_{f,\Phi}>0$ such that for each solution $y\in C(\mathbb{I},\mathbb{R})$ of the inequality (3.3) there exists a solution $z\in C(\mathbb{I},\mathbb{R})$ of (1.1)-(1.3) with
$$|y(t)-z(t)>|\leqslant c_{f,\Phi}\Phi(t),\quad t\in\mathbb{I}.$$

Next, we present the following Ulam-Hyers stablity result.

Theorem 3.5. *Assume that the assumptions of Theorem 2.5 hold. Then ISDP (1.1)-(1.3) is Ulam-Hyers stable.*

Proof. Let $\epsilon > 0$ and let $z \in C(\mathbb{I}, \mathbb{R})$ be a function that meets inequality (3.1), i.e.,

$$|\frac{d^2}{d\tau^2}z(\tau) - \theta\big(\tau, z(\tau), ^c D^{\beta;\phi} z(\tau), \int_0^\tau k(\tau,\sigma)^c D^{\alpha;\phi} z(\sigma) ds\big)| \leqslant \epsilon, \quad \tau \in \mathbb{I},$$

and allow a unique solution of ISDP (1.1)-(1.3) be $y \in C(\mathbb{I}, \mathbb{R})$, which is by Lemma 2.2, the ISDP (1.1)-(1.3) equivalence to fractional order integral equation

$$y(\tau) = \mathcal{F}(\tau, y(\tau)) + \int_0^1 G(\tau,\sigma) u(\sigma) d\sigma,$$

where u is the solution of the functional integral equation

$$u(\tau) = \theta\big(\tau, \mathcal{F}(\tau, y(\tau)) + \int_0^1 G(\tau,\sigma)u(\sigma)d\sigma, I^{2-\beta;\phi} u(\tau), \int_0^\tau k(\tau,\sigma)\ I^{2-\alpha;\phi}\ u(\sigma) d\sigma\big).$$

Operating by I^2 on both sides of (3.1), and then integrating, we get

$$\left| z(\tau) - \mathcal{F}(\tau, z(\tau)) - \int_0^1 G(\tau,\sigma) v(\sigma) d\sigma \right| \leqslant \frac{\epsilon}{2},$$

where v is the solution of the functional integral equation

$$v(\tau) = \theta\big(\tau, \mathcal{F}(\tau, z(\tau)) + \int_0^1 G(\tau,\sigma)v(\sigma)d\sigma, I^{2-\beta;\phi} v(\tau), \int_0^\tau k(\tau,\sigma)\ I^{2-\alpha;\phi}\ v(\sigma) d\sigma\big).$$

For each $\tau \in \mathbb{I}$, we have

$$\begin{aligned}
|z(\tau) - y(\tau)| &= \big|z(\tau) - \mathcal{F}(\tau, y(\tau) - \int_0^1 G(\tau,\sigma)u(\sigma)d\sigma\big| \\
&\leqslant \big|z(\tau) - \mathcal{F}(\tau, z(\tau) + \int_0^1 G(\tau,\sigma)v(\sigma)d\sigma\big| \\
&\quad + \big|\mathcal{F}(\tau, z(\tau) + \int_0^1 G(\tau,\sigma)v(\sigma)ds - \mathcal{F}(\tau, y(\tau) - \int_0^1 G(\tau,\sigma)u(\sigma)d\sigma\big| \\
&\leqslant \frac{\epsilon}{2} + \big|\mathcal{F}(\tau, z(\tau)) - \mathcal{F}(\tau, y(\tau))\big| + \int_0^1 G(\tau,\sigma)|v(\sigma) - u(\sigma)|d\sigma \\
&\leqslant \frac{\epsilon}{2} + c\ |z(\tau) - y(\tau)| + \int_0^1 G(\tau,\sigma)|v(\sigma) - u(\sigma)|d\sigma \\
&\leqslant \frac{\epsilon}{2} + c\ \|z - y\| + G_\circ \|u - v\|\ .
\end{aligned}$$

In fact, the proof of Theorem 2.5 provides us with

$$\|u - v\| \leqslant \frac{\|\psi\|}{1 - \|\psi\|\big(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{K\ (\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\big)}\|z - y\|.$$

Then, for each $\tau \in \mathbb{I}$,

$$\|z - y\| \leqslant \frac{\epsilon}{2} + c\ \|z - y\| + \frac{G_\circ\ \|\psi\|}{1 - \|\psi\|\big(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{K\ (\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\big)}\|z - y\|.$$

Thus

$$\|z-y\| \leqslant \frac{\epsilon}{2}\left[1-\Big(c+\frac{G_\circ\ \|\psi\|}{1-\|\psi\|\big(\frac{(\varphi(1)-\varphi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\ (\varphi(1)-\varphi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\big)}\Big)\right]^{-1}=\sigma\ \epsilon,$$

Let $\sigma=\left[1-\Big(c\ +\frac{G_\circ\ \|\psi\|}{1-\|\psi\|\big(\frac{(\varphi(1)-\varphi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\ (\varphi(1)-\varphi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\big)}\Big)\right]^{-1}$. So, the ISDP (1.1)-(1.3) is Ulam-Hyers stable. □

It is revealed that the ISDP (1.1)-(1.3) is generalized Ulam-Hyers stable by setting

$$\Phi(\epsilon)=\sigma\ \epsilon,\ \ \Phi(0)=0.$$

3.1. Ulam-Hyers-Rassias stability.

Now, we state the following Ulam-Hyers-Rassias stable result under the following assumption.

(H_4) The function $\Theta \in C(\mathbb{I},\mathbb{R}_+)$ is increasing and there exists $\lambda_\Phi > 0$ such that, for each $\mathfrak{r}\in\mathbb{I}$, we have

$$I^2\ \Theta(\mathfrak{r}) \leqslant \lambda_\Theta\ \Theta(\mathfrak{r}).$$

Theorem 3.6. *Assume that assumptions* (H_1)-(H_4) *and are satisfied. Then ISDP* (1.1)-(1.3) *is Ulam-Hyers-Rassias stable with respect to* Θ.

Proof. Let $z \in C(\mathbb{I},\mathbb{R})$ be a solution of the inequation (3.3), i.e.,

$$\Big|\frac{d^2}{d\mathfrak{r}^2}z(\mathfrak{r})-f\big(\mathfrak{r},z(\mathfrak{r}),{}^cD^{\beta;\varphi}z(\mathfrak{r}),\int_0^{\mathfrak{r}}k(\mathfrak{r},\sigma)\,{}^cD^{\alpha;\varphi}z(\sigma)d\sigma\big)\Big| \leqslant \epsilon\ \Theta,\quad \mathfrak{r}\in\mathbb{I},$$

and take y as a solution of the problem (1.1)-(1.3). Consequently, we get

$$y(\mathfrak{r})=\digamma(\mathfrak{r},y(\mathfrak{r}))+\int_0^1 G(\mathfrak{r},\sigma)u(\sigma)d\sigma,$$

where $u\in C(\mathbb{I},\mathbb{R})$ such that

$$u(\mathfrak{r})=\theta\big(\mathfrak{r},x(\mathfrak{r}),I^{2-\beta;\varphi}u(\mathfrak{r}),\int_0^{\mathfrak{r}}k(\mathfrak{r},\sigma)\ I^{2-\alpha;\varphi}\ u(\sigma)d\sigma\big).$$

Using I^2 on both sides of the inequality (3.3) and integrating after that, we obtain

$$\left|z(\mathfrak{r})-\digamma(\mathfrak{r},z(\mathfrak{r}))-\int_0^1 G(\mathfrak{r},\sigma)v(\sigma)d\sigma\right| \leqslant \frac{\epsilon}{2},$$

where $v\in C(\mathbb{I},\mathbb{R})$ with

$$v(\mathfrak{r})=\theta\big(\mathfrak{r},y(\mathfrak{r}),I^{2-\beta;\varphi}v(\mathfrak{r}),\int_0^{\mathfrak{r}}k(\mathfrak{r},\sigma)\ I^{2-\alpha;\varphi}\ v(\sigma)d\sigma\big).$$

For each $\mathfrak{r}\in\mathbb{I}$, we obtain

$$\begin{aligned}|z(\mathfrak{r})-y(\mathfrak{r})| &= \Big|z(\mathfrak{r})-\digamma(\mathfrak{r},y(\mathfrak{r}))+\int_0^1 G(\mathfrak{r},\sigma)u(\sigma)d\sigma\Big|\\ &\leqslant \Big|z(\mathfrak{r})-\digamma(\mathfrak{r},z(\mathfrak{r}))+\int_0^1 G(\mathfrak{r},\sigma)v(\sigma)d\sigma\Big|\end{aligned}$$

$$+\left|\mathcal{F}(\mathfrak{r},z(\mathfrak{r}))+\int_0^1 G(\mathfrak{r},\sigma)\nu(\sigma)ds-\mathcal{F}(\mathfrak{r},y(\mathfrak{r}))-\int_0^1 G(\mathfrak{r},\sigma)u(\sigma)d\sigma\right|$$
$$\leqslant \frac{\epsilon}{2}+\left|\mathcal{F}(\mathfrak{r},z(\mathfrak{r}))-\mathcal{F}(\mathfrak{r},y(\mathfrak{r}))\right|+\int_0^1 G(\mathfrak{r},\sigma)|\nu(\sigma)-u(\sigma)|d\sigma$$
$$\leqslant \frac{\epsilon}{2}+c\ |z(\mathfrak{r}))-y(\mathfrak{r})|+\int_0^1 G(\mathfrak{r},\sigma)|\nu(\sigma)-u(\sigma)|d\sigma$$
$$\leqslant \frac{\epsilon}{2}+c\ \|z-y\|+G_\circ\|\nu-u\|.$$

Indeed, from proof of Theorem 2.5, we have

$$\|u-\nu\|\leqslant\frac{\|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\ (\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\|z-y\|.$$

Then, for each $\mathfrak{r}\in\mathbb{I}$,

$$\|z-y\|\leqslant\frac{\epsilon}{2}\ +c\ \|z-y\|+\frac{G_\circ\ \|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\ (\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\|z-y\|.$$

Thus

$$\|z-y\|\leqslant\left[1-\left(c+\frac{G_\circ\ \|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\ (\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\right)\right]\ \epsilon\ \lambda_\Theta\ \Theta(t)=c_\Theta\epsilon\ \Theta(t),$$

where

$$c_\Theta=\left[1-\left(c+\frac{G_\circ\ \|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)}+\frac{K\ (\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)}\right)\right]\ \lambda_\Theta.$$

So, the ISDP (1.1)-(1.3) is Ulam-Hyers-Rassias stable with respect to Θ. □

4. Discussion and examples

As special examples of our main result, we present certain existence results for some boundary value problems in this section.

• Setting $\Upsilon_1(\sigma)=\frac{\phi'(\sigma)(\phi(\eta)-\phi(\sigma))^\lambda}{\Gamma(\lambda+1)}$, $\Upsilon_2(\sigma)=\frac{\phi'(\sigma)(\phi(\eta)-\phi(\sigma))^\lambda}{\Gamma(\lambda+1)}$, then using Riemann-Stieltjes integrals, we construct implicit ϕ-Caputo fractional differential equation with nonlocal boundary conditions:

$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r})=\theta\left(\mathfrak{r},y(\mathfrak{r}),{}^cD^{\beta;\phi}y(\mathfrak{r}),\int_0^{\mathfrak{r}}k(\mathfrak{r},\sigma){}^cD^{\alpha;\phi}y(\sigma)d\sigma\right),\quad \mathfrak{r}\in(0,1).$$

Given the boundary conditions

$$y(0)=\int_0^1\sigma_1(\sigma,y(\sigma))d\Upsilon_1(\sigma),\qquad y'(1)=\int_0^1\ \sigma_2(\sigma,y(\sigma))\ d\Upsilon_2(\sigma).$$

Many studies, for example [31], have investigated into this form of such boundary conditions.

• Letting $\gamma\to1$, $\phi(t)=t$, $\theta(\mathfrak{r},y(\mathfrak{r}),u(\mathfrak{r}),\nu(\mathfrak{r}))=-\delta\ m(\mathfrak{r})\theta(y(\mathfrak{r}))$, $\mathcal{F}_1(\sigma,y(\sigma))=\mathcal{F}_2(\sigma,y(\sigma))=y(\sigma)$, we get a nonlocal value problem including integral condition

$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r})+\delta\ m(\mathfrak{r})\theta(y(\mathfrak{r}))=0,\quad \mathfrak{r}\in(0,1),\quad y(0)=\int_0^1\ y(\sigma)d\sigma \text{ and }\ y'(1)=\int_0^1\ y(\sigma)d\sigma,$$

which this kind of problem is investigated in [15]. Furthermore, some examples are provided to support the main findings.

Example 4.1. Consider the following ISDP:

$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) = \frac{e^{-\mathfrak{r}}}{e^{\mathfrak{r}}+8}\left(\frac{|y(\mathfrak{r})|}{1+|y(\mathfrak{r})|} - \frac{|{}^cD^{\frac{5}{4};\ln(1+\mathfrak{r})}y(\mathfrak{r})|}{1+|{}^cD^{\frac{5}{4};\ln(1+\mathfrak{r})}y(\mathfrak{r})|} - \frac{|\int_0^1 \ln(\mathfrak{r}+\sigma)\,{}^cD^{\frac{5}{4};\ln(1+\mathfrak{r})}y(\mathfrak{r})d\sigma|}{1+|\int_0^1 \ln(\mathfrak{r}+\sigma)\,{}^cD^{\frac{5}{4};\ln(1+\mathfrak{r})}y(\mathfrak{r})d\sigma|}\right), \tag{4.1}$$

$$y(0) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (\ln(2)-\ln(1+\sigma))^{\frac{1}{2}}\,\frac{\sin y(\sigma)}{20(1+\sigma)}\,d\sigma, \tag{4.2}$$

$$y'(1) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (\ln(2)-\ln(1+\sigma))^{\frac{1}{2}}\,\frac{e^{-y(\sigma)}}{30(1+\sigma)}\,d\sigma. \tag{4.3}$$

Set

$$\theta(\mathfrak{r},u,v,w) = \frac{e^{-\mathfrak{r}}}{e^{\mathfrak{r}}+8}\left(\frac{|u(\mathfrak{r})|}{1+|u(\mathfrak{r})|} - \frac{|v(\mathfrak{r})|}{1+|v(\mathfrak{r})|} - \frac{|w(\mathfrak{r})|}{1+|w(\mathfrak{r})|}\right).$$

Obviously, the function θ is mutually continuous. Currently, for any $u_1.v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$ and $\mathfrak{r} \in [0,1]$,

$$\begin{aligned}|\theta(\mathfrak{r},u_1,v_1,w_1)-\theta(\mathfrak{r},u_2,v_2,w_2)| &\leqslant \frac{e^{-\mathfrak{r}}}{e^{\mathfrak{r}}+8}\big(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|\big)\\ &\leqslant \frac{1}{9}\big(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|\big).\end{aligned}$$

Hence the condition (H_2) holds with $\|\psi\| = \frac{1}{9}$. On the other hand, we have

$$\begin{aligned}|\digamma(\mathfrak{r},x(\mathfrak{r}))-\digamma(\mathfrak{r},y(\mathfrak{r}))| \leqslant{}& \frac{1}{1.772453}\int_0^1 (\ln(2)-\ln(1+\sigma))^{\frac{1}{2}}\frac{|\sin x(\sigma)-\sin y(\sigma)|}{20(1+\sigma)}\,d\sigma\\ &+\frac{1}{1.772453}\int_0^1 (\ln(2)-\ln(1+\sigma))^{\frac{1}{2}}\,\frac{|e^{-x(\sigma)}-e^{-y(\sigma)}|}{30(1+\sigma)}\,d\sigma.\end{aligned}$$

Also, we meet Lemma 2.3, then $\digamma$ is lipschitz with constant $c = 0.031348808$.

We shall check condition 2.7.

$$c + \frac{G_\circ\,\|\psi\|}{1-\|\psi\|\left(\frac{(\phi(1)-\phi(0))^{2-\beta}}{\Gamma(3-\beta)} + \frac{K\,(\phi(1)-\phi(0))^{2-\alpha}}{\Gamma(3-\alpha)}\right)} \simeq 0.4297118717 < 1.$$

Indeed $\alpha = \frac{4}{3}$, $\beta = \frac{5}{4}$, $c = 0.03134$, $\|\psi\| = \frac{1}{9}$, $K = \ln(2)$ and $G_0 < 3$. Theorem 2.5 guarantees a unique mild solution on $\mathbb{I}$ for ISDP (4.1)-(4.3).

Example 4.2. Consider the following ISDP:

$$\frac{d^2}{d\mathfrak{r}^2}y(\mathfrak{r}) = \frac{2+|y(\mathfrak{r})|+|{}^cD^{\frac{4}{3};\mathfrak{r}}y(\mathfrak{r})|+|\int_0^1 e^{\mathfrak{r}-\sigma}\,{}^cD^{\frac{3}{2};\mathfrak{r}}y(\sigma)d\sigma|}{2e^{\mathfrak{r}+1}\big(1+|y(\mathfrak{r})|+|{}^cD^{\frac{4}{3};\mathfrak{r}}y(\mathfrak{r})|+|\int_0^1 e^{\mathfrak{r}-\sigma}\,{}^cD^{\frac{3}{2};\mathfrak{r}}y(\sigma)d\sigma|\big)}, \qquad \mathfrak{r}\in[0,1], \tag{4.4}$$

$$y(0) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-\sigma)^{\frac{1}{2}}\,\frac{y}{20e^{-\sigma+2}(1+y)}\,d\sigma, \tag{4.5}$$

$$y'(1) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-\sigma)^{\frac{1}{2}}\,\frac{\cos y}{30(\sigma+2)}\,d\sigma. \tag{4.6}$$

Set

$$\theta(\mathfrak{r},u,v,w) = \frac{2+|u|+|v|+|w|}{2e^{\mathfrak{r}+1}\big(1+|u|+|v|+|w|\big)}.$$

The function θ is clearly jointly continuous, for any $u_1.v_1, w_1, u_2.v_2, w_2 \in R$ and $\mathfrak{t} \in [0,1]$,

$$|\theta(\mathfrak{t}, u_1, v_1, w_1) - \theta(\mathfrak{t}, u_2, v_2, w_2)| \leqslant \frac{1}{2e}\big(|u_1 - u_2| + |v_1 - v_2| + |w_1 - w_2|\big).$$

Therefore, condition (H_2) is hold with $\psi(\mathfrak{t}) = \frac{1}{2e^{\mathfrak{t}+1}}$. Moreover, we have

$$|\theta(\mathfrak{t}, u, v, w)| = \frac{1}{2e^{\mathfrak{t}+1}}\big(2 + |u| + |v| + |w|\big),$$

where $\theta(\mathfrak{t}, 0, 0, 0) = \frac{1}{e^{\mathfrak{t}+1}}$, and $\|\psi\| = \frac{1}{2e}$. Setting $F_1(\mathfrak{t}, x(\mathfrak{t})) = \frac{\cos x}{30(\mathfrak{t}+2)}$ and $F_2(\mathfrak{t}, x(\mathfrak{t})) = \frac{x}{20e^{-\mathfrak{t}+2}(1+x)}$,

$$|F_1(\mathfrak{t}, x(\mathfrak{t})) - F_1(\mathfrak{t}, y(\mathfrak{t}))| \leqslant \Big|\frac{\cos x(\mathfrak{t})}{30(\mathfrak{t}+2)} - \frac{\cos y(\mathfrak{t})}{30(\mathfrak{t}+2)}\Big| \leqslant \frac{1}{30}\,|x(\mathfrak{t}) - y(\mathfrak{t})|,$$

and

$$|F_2(\mathfrak{t}, x(\mathfrak{t})) - F_2(\mathfrak{t}, y(\mathfrak{t}))| \leqslant \Big|\frac{x(\mathfrak{t})}{20e^{-\mathfrak{t}+2}(1+x)} - \frac{y(\mathfrak{t})}{20e^{-\mathfrak{t}+2}(1+y)}\Big| \leqslant \frac{1}{20e}\,|x(\mathfrak{t}) - y(\mathfrak{t})|.$$

Hence, the condition (H_1) is satisfied with $k_1 = \frac{1}{30}$ and $k_2 = \frac{1}{20e}$. From (2.1) clearly for $\alpha = \frac{3}{2}$, then $G_0 < 1$. Thus condition

$$\frac{k_1 + \mathfrak{t}k_2}{\Gamma(\gamma+1)} + \frac{G_\circ\,\|\psi\|}{1 - \mathfrak{M}} \simeq 0.4103216862 < 1,$$

where $\mathfrak{M} = \frac{\|\psi\|}{\Gamma(3-\beta)} + \frac{\|\psi\|K}{\Gamma(3-\alpha)} = 0.2568803025$ is satisfied with $\alpha = \frac{3}{2}$, $\beta = \frac{4}{3}$, $\Theta = \frac{1}{e}$, $\|\psi\| = \frac{1}{2e}$ $k_1 = \frac{1}{30}$, $k_2 = \frac{1}{20e}$, and $K = e$. It follows that from Theorem 2.4, the ISDP (4.4)-(4.6) has at least one mild solution on $\mathbb{I}$.

5. Conclusion

Our purpose in this paper is to study the existence and uniqueness of mild solutions for boundary value problems of implicit second-order differential equation (1.1)-(1.3) based on Krasnoselskii's fixed point theorem and Banach contraction principle. In addition, stability analysis in the Ulam-Hyers sense of a given implicit differential equation of ϕ-fractional order, supplemented with fractional integral type boundary conditions was considered. Finally, we end the article with illustrations were provided to confirm the results applicability.

References

[1] S. Abbas, M. Benchohra, *On the generalized Ulam-Hyers-Rassias stability for Darboux problem for partial fractional implicit differential equations*, Appl. Math. E-Notes, **14** (2014), 20–28. 1

[2] S. Abbas, M. Benchohra, G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, **27** (2012). 1

[3] O. P. Agrawal, *Some generalized fractional calculus operators and their applications in integral equations*, Fract. Calc. Appl. Anal., **15** (2012), 700–711. 1

[4] R. P. Agarwal, S. Arshad, D. O'Regan, V. Lupulescu, *Fuzzy fractional integral equations under compactness type condition*, Fract. Calc. Appl. Anal., **15** (2012), 572–590. 1

[5] R. P Agarwal, M. Benchohra, S. Hamani, *Boundary value problems for fractional differential equations*, Georg. Math. J., **16** (2009), 401-411. 1

[6] Sh. M. Al-Issa, A. M. A. El-Sayed, Y. M. Y. Omar, *On some characterizations of the solutions of a hybrid nonlinear functional integral inclusion and applications*, J. Math. Comput. Sci., **28** (2023), 21–36. 2.1

[7] Sh. M. Al-Issa, N. M. Mawed, *Results on solvability of nonlinear quadratic integral equations of fractional orders in Banach algebra*, J. Nonlinear Sci. Appl., **14** (2021), 181–195. 1

[8] R. Almeida, *A Caputo fractional derivative of a function with respect to another function*, Commun. Nonlinear Sci. Numer. Simul., **44** (2017), 460–481. 1, 1.1, 1.2

[9] T. A. Burton, C. Kirk, *A fixed point theorem of Krasnoselskii Schaefer type*, Math. Nachr., **189** (1998), 23–31. 2.1

[10] A. Babakhani, V. Daftardar-Gejji, *Existence of positive solutions for multi-term non-autonomous fractional differential equations with polynomial coefficients*, Electron. J. Differ. Equ., **2006** (2006), 1–12. 1

[11] M. Belmekki, M. Benchohra, *Existence results for fractional order semilinear functional differential equations*, Proc. A. Razmadze Math. Inst., **146** (2008), 9–20. 1

[12] M. Benchohra, J. R. Graef, S. Hamani, *Existence results for boundary value problems with nonlinear fractional differential equations*, Appl. Anal., **87** (2008), 851–863. 1

[13] M. Benchohra, S. Hamani, S. K. Ntouyas, *Boundary value problems for differential equations with fractional order*, Surv. Math. Appl., **3** (2008), 1–12.

[14] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay*, J. Math. Anal. Appl. **338** (2008), 1340–1350. 1

[15] S. Chasreechai, J. Tariboon, *Positive solutions to generalized second-order three-point integral boundary-value problems*, Electron. J. Differ. Equ. **2011** (2011), 14 pages. 1, 4

[16] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, New York, (2012). 1

[17] A. M. A. El-Sayed, *Nonlinear functional differential equations of arbitrary orders*, Nonlinear Anal., **33** (1998), 181–186. 1

[18] A. M. A. El-Sayed, Sh. M Al-Issa, *Existence of integrable solutions for integro-differential inclusions of fractional order; coupled system approach*, J. Nonlinear Sci. Appl., **13** (2020), 180–186.

[19] A. M. A. El-Sayed, Sh. M. Al-Issa, M. H. Hijazi, *Existence results for a functional integro-differential inclusions with Riemann-Stieltjes integral or infinite-point boundary conditions*, Surv. Math. Appl., **16** (2021), 301–325.

[20] A. El-Sayed, H. Hashem, Sh. Al-Issa, *Analysis of a hybrid integro-differential inclusion*, Bound. Value Probl., **2022** (2022), 1–21. 1

[21] M. Hu, L. Wang, *Existence of solutions for a nonlinear fractional differential equation with integral boundary condition*, Int. J. Math. Comput. Sci., **5** (2011), 55–58. 1

[22] S.-M. Jung, K.-S. Lee, *Hyers-Ulam stability of first order linear partial differential equations with constant coeficients*, Math. Inequal. Appl., **10** (2007), 261–266. 1

[23] A. A. Kilbas, S. A. Marzan, *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Differ. Equ., **41** (2005), 84–89. 1

[24] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, (2006). 1

[25] V. Kiryakova, *Generalized Fractional Calculus and Applications*, J. Wiley & Sons Inc., New York, (1994). 1

[26] V. Lakshmikantham, S. Leela, J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, (2009). 1

[27] S. A. Murad, S. Hadid, *An existence and uniqueness theorem for fractional differential equation with integral boundary condition*, J. Frac. Calc. Appl., **3** (2012), 1–9. 1

[28] M. Obłoza, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat. **13** (1993), 259–270. 1, 3.1, 3.2, 3.3, 3.4

[29] I. Podlubny, A. M. A. EL-Sayed, *On two defintions of fractional calculus*, Preprint UEF 03-69, Solvak Academy of science-Institute of Experimental phys., (1996). 1

[30] S. G. Samko, A. A. Kilbas, O. I. Mariche, *Fractional integrals and derivatives*, Gordon and Breach Science Publishers, Yverdon, (1993). 1

[31] J. R. L. Webb, G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc., **74** (2006), 673–693. 4