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The soft generalized closure operator and the soft topology it generates

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Abstract

In this paper, we use soft g-closed subsets of a soft topological space (M, λ, B) to define a new soft closure operator and, thus, a new soft topology $\lambda^{\times}$ on M relative to B. We show that $\lambda^{\times}$ contains the class of soft g-open sets, and thus $\lambda^{\times}$ contains λ. We also show that $\lambda^{\times} = \lambda$ if and only if (M, λ, B) is soft $T_{1/2}$. Furthermore, we show that $(M, \lambda^{\times}, B)$ is always soft $T_{1/2}$, and as a result, $(\lambda^{\times})^{\times} = \lambda^{\times}$; and we give conditions equivalent to the soft discretness of $(M, \lambda^{\times}, B)$. Furthermore, with emphasis on the transfer of "soft regularity" conditions on (M, λ, B) to "soft separation" conditions on $(M, \lambda^{\times}, B)$. We have also demonstrated by examples that each of soft compactness, soft connectedness, and soft second countability of (M, λ, B) does not transfer to $(M, \lambda^{\times}, B)$ in general. In addition to these, we provide new properties and characterizations of the well-known concept of "soft g-continuity". Finally, we investigate the correspondences between the novel soft topological concepts and their general topological analogs.

Keywords: Soft g-closed sets, soft regularity, soft $T_{1/2}$, soft g-continuity.

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1. Introduction and Preliminaries

The application of classical mathematical concepts to various domains, including engineering, the environment, economics, medical science, and social science, presents particular difficulties. Methods for managing uncertainty include fuzzy sets, rough sets, intuitionistic fuzzy sets, and vague sets [16, 22, 32, 35]. According to Molodtsov [29], each of these methods has its own problems. The limitations of the parameterization tool are mostly to blame for these issues. To address these problems and deal with ambiguity, Molodtsov [29] proposed soft sets. Several authors have examined and explored the ideas of soft sets (see [24, 28]). Soft sets were employed by the authors [29, 30] in a variety of disciplines, including operation research, game theory, smoothness of function, probability, and measurement theory.

Many researchers have studied many mathematical structures using soft set theory. Soft topology is one of the significant expansions of classical topology, according to Shabir and Naz [34]. There is still space for significant contributions despite the fact that many traditional topological ideas, like generalized open sets, separation axioms, covering characteristics, etc. [1–6, 9–14, 18, 31] have been extended and advanced in soft set settings. Thus, a current trend among topological researchers is the study of soft topology.

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Levine [26] presented generalized closed (g-closed) sets in a topological space to extend many of the fundamental properties of closed sets to a larger family. For example, it has been demonstrated that g-closed subsets inherit compactness, normality, and completeness in a uniform space. Kannan [23], presented soft g-closed sets. The soft g-closed sets were used in research papers by several authors.

In this paper, we use soft g-closed subsets of a soft topological space (M, λ, B) to define a new soft closure operator and, thus, a new soft topology $\lambda^{\times}$ on M relative to B. We show that $\lambda^{\times}$ contains the class of soft g-open sets, and thus $\lambda^{\times}$ contains λ. We also show that $\lambda^{\times} = \lambda$ if and only if (M, λ, B) is soft $T_{1/2}$. Furthermore, we show that $(M, \lambda^{\times}, B)$ is always soft $T_{1/2}$, and as a result, $\left(\lambda^{\times}\right)^{\times} = \lambda^{\times}$; and we give conditions equivalent to the soft discreetness of $(M, \lambda^{\times}, B)$. Furthermore, with emphasis on the transfer of "soft regularity" conditions on (M, λ, B) to "soft separation" conditions on $(M, \lambda^{\times}, B)$. We have also demonstrated by examples that each of the soft compactness, soft connectedness, and soft second countability of (M, λ, B) does not transfer to $(M, \lambda^{\times}, B)$ in general. In addition to these, we provide new properties and characterizations of the well-known concept of "soft g-continuity." Finally, we investigate the correspondences between the novel soft topological concepts and their general topological analogs.

In this paper, topological space and soft topological space (STS and TS, respectively) will be used alternatively. Throughout this paper, concepts and terminology from [7, 8] will be used.

Let (M, λ, B) be an STS and (M, μ) be a TS. Let $H \in SS(M, B)$ and $U \subseteq M$. Throughout this paper, λ^c will denote the collection of all soft closed sets of (M, λ, B), and μ^c will denote the collection of all closed sets of (M, μ), with $Cl_{\lambda}(H)$ and $Cl_{\mu}(U)$ denoting the soft closure of H in (M, λ, B) and the closure of U in (M, μ).

Now we recall some preliminaries that will be used in the sequel.

Definition 1.1 ([26]). Let (M, μ) be a TS, and let $X \subseteq M$. Then

(a) X is said to be a generalized closed (briefly: g-closed) set in (M, μ) if $Cl_{\mu}(X) \subseteq U$ whenever $U \in \mu$ and $X \subseteq U$, the collection of all g-closed sets in (M, μ) will be denoted by $GC(M, \mu)$.

(b) X is said to be a generalized open (briefly: g-open) set in (M, μ) if $M - X \in GC(M, \mu)$, the collection of all g-open sets in (M, μ) will be denoted by $GO(M, \mu)$.

Definition 1.2 ([21]). Let (M, μ) be a STS and let $X \subseteq M$. The generalized closure of X in $X \subseteq M$ is denoted by $Cl^{g}_{\mu}(X)$ and defined as follows:

$$Cl^{g}_{\mu}(X) = \widetilde{\cap}\{Y : X \subseteq Y \in GC(M, \mu)\}.$$

Theorem 1.3 ([21]). *For any STS (M, μ), denote the family $\left\{X \in SS(M, B) : Cl^{g}_{\lambda}(M - X) = M - X\right\}$ forms a topology on M. This topology will be denoted by $\mu^{\times}$.*

Definition 1.4 ([19]). A function $p : (M, \mu) \longrightarrow (N, \delta)$ between TSs (M, μ) and (N, δ) is called g-continuous if $p^{-1}(U) \in GC(M, \mu)$ for every $U \in \delta^{c}$.

Definition 1.5 ([23]). Let (M, λ, B) be a STS and let $H \in SS(M, B)$. Then, H is called a soft generalized closed (briefly: soft g-closed) set in (M, λ, B) if $Cl_{\lambda}(H) \widetilde{\subseteq} K$ whenever $K \in \lambda$ and $H \widetilde{\subseteq} K$. The collection of all soft g-closed sets in (M, λ, B) will be denoted by $GC(M, \lambda, B)$. Soft complements of soft g-closed sets in (M, λ, B) are called soft g-open sets (M, λ, B). The collection of all soft g-open sets in (M, λ, B) will be denoted by $GO(M, \lambda, B)$.

Definition 1.6. A STS (M, λ, B) is called

(1) soft compact if for every $\mathcal{A} \subseteq \geqslant$ such that $\widetilde{\cup}_{A \in \mathcal{A}} A = 1_B$, there exists a finite subcollection $\mathcal{A}_1 \subseteq \mathcal{A}$ such that $\widetilde{\cup}_{A \in \mathcal{A}_1} = 1_B$ [17];

(2) soft connected if there does not exist $F, G \in \lambda - \{0_B, 1_B\}$ such that $F \widetilde{\cap} G = 0_B$ and $F \widetilde{\cup} G = 1_B$ [27];

(3) soft second countable if it has a countable soft base [33];

(4) soft $T_{1/2}$ if $GC(M, \lambda, B) \subseteq \lambda^c$ [23];

(5) soft regular if whenever $G \in \lambda^c$ and $b_t \tilde{\in} 1_B - G$, then there exists $L, N \in \lambda$ such that $b_t \tilde{\in} L$, $G \tilde{\subseteq} N$, and $L \tilde{\cap} N = 0_B$ [20];
(6) soft T_1 if for any two soft points $b_m, d_n \in SP(M, B)$ with $b_m \neq d_n$, there exist $G, F \in \lambda$ such that $b_m \tilde{\in} G - F$ and $d_n \tilde{\in} F - G$ [20];
(7) soft T_0 if for any two soft points $b_m, d_n \in SP(M, B)$ with $b_m \neq d_n$, there exists $G \in \lambda$ such that $b_m \tilde{\in} G$, $d_n \tilde{\notin} G$ or $b_m \tilde{\notin} G$, $d_n \tilde{\in} G$ [20];
(8) soft T_3 if it is soft soft regular and soft T_1 [20];
(9) soft R_0 if for every $b_m \in SP(M, B)$ and every $G \in \lambda$ such that $b_m \tilde{\in} G$, we have $Cl_\lambda (b_m) \tilde{\subseteq} G$ [25].

Definition 1.7 ([15])**.** A soft function $f_{pu} : (M, \lambda, B) \longrightarrow (N, \gamma, D)$ $GC(M, \lambda, B)$ is called soft g-continuous if $f_{pu}^{-1}(H) \in GC(M, \lambda, B)$ for every $H \in \gamma^c$.

2. The soft generalized closure operator

In this section, we use soft g-closed subsets of a STS to define a new soft closure operator and, thus, a new soft topology finer than the original soft topology. Furthermore, with emphasis on the transfer of "soft regularity" conditions on the original STS to "soft separation" conditions on the new STS.

Definition 2.1. Let (M, λ, B) be a STS and let $H \in SS(M, B)$. The soft generalized closure of H in (M, λ, B) is denoted by $Cl_\lambda^g(H)$ and defined as follows:

$$Cl_\lambda^g(H) = \tilde{\cap} \left\{ K : H \tilde{\subseteq} K \in GC(M, \lambda, B) \right\}.$$

Theorem 2.2. *Let (M, λ, B) be a STS and let $H \in SS(M, B)$. Then $b_m \tilde{\in} Cl_\lambda^g(H)$ if and only if for any $K \in GO(M, \lambda, B)$ with $b_m \tilde{\in} K$, $K \tilde{\cap} H \neq 0_B$.*

Proof.

Necessity. Suppose that $b_m \tilde{\in} Cl_\lambda^g(H)$ and suppose to the contrary that there exists $K \in GO(M, \lambda, B)$ such that $b_m \tilde{\in} K$ and $K \tilde{\cap} H = 0_B$. Then we have $H \tilde{\subseteq} 1_B - K \in GC(M, \lambda, B)$ and so, $Cl_\lambda^g(H) \tilde{\subseteq} 1_B - K$. Since $b_m \tilde{\in} Cl_\lambda^g(H)$, then $b_m \tilde{\in} 1_B - K$. But $b_m \tilde{\in} K$, a contradiction.

Sufficiency. Suppose that for every $K \in GO(M, \lambda, B)$ with $b_m \tilde{\in} K$, $K \tilde{\cap} H \neq 0_B$. Suppose to the contrary that $b_m \tilde{\notin} Cl_\lambda^g(H)$. Then there exists $S \in GC(M, \lambda, B)$ such that $H \tilde{\subseteq} S$ and $b_m \tilde{\in} 1_B - S \in GO(M, \lambda, B)$. By assumption we must have $(1_B - S) \tilde{\cap} H \neq 0_B$. But $H \tilde{\subseteq} S$, a contradiction. □

Theorem 2.3. *Let $\{(M, \mu_b) : b \in B\}$ be an indexed family of TSs. Let $H \in SS(M, B)$. Then for every $b \in B$,*

$$\left(Cl^g_{\oplus_{b \in B} \mu_b}(H) \right)(b) = Cl^g_{\mu_b}(H(b)).$$

Proof. Let $b \in B$. To see that $\left(Cl^g_{\oplus_{b \in B} \mu_b}(H) \right)(b) \subseteq Cl^g_{\mu_b}(H(b))$, let $m \in \left(Cl^g_{\oplus_{b \in B} \mu_b}(H) \right)(b)$ and let $S \in GC(M, \mu_b)$ such that $H(b) \subseteq S$. Then $b_m \tilde{\in} Cl^g_{\oplus_{b \in B} \mu_b}(H)$. Define $K \in SS(M, B)$ by $K(a) = S$ when $a = b$ and $K(a) = M$ when $a \neq b$. Then $H \tilde{\subseteq} K$ and by Theorem 15 of [3], $K \in GC(M, \oplus_{b \in B} \mu_b, B)$. Since $b_m \tilde{\in} Cl_\lambda^g(H)$, then $b_m \tilde{\in} K$. Thus, $m \in K(b) = S$. Hence, $m \in Cl^g_{\mu_b}(H(b))$.

To see that $Cl^g_{\mu_b}(H(b)) \subseteq \left(Cl^g_{\oplus_{b \in B} \mu_b}(H) \right)(b)$, let $m \in Cl^g_{\mu_b}(H(b))$. We are going to show that $b_m \tilde{\in} Cl^g_{\oplus_{b \in B} \mu_b}(H)$. Let $K \in GC(M, \oplus_{b \in B} \mu_b, B)$ such that $H \tilde{\subseteq} K$. Then $H(b) \subseteq K(b)$ and by Theorem 15 of [3], $K(b) \in GC(M, \mu_b)$. Since $m \in Cl^g_{\mu_b}(H(b))$, then $m \in K(b)$ and so, $b_m \tilde{\in} K$. Hence, $b_m \tilde{\in} Cl^g_{\oplus_{b \in B} \mu_b}(H)$. Therefore, for every $b \in B$,

$$\left(Cl^g_{\oplus_{b \in B} \mu_b}(H) \right)(b) = Cl^g_{\mu_b}(H(b)).$$

□

Corollary 2.4. *Let (M,μ) be a TS and B be any set of parameters. Let $H \in SS(M,B)$. Then for every $b \in B$,*

$$\left(Cl^{g}_{\tau(\mu)}(H)\right)(b) = Cl^{g}_{\mu}(H(b)).$$

Proof. For each $b \in B$, put $\mu_b = \mu$. Then $\tau(\mu) = \oplus_{b\in B}\mu_b$, and by Theorem 2.3 we get the result. □

Theorem 2.5. *Let (M,λ,B) be a STS. Then for any $H \in SS(M,B)$, $H\widetilde{\subseteq}Cl^{g}_{\lambda}(H)\widetilde{\subseteq}Cl_{\lambda}(H)$.*

Proof. The soft inclusion $H\widetilde{\subseteq}Cl^{g}_{\lambda}(H)$ follows directly from the definition of $Cl^{g}_{\lambda}(H)$. The soft inclusion $Cl^{g}_{\lambda}(H)\widetilde{\subseteq}Cl_{\lambda}(H)$ follows because $\lambda^{c} \subseteq GC(M,\lambda,B)$. □

The following example will show that each of the soft inclusions in Theorem 2.5 cannot be replaced by equality in general.

Example 2.6. Let $M=\{1,2,3\}$, $\mu = \{\emptyset, M, \{1\}, \{1,2\}\}$, and $B = \mathbb{R}$. Then $Cl_{\tau(\mu)}(C_{\{1\}}) = 1_B$ and $Cl^{g}_{\tau(\mu)}(C_{\{1\}}) = C_{\{1,2\}}$.

Theorem 2.7. *Let (M,λ,B) be a STS. Then*

(a) *$Cl^{g}_{\lambda}(0_B) = 0_B$ and $Cl^{g}_{\lambda}(1_B) = 1_B$;*
(b) *if $H \in GC(M,\lambda,B)$, then $H = Cl^{g}_{\lambda}(H)$;*
(c) *if $H, G \in SS(M,B)$ such that $H\widetilde{\subseteq}G$, then $Cl^{g}_{\lambda}(H)\widetilde{\subseteq}Cl^{g}_{\lambda}(G)$;*
(d) *for any $H, G \in SS(M,B)$, $Cl^{g}_{\lambda}(H\widetilde{\cup}G) = Cl^{g}_{\lambda}(H)\widetilde{\cup}Cl^{g}_{\lambda}(G)$;*
(e) *if $\{H_\alpha : \alpha \in \Delta\} \subseteq SS(M,B)$ such that $Cl^{g}_{\lambda}(H_\alpha) = H_\alpha$ for all $\alpha \in \Delta$, then $Cl^{g}_{\lambda}\left(\widetilde{\cap}_{\alpha\in\Delta}H_\alpha\right) = \widetilde{\cap}_{\alpha\in\Delta}H_\alpha$;*
(f) *$Cl^{g}_{\lambda}\left(Cl^{g}_{\lambda}(H)\right) = Cl^{g}_{\lambda}(H)$.*

Proof.

(a) By Theorem 2.5, $0_B\widetilde{\subseteq}Cl^{g}_{\lambda}(0_B)\widetilde{\subseteq}Cl_{\lambda}(0_B) = 0_B$, and thus, $Cl^{g}_{\lambda}(0_B) = 0_B$. Since $Cl^{g}_{\lambda}(1_B)\widetilde{\subseteq}1_B$ and by Theorem 2.5, $1_B\widetilde{\subseteq}Cl^{g}_{\lambda}(1_B)$, then $1_B = Cl^{g}_{\lambda}(1_B)$.

(b) If $H \in GC(M,\lambda,B)$, then $Cl^{g}_{\lambda}(H) = \widetilde{\cap}\left\{K : H\widetilde{\subseteq}K \in GC(M,\lambda,B)\right\}\widetilde{\subseteq}H$. On the other hand, by Theorem 2.5 we have $H\widetilde{\subseteq}Cl^{g}_{\lambda}(H)$. Hence, $H = Cl^{g}_{\lambda}(H)$.

(c) Suppose that $H\widetilde{\subseteq}G$. We will apply Theorem 2.2. Let $b_m\widetilde{\in}Cl^{g}_{\lambda}(H)$ and let $K \in GO(M,\lambda,B)$ such that $b_m\widetilde{\in}K$. Then $K\widetilde{\cap}H \neq 0_B$. Since $H\widetilde{\subseteq}G$, then $K\widetilde{\cap}H\widetilde{\subseteq}K\widetilde{\cap}G$, and so $K\widetilde{\cap}G \neq 0_B$.

(d) Since $H\widetilde{\subseteq}H\widetilde{\cup}G$ and $G\widetilde{\subseteq}H\widetilde{\cup}G$, then by (c), $Cl^{g}_{\lambda}(H)\widetilde{\subseteq}Cl^{g}_{\lambda}\left(H\widetilde{\cup}G\right)$ and $Cl^{g}_{\lambda}(G)\widetilde{\subseteq}Cl^{g}_{\lambda}\left(H\widetilde{\cup}G\right)$. Hence, $Cl^{g}_{\lambda}(H)\widetilde{\cup}Cl^{g}_{\lambda}(G)\widetilde{\subseteq}Cl^{g}_{\lambda}\left(H\widetilde{\cup}G\right)$. To see that $Cl^{g}_{\lambda}\left(H\widetilde{\cup}G\right)\widetilde{\subseteq}Cl^{g}_{\lambda}(H)\widetilde{\cup}Cl^{g}_{\lambda}(G)$, suppose to the contrary that there exists $b_m\widetilde{\in}Cl^{g}_{\lambda}\left(H\widetilde{\cup}G\right)$ such that $b_m\widetilde{\notin}Cl^{g}_{\lambda}(H)$ and $b_m\widetilde{\notin}Cl^{g}_{\lambda}(G)$. Since $b_m\widetilde{\notin}Cl^{g}_{\lambda}(H)$ and $b_m\widetilde{\notin}Cl^{g}_{\lambda}(G)$, then by Theorem 2.2, there exist $S, T \in GO(M,\lambda,B)$ such that $b_m\widetilde{\in}S\widetilde{\cap}T$ and $S\widetilde{\cap}H = T\widetilde{\cap}G = 0_B$. Since $S, T \in GO(M,\lambda,B)$, then by Theorem 4.5 of [33], $S\widetilde{\cap}T \in GO(M,\lambda,B)$. Since

$$\left(S\widetilde{\cap}T\right)\widetilde{\cap}\left(H\widetilde{\cup}G\right) = \left(\left(S\widetilde{\cap}T\right)\widetilde{\cap}H\right)\widetilde{\cup}\left(\left(S\widetilde{\cap}T\right)\widetilde{\cap}G\right)\widetilde{\subseteq}\left(S\widetilde{\cap}H\right)\widetilde{\cup}\left(T\widetilde{\cap}G\right) = 0_B,$$

then by Theorem 2.2, $b_m\widetilde{\notin}Cl^{g}_{\lambda}\left(H\widetilde{\cup}G\right)$, a contradiction.

(e) For every $\beta \in \Delta$, we have $\widetilde{\cap}_{\alpha\in\Delta}H_\alpha\widetilde{\subseteq}H_\beta$ and by (c), $Cl^{g}_{\lambda}\left(\widetilde{\cap}_{\alpha\in\Delta}H_\alpha\right)\widetilde{\subseteq}Cl^{g}_{\lambda}\left(H_\beta\right) = H_\beta$. Hence, $Cl^{g}_{\lambda}\left(\widetilde{\cap}_{\alpha\in\Delta}H_\alpha\right)\widetilde{\subseteq}\widetilde{\cap}_{\alpha\in\Delta}H_\alpha$. On the other hand, by Theorem 2.5, we have $\widetilde{\cap}_{\alpha\in\Delta}H_\alpha\widetilde{\subseteq}Cl^{g}_{\lambda}\left(\widetilde{\cap}_{\alpha\in\Delta}H_\alpha\right)$. Therefore, $Cl^{g}_{\lambda}\left(\widetilde{\cap}_{\alpha\in\Delta}H_\alpha\right) = \widetilde{\cap}_{\alpha\in\Delta}H_\alpha$.

(f) By Theorem 2.5, $Cl^{g}_{\lambda}(H)\widetilde{\subseteq}Cl^{g}_{\lambda}\left(Cl^{g}_{\lambda}(H)\right)$. To see that $Cl^{g}_{\lambda}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl^{g}_{\lambda}(H)$, let $K \in GC(M,\lambda,B)$ such that $H\widetilde{\subseteq}K$, then by (c) and (b), $Cl^{g}_{\lambda}(H)\widetilde{\subseteq}Cl^{g}_{\lambda}(K) = K$. Thus, again by (c) and (b), $Cl^{g}_{\lambda}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl^{g}_{\lambda}(K) = K$. Therefore, $Cl^{g}_{\lambda}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}\widetilde{\cap}\left\{K : H\widetilde{\subseteq}K \in GC(M,\lambda,B)\right\} = Cl^{g}_{\lambda}(H)$. □

Theorem 2.7 (b) is not generally reversible, as demonstrated by the following example.

Example 2.8. Let $M = \{1, 2, 3\}$, $B = \{b\}$, and $\lambda = \{0_B, 1_B, b_1\}$. Then $Cl^g_\lambda(b_1) = b_1 \notin GC(M, \lambda, B)$.

Theorem 2.9. *For any STS* (M, λ, B), *the following are equivalent:*

(a) $GC(M, \lambda, B)$ *is closed under soft intersection;*
(b) $H \in GC(M, \lambda, B)$ *if and only if* $H = Cl^g_\lambda(H)$.

Proof.
(a) $\longrightarrow$ (b): If $H \in GC(M, \lambda, B)$, then by Theorem 2.7 (b), $H = Cl^g_\lambda(H)$. Conversely, suppose that $H = Cl^g_\lambda(H)$. Since by (a), $Cl^g_\lambda(H) \in GC(M, \lambda, B)$, then $H \in GC(M, \lambda, B)$.

(b) $\longrightarrow$ (a): Let $\{H_\alpha : \alpha \in \Delta\} \subseteq GC(M, \lambda, B)$. Then by (b), $H_\alpha = Cl^g_\lambda(H_\alpha)$ for all $\alpha \in \Delta$. So, by Theorem 2.7 (e), $Cl^g_\lambda\left(\widetilde{\cap}_{\alpha\in\Delta} H_\alpha\right) = \widetilde{\cap}_{\alpha\in\Delta} H_\alpha$. Hence, by (b), $\widetilde{\cap}_{\alpha\in\Delta} H_\alpha \in GC(M, \lambda, B)$. □

Definition 2.10. For any STS (M, λ, B), denote the family $\left\{S \in SS(M, B) : Cl^g_\lambda(1_B - S) = 1_B - S\right\}$ by $\lambda^\times$.

Theorem 2.11. *For any STS* (M, λ, B), $\lambda^\times$ *is a soft topology on* M *relative to* B.

Proof.
(a) By Theorem 2.7 (a), $Cl^g_\lambda(1_B - 0_B) = 1_B - 0_B$ and $Cl^g_\lambda(1_B - 1_B) = 1_B - 1_B$. Thus, $0_B, 1_B \in \lambda^\times$.
(b) Let $S, T \in \lambda^\times$. Then $Cl^g_\lambda(1_B - S) = 1_B - S$ and $Cl^g_\lambda(1_B - T) = 1_B - T$. So, by Theorem 2.7 (d),

$$\begin{aligned} Cl^g_\lambda\left(1_B - \left(S\widetilde{\cap}T\right)\right) &= Cl^g_\lambda\left((1_B - S)\,\widetilde{\cup}\,(1_B - T)\right) \\ &= Cl^g_\lambda(1_B - S)\,\widetilde{\cup} Cl^g_\lambda(1_B - T) = (1_B - S)\,\widetilde{\cup}\,(1_B - T) = 1_B - \left(S\widetilde{\cap}T\right). \end{aligned}$$

Hence, $S\widetilde{\cap}T \in \lambda^\times$.

(c) Let $\{S_\alpha : \alpha \in \Delta\} \subseteq \lambda^\times$. Then for every $\alpha \in \Delta$, $Cl^g_\lambda(1_B - S_\alpha) = 1_B - S_\alpha$. So, by Theorem 2.7 (e), $Cl^g_\lambda\left(\widetilde{\cap}_{\alpha\in\Delta}(1_B - S_\alpha)\right) = \widetilde{\cap}_{\alpha\in\Delta}(1_B - S_\alpha)$. Thus

$$Cl^g_\lambda\left(1_B - \left(\widetilde{\cup}_{\alpha\in\Delta} S_\alpha\right)\right) = Cl^g_\lambda\left(\widetilde{\cap}_{\alpha\in\Delta}(1_B - S_\alpha)\right) = \widetilde{\cap}_{\alpha\in\Delta}(1_B - S_\alpha) = 1_B - \left(\widetilde{\cup}_{\alpha\in\Delta} S_\alpha\right).$$

Hence, $\widetilde{\cup}_{\alpha\in\Delta} S_\alpha \in \lambda^\times$. □

Theorem 2.12. *For any STS* (M, λ, B), $GO(M, \lambda, B) \subseteq \lambda^\times$.

Proof. Let $S \in GO(M, \lambda, B)$. Then $1_B - S \in GC(M, \lambda, B)$. So, by Theorem 2.7 (b), $1_B - S = Cl^g_\lambda(1_B - S)$. Hence, $S \in \lambda^\times$. □

Corollary 2.13. *For any STS* (M, λ, B), $\lambda \subseteq \lambda^\times$.

Proof. Follows from Theorem 2.12 and the fact that $\lambda \subseteq GO(M, \lambda, B)$. □

Theorem 2.14. *For any STS* (M, λ, B), $\lambda = \lambda^\times$ *if and only if* (M, λ, B) *is soft* $T_{1/2}$.

Proof.
Necessity. Suppose that $\lambda = \lambda^\times$. Let $H \in GC(M, \lambda, B)$. Then by Theorem 2.7 (b), $H = Cl^g_\lambda(H)$ and so, $1_B - H \in \lambda^\times = \lambda$. Hence, $H \in \lambda^c$. This show that $GC(M, \lambda, B) \subseteq \lambda^c$. Therefore, (M, λ, B) is soft $T_{1/2}$.

Sufficiency. Suppose that (M, λ, B) is soft $T_{1/2}$. We will show that $\lambda^\times \subseteq \lambda$. Let $S \in \lambda^\times$. Then $Cl^g_\lambda(1_B - S) = 1_B - S$. Since (M, λ, B) is soft $T_{1/2}$, then $GC(M, \lambda, B) = \lambda^c$ and so $Cl^g_\lambda(1_B - S) = Cl_\lambda(1_B - S)$. Thus, $Cl_\lambda(1_B - S) = 1_B - S$ and hence, $1_B - S \in \lambda^c$. Therefore, $S \in \lambda$. □

Theorem 2.15. *Let* $\{(M, \mu_b) : b \in B\}$ *be an indexed family of TSs. Then* $(\oplus_{b\in B}\mu_b)^\times = \oplus_{b\in B}(\mu_b)^\times$.

Proof. To see that $(\oplus_{b\in B}\mu_b)^{\times} \subseteq \oplus_{b\in B}(\mu_b)^{\times}$, let $S \in (\oplus_{b\in B}\mu_b)^{\times}$. Then $Cl^g_{\oplus_{b\in B}\mu_b}(1_B - S) = 1_B - S$. Thus, by Theorem 2.3, for every $b \in B$,

$$M - S(b) = (1_B - S)(b) = \left(Cl^g_{\oplus_{b\in B}\mu_b}(1_B - S)\right)(b) = Cl^g_{\mu_b}((1_B - S)(b)) = Cl^g_{\mu_b}((M - S(b))).$$

Therefore, for every $b \in B$, $S(b) \in (\mu_b)^{\times}$ and hence $S \in \oplus_{b\in B}(\mu_b)^{\times}$.

To see that $\oplus_{b\in B}(\mu_b)^{\times} \subseteq (\oplus_{b\in B}\mu_b)^{\times}$, let $S \in \oplus_{b\in B}(\mu_b)^{\times}$. Then for each $b \in B$, $S(b) \in (\mu_b)^{\times}$ and so $Cl^g_{\mu_b}((M - S(b))) = M - S(b)$. Thus, by Theorem 2.3, for every $b \in B$,

$$\left(Cl^g_{\oplus_{b\in B}\mu_b}(1_B - S)\right)(b) = Cl^g_{\mu_b}((1_B - S)(b)) = Cl^g_{\mu_b}((1_B - S(b)) = M - S(b) = (1_B - S)(b).$$

Therefore, $Cl^g_{\oplus_{b\in B}\mu_b}(1_B - S) = 1_B - S$. Hence, $S \in (\oplus_{b\in B}\mu_b)^{\times}$. □

Corollary 2.16. *Let (M, μ) be a TS and B be any set of parameters. Let $H \in SS(M, B)$. Then $(\tau(\mu))^{\times} = \tau(\mu^{\times})$.*

Proof. For each $b \in B$, put $\mu_b = \mu$. Then $\tau(\mu) = \oplus_{b\in B}\mu_b$, and by Theorem 2.15, $(\tau(\mu))^{\times} = \oplus_{b\in B}(\mu_b)^{\times} = \tau(\mu^{\times})$. □

Theorem 2.17. *Let (M, λ, B) be a STS and let $H \in SS(M, B)$. Then $Cl^g_{\lambda}(H) = Cl_{\lambda^{\times}}(H)$.*

Proof. To see that $Cl^g_{\lambda}(H)\,\widetilde{\subseteq}\,Cl_{\lambda^{\times}}(H)$, let $b_m \widetilde{\in} Cl^g_{\lambda}(H)$ and let $S \in \lambda^{\times}$ such that $b_m \widetilde{\in} S$. Since $S \in \lambda^{\times}$, then $Cl^g_{\lambda}(1_B - S) = 1_B - S$, and so $b_m \widetilde{\notin} Cl^g_{\lambda}(1_B - S)$. So, there exists $K \in RC(M, \lambda, B)$ such that $1_B - S\,\widetilde{\subseteq}\,K$ and $b_m \widetilde{\notin} K$. Since $b_m \widetilde{\in} Cl^g_{\lambda}(H)$ and $b_m \widetilde{\in} 1_B - K \in GO(M, \lambda, B)$, then by Theorem 2.2, we have $(1_B - K)\,\widetilde{\cap}\,H \neq 0_B$. Since $1_B - S\,\widetilde{\subseteq}\,K$, then $1_B - K\,\widetilde{\subseteq}\,S$. Therefore, $S\,\widetilde{\cap}\,H \neq 0_B$. Hence, $b_m \widetilde{\in} Cl_{\lambda^{\times}}(H)$.

To see that $Cl_{\lambda^{\times}}(H)\,\widetilde{\subseteq}\,Cl^g_{\lambda}(H)$, let $b_m \widetilde{\in} Cl_{\lambda^{\times}}(H)$ and let $S \in GO(M, \lambda, B)$ such that $b_m \widetilde{\in} S$. Since $S \in GO(M, \lambda, B)$, then by Theorem 2.12, $S \in \lambda^{\times}$ and thus $S\,\widetilde{\cap}\,H \neq 0_B$. Hence, $b_m \widetilde{\in} Cl^g_{\lambda}(H)$. □

Theorem 2.18. *For any STS (M, λ, B) and any $b_m \in SP(M, B)$, $b_m \in \lambda^c$ or $1_B - b_m \in GC(M, \lambda, B)$.*

Proof. Suppose that $b_m \notin \lambda^c$, then $1_B - b_m \notin \lambda$. To see that $1_B - b_m \in GC(M, \lambda, B)$, let $T \in \lambda$ such that $1_B - b_m\,\widetilde{\subseteq}\,T$. Since $1_B - b_m \notin \lambda$, then $T = 1_B$ and so $Cl_{\lambda}(1_B - b_m)\,\widetilde{\subseteq}\,T$. □

Theorem 2.19. *For any STS (M, λ, B) and any $b_m, d_n \in SP(M, B)$, $b_m \neq d_n$ implies $Cl^g_{\lambda}(b_m) \neq Cl^g_{\lambda}(d_n)$.*

Proof. By Theorem 2.18, we have two cases.

Case 1: $b_m \in \lambda^c$. Then we have and $d_n \widetilde{\notin} b_m = Cl_{\lambda}(b_m) = Cl^g_{\lambda}(b_m)$ and $d_n \widetilde{\in} Cl^g_{\lambda}(d_n)$. This implies that $Cl^g_{\lambda}(b_m) \neq Cl^g_{\lambda}(d_n)$.

Case 2: $1_B - b_m \in GC(M, \lambda, B)$. Then we have $d_n \widetilde{\in} 1_B - b_m \in GC(M, \lambda, B)$, and thus $d_n \widetilde{\in} Cl^g_{\lambda}(d_n)\,\widetilde{\subseteq}\,1_B - b_m$. So, we have $b_m \widetilde{\in} Cl^g_{\lambda}(b_m) - Cl^g_{\lambda}(d_n)$, and hence $Cl^g_{\lambda}(b_m) \neq Cl^g_{\lambda}(d_n)$. □

Corollary 2.20. *For any STS (M, λ, B), $(M, \lambda^{\times}, B)$ is always soft T_0.*

Theorem 2.21. *For any STS (M, λ, B), $(M, \lambda^{\times}, B)$ is soft $T_{1/2}$.*

Proof. We will apply Theorem 23 of [3]. Let $b_m \in SP(M, B)$. Then by Theorem 2.18, we have two cases.

Case 1: $b_m \in \lambda^c$. Then by Corollary 2.13, $b_m \in \left(\lambda^{\times}\right)^c$.

Case 2: $b_m \in GO(M, \lambda, B)$. Then by Theorem 2.12, $b_m \in \lambda^{\times}$. □

Corollary 2.22. *For any STS (M, λ, B), $\left(\lambda^{\times}\right)^{\times} = \lambda^{\times}$.*

Proof. Follows from Theorems 2.14 and 2.21. □

Theorem 2.23. *For any STS (M, λ, B), the following are equivalent:*

(a) $(M,\lambda^{\times},B)$ *is soft discrete;*
(b) *for each* $b_m \in SP(M,B)$*,* $1_B - b_m \in GC(M,\lambda,B)$*;*
(c) *if* $b_m \in \lambda^c$*, then* $b_m \in \lambda$*.*

Proof.

(a) $\longrightarrow$ (b): Suppose to the contrary that there exists $b_m \in SP(M,B)$ with $1_B - b_m \notin GC(M,\lambda,B)$. Then $\left\{K : 1_B - b_m \widetilde{\subseteq} K \in GC(M,\lambda,B)\right\} = \{1_B\}$, and so $Cl^g_\lambda(1_B - b_m) = 1_B$. But by (a) and Theorem 2.17 $Cl^g_\lambda(1_B - b_m) = Cl_{\lambda^\times}(1_B - b_m) = 1_B - b_m$. This is a contradiction.

(b) $\longrightarrow$ (c): Suppose that $b_m \in \lambda^c$. Then $1_B - b_m \in \lambda$. Since by (b), $1_B - b_m \in GC(M,\lambda,B)$, $1_B - b_m \in \lambda$, and $1_B - b_m \widetilde{\subseteq} 1_B - b_m$, then $Cl_\lambda(1_B - b_m) \widetilde{\subseteq} 1_B - b_m$ and so, $1_B - b_m \in \lambda^c$. Hence, $b_m \in \lambda$.

(c) $\longrightarrow$ (a): We will show that $SP(M,B) \subseteq \lambda^\times$. Let $b_m \in SP(M,B)$. Then by Theorem 2.18, $b_m \in \lambda^c$ or $1_B - b_m \in GC(M,\lambda,B)$. If $b_m \in \lambda^c$, then by (c), $b_m \in \lambda \subseteq \lambda^\times$. If $1_B - b_m \in GC(M,\lambda,B)$, then $b_m \in GO(M,\lambda,B)$ and by Theorem 2.12, $b_m \in \lambda^\times$. □

Corollary 2.24. *If* (M,λ,B) *is a STS such that* $\lambda^c \cap SP(M,B) = \emptyset$*, then* $(M,\lambda^\times,B)$ *is soft discrete.*

Corollary 2.25. *If* (M,λ,B) *is a soft indiscrete STS (i.e.,* $\lambda = \{0_B,1_B\}$*), then* $(M,\lambda^\times,B)$ *is soft discrete.*

Proof. We may assume that $SP(M,B)$ is not a singleton. Then $\lambda^c \cap SP(M,B) = \emptyset$, and by Corollary 2.24, $(M,\lambda^\times,B)$ is soft discrete. □

The following question is natural.

Let (M,λ,B) and (M,σ,B) be two STSs such that $\lambda \subseteq \sigma$. Is it true that $\lambda^\times \subseteq \sigma^\times$?

The following example gives a negative answer for the above question.

Example 2.26. Let $M = \{1,2\}$, $B = \{b\}$, $\lambda = \{0_B,1_B\}$, and $\sigma = \{0_B,1_B,b_1\}$. Then by Corollary 2.25, $\lambda^\times = \{0_B,1_B,b_1,b_2\}$. Since $b_2 = 1_B - b_1 \in \sigma^c$ while $b_2 \notin \sigma$, then by Theorem 2.23, $\sigma^\times \neq \{0_B,1_B,b_1,b_2\}$. Therefore, we have $\lambda \subseteq \sigma$ while $\lambda^\times \nsubseteq \sigma^\times$.

The following three questions are natural.

Is it true that $(M,\lambda^\times,B)$ is soft compact (resp. connected, second countable) if (M,λ,B) is soft compact (resp. connected, second countable)?

The following example gives negative answers to the above three questions.

Example 2.27. Let $M = \mathbb{R}$, $B = \mathbb{Z}$, and $\lambda = \{0_B,1_B\}$. Then (M,λ,B) is soft compact, soft connected, and second countable. On the other hand, by Theorem 2.23, $(M,\lambda^\times,B)$ is soft discrete, and so $(M,\lambda^\times,B)$ is not soft compact, not soft connected, and not soft second countable.

Theorem 2.28. *If* (M,λ,B) *is soft* R_0*, then* $(M,\lambda^\times,B)$ *is soft* T_1*.*

Proof. Let $b_m \in SP(M,B)$. Since (M,λ,B) is soft R_0, then $b_m \in RC(M,\lambda,B)$. So, by Theorem 2.7 (b), $b_m = Cl^g_\lambda(b_m) = Cl_{\lambda^\times}(b_m)$. Therefore, $b_m \in \left(\lambda^\times\right)^c$. Hence, $(M,\lambda^\times,B)$ is soft T_1. □

Theorem 2.29. *If* (M,λ,B) *is soft regular, then* $(M,\lambda^\times,B)$ *is soft regular.*

Proof. Let $b_m \in SP(M,B)$ and let $H \in \left(\lambda^\times\right)^c$ such that $b_m \widetilde{\notin} H$. By Theorem 2.18, $b_m \in \lambda^c$ or $1_B - b_m \in GC(M,\lambda,B)$.

Case 1: $b_m \in \lambda^c$. Since $b_m \widetilde{\notin} H = Cl_{\lambda^\times}(b_m) = Cl^g_\lambda(b_m)$, then there exists $K \in GC(M,\lambda,B)$ such that $H \widetilde{\subseteq} K$ and $b_m \widetilde{\notin} K$. Since we have $K \widetilde{\subseteq} 1_B - b_m \in \lambda$, then $Cl_\lambda(K) \widetilde{\subseteq} 1_B - b_m$. Thus, we have $b_m \widetilde{\notin} Cl_\lambda(K) \in \lambda^c$ in the soft regular STS (M,λ,B). Hence, there exist $S,T \in \lambda$ such that $b_m \widetilde{\in} S$, $Cl_\lambda(K) \widetilde{\subseteq} T$, and $S \widetilde{\cap} T = 0_B$. Since by Corollary 2.13, $\lambda \subseteq \lambda^\times$, then $S,T \in \lambda^\times$. Also, we have $b_m \widetilde{\in} S$ and $H \widetilde{\subseteq} K \widetilde{\subseteq} Cl_\lambda(K) \widetilde{\subseteq} T$.

Case 2: $1_B - b_m \in GC(M,\lambda,B)$. Then $b_m \in GO(M,\lambda,B)$, and by Theorem 2.12, $b_m \in \lambda^\times$.

Claim: $H\widetilde{\subseteq}1_B - Cl_{\lambda^\times}(b_m)$.

Proof of Claim. Let $d_n\widetilde{\in}H$. By Theorem 2.18, $d_n \in \lambda^c$ or $1_B - d_n \in GC(M,\lambda,B)$.

Suppose that $d_n \in \lambda^c$. Then we have $b_m\widetilde{\notin}d_n \in \lambda^c$ and by soft regularity of (M,λ,B), there exist $S,T \in \lambda \subseteq \lambda^\times$ such that $b_m\widetilde{\in}S$, $d_n\widetilde{\subseteq}T$, and $S\widetilde{\cap}T = 0_B$. So we have $d_n\widetilde{\in}T \in \lambda^\times$ and $b_m\widetilde{\cap}T = 0_B$. Thus, $d_n\widetilde{\in}1_B - Cl_{\lambda^\times}(b_m)$.

Suppose that $1_B - d_n \in GC(M,\lambda,B)$. Then $d_n \in GO(M,\lambda,B)$ and by Theorem 2.12, $d_n \in \lambda^\times$. Thus, we have $d_n\widetilde{\in}d_n \in \lambda^\times$ and $b_m\widetilde{\cap}d_n = 0_B$. Hence, $d_n\widetilde{\in}1_B - Cl_{\lambda^\times}(b_m)$.

Therefore, we have $b_m\widetilde{\in}b_m \in \lambda^\times$, $H\widetilde{\subseteq}1_B - Cl_{\lambda^\times}(b_m) \in \lambda^\times$, and $b_m\widetilde{\cap}(1_B - Cl_{\lambda^\times}(b_m)) = 0_B$.

By cases (1) and (2), $(M,\lambda^\times,B)$ is soft regular. □

□

Corollary 2.30. *If (M,λ,B) is soft regular, then $(M,\lambda^\times,B)$ is soft T_3.*

Proof. Follows from Theorems 2.21 and 2.29, and the fact that soft $T_{1/2}$ soft regular STSs are soft T_3. □

The following example shows that the converse of each of Theorem 2.28 and Corollary 2.30 need not be true in general.

Example 2.31. Let $M = \{1,2,3\}$, $B = \{b\}$, and $\lambda = \{0_B, 1_B, b_1\}$. Since $b_1 \notin GC(M,\lambda,B)$, then (M,λ,B) is not soft R_0 and so, it is neither soft regular nor soft T_1. On the other hand, since $\lambda^c \cap SP(M,B) = \emptyset$, then by Corollary 2.24, $(M,\lambda^\times,B)$ is soft discrete.

3. Soft g-continuity

In this section, we provide new properties and characterizations of the well-known concept of "soft g-continuity".

Theorem 3.1. *Let $\{(M,\lambda_t) : t \in \Lambda\}$ and $\{(N,\gamma_s) : s \in \Gamma\}$ be two families of TSs. Let $p : M \longrightarrow N$ be a function and $u : \Lambda \longrightarrow \Gamma$ be a bijective function. Then $f_{pu} : (M, \oplus_{t\in\Lambda}\lambda_t, \Lambda) \longrightarrow (N, \oplus_{s\in\Gamma}\gamma_s, \Gamma)$ is soft g-continuous if and only if $p : (M,\lambda_t) \longrightarrow (N,\gamma_{u(t)})$ is g-continuous for all $t \in \Lambda$.*

Proof.

Necessity. Suppose that $f_{pu} : (M, \oplus_{t\in\Lambda}\lambda_t, \Lambda) \longrightarrow (N, \oplus_{s\in\Gamma}\gamma_s, \Gamma)$ is soft g-continuous. Let $t \in \Lambda$ and let $W \in (\gamma_{u(t)})^c$. Then $(u(t))_W \in (\oplus_{s\in\Gamma}\gamma_s)^c$ and so, $f_{pu}^{-1}((u(t))_W) \in GC(M, \oplus_{t\in\Lambda}\lambda_t, \Lambda)$. Since $u : \Lambda \longrightarrow \Gamma$ is bijective, then $f_{pu}^{-1}((u(t))_W) = t_{p^{-1}(W)}$. Thus, $t_{p^{-1}(W)} \in GC(M, \oplus_{t\in\Lambda}\lambda_t, \Lambda)$. Hence, by Theorem 15 of [3], $\left(t_{p^{-1}(W)}\right)(t) = p^{-1}(W) \in GC(M,\lambda_t)$. Therefore $p : (M,\lambda_t) \longrightarrow (N,\gamma_{u(t)})$ is g-continuous.

Sufficiency. Suppose that $p : (M,\lambda_t) \longrightarrow (N,\gamma_{u(t)})$ is g-continuous for all $t \in \Lambda$. Let $G \in (\oplus_{s\in\Gamma}\gamma_s)^c$. Then $G(s) \in (\gamma_s)^c$ for all $s \in \Gamma$. Since $u : \Lambda \longrightarrow \Gamma$ is bijective, then $p : \left(M, \lambda_{u^{-1}(s)}\right) \longrightarrow (N,\gamma_s)$ is g-continuous for all $s \in \Gamma$. Thus, $p^{-1}(G(s)) = \left(\left(f_{pu}^{-1}(G)\right)\right)(u^{-1}(s)) \in GC\left(M, \lambda_{u^{-1}(s)}\right)$ for all $s \in \Gamma$. So, $\left(f_{pu}^{-1}(G)\right)(t) \in GC(M,\lambda_t)$ for all $t \in \Lambda$. Therefore, by Theorem 15 of [3], $f_{pu}^{-1}(G) \in GC(M, \oplus_{t\in\Lambda}\lambda_t, \Lambda)$. It follows that $f_{pu} : (M, \oplus_{t\in\Lambda}\lambda_t, \Lambda) \longrightarrow (N, \oplus_{s\in\Gamma}\gamma_s, \Gamma)$ is soft g-continuous. □

Corollary 3.2. *Let $p : (M,\mu) \longrightarrow (N,\delta)$ be a function between two TSs and let $u : \Lambda \longrightarrow \Gamma$ be a bijective function. Then $p : (M,\mu) \longrightarrow (N,\delta)$ is g-continuous if and only if $f_{pu} : (M, \tau(\mu), \Lambda) \longrightarrow (N, \tau(\delta), \Gamma)$ is soft g-continuous.*

Proof. For each $t \in \Lambda$ and $s \in \Gamma$, put $\lambda_t = \mu$ and $\gamma_s = \delta$. Then $\tau(\mu) = \oplus_{t\in\Lambda}\lambda_t$ and $\tau(\delta) = \oplus_{s\in\Gamma}\gamma_s$. Thus, by Theorem 3.1, we get the result. □

Theorem 3.3. *If* $f_{pu}:(M,\lambda,B)\longrightarrow(N,\gamma,D)$ *is soft* g*-continuous, then* $f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl_{\gamma}(f_{pu}(H))$ *for every* $H\in SS(M,B)$.

Proof. Suppose that $f_{pu}:(M,\lambda,B)\longrightarrow(N,\gamma,D)$ is soft g-continuous. Let $H\in SS(M,B)$. Then $Cl_{\gamma}(f_{pu}(H))\in\gamma^{c}$. Since $f_{pu}:(M,\lambda,B)\longrightarrow(N,\gamma,D)$ is soft g-continuous, then $f^{-1}_{pu}(Cl_{\gamma}(f_{pu}(H)))\in GC(M,\lambda,B)$. Since $H\widetilde{\subseteq}f^{-1}_{pu}(Cl_{\gamma}(f_{pu}(H)))\in GC(M,\lambda,B)$, then $Cl^{g}_{\lambda}(H)\widetilde{\subseteq}f^{-1}_{pu}(Cl_{\gamma}(f_{pu}(H)))$, and so, $f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}f_{pu}\left(f^{-1}_{pu}(Cl_{\gamma}(f_{pu}(H)))\right)\widetilde{\subseteq}Cl_{\gamma}(f_{pu}(H))$. □

The converse of Theorem 3.3 need not to be true in general.

Example 3.4. Let $M=\{1,2,3\}$, $B=\{b\}$, $\lambda=\{0_B,1_B,b_1\}$, and $\lambda=\left\{0_B,1_B,b_{\{1,3\}}\right\}$. Define $p:M\longrightarrow M$ and $u:B\longrightarrow B$ as follows: $p(1)=2$, $p(2)=1$, $p(3)=3$, and $u(b)=b$. Then for every $H\in SS(M,B)$, $f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl_{\gamma}(f_{pu}(H))$ holds but $f_{pu}:(M,\lambda,B)\longrightarrow(M,\gamma,B)$ is not soft g-continuous.

The following result shows that the converse of Theorem 3.3 is true if we add the condition $GC(M,\lambda,B)$ is closed under soft intersection.

Theorem 3.5. *Let* $f_{pu}:(M,\lambda,B)\longrightarrow(N,\gamma,D)$ *be a soft function such that* $GC(M,\lambda,B)$ *is closed under arbitrary soft intersection. Then* f_{pu} *is soft* g*-continuous if and only if* $f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl_{\gamma}(f_{pu}(H))$ *for every* $H\in SS(M,B)$.

Proof.

Necessity. Follows from Theorem 3.3.

Sufficiency. Suppose that $f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl_{\gamma}(f_{pu}(H))$ for every $H\in SS(M,B)$. Let $K\in\gamma^{c}$. To see that $f^{-1}_{pu}(K)\in GC(M,\lambda,B)$, by Theorem 2.9, it is sufficient to show that $f^{-1}_{pu}(K)=Cl^{g}_{\lambda}(f^{-1}_{pu}(K))$. By assumption, $f_{pu}\left(Cl^{g}_{\lambda}\left(f^{-1}_{pu}(K)\right)\right)\widetilde{\subseteq}Cl_{\gamma}\left(f_{pu}\left(f^{-1}_{pu}(K)\right)\right)\widetilde{\subseteq}Cl_{\gamma}(K)=K$, and so, $Cl^{g}_{\lambda}\left(f^{-1}_{pu}(K)\right)\widetilde{\subseteq}f^{-1}_{pu}\left(f_{pu}\left(Cl^{g}_{\lambda}\left(f^{-1}_{pu}(K)\right)\right)\right)\widetilde{\subseteq}f^{-1}_{pu}(K)$. On the other hand, by Theorem 2.5, $f^{-1}_{pu}(K)\widetilde{\subseteq}Cl^{g}_{\lambda}(f^{-1}_{pu}(K))$. Hence, $f^{-1}_{pu}(K)=Cl^{g}_{\lambda}(f^{-1}_{pu}(K))$. It follows that f_{pu} is soft g-continuous. □

Theorem 3.6. *For any soft function* $f_{pu}:(M,\lambda,B)\longrightarrow(N,\gamma,D)$*, the following are equivalent:*

(a) *for each* $b_m\in SP(M,B)$ *and each* $G\in\gamma$ *such that* $f_{pu}(b_m)\widetilde{\in}G$*, there exists* $K\in GO(M,\lambda,B)$ *such that* $b_m\widetilde{\in}K$ *and* $f_{pu}(K)\widetilde{\subseteq}G$;
(b) *for every* $H\in SS(M,B)$, $f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl_{\gamma}(f_{pu}(H))$ *holds;*
(c) *for every* $K\in SS(N,D)$, $Cl^{g}_{\lambda}\left(f^{-1}_{pu}(K)\right)\widetilde{\subseteq}f^{-1}_{pu}(Cl_{\gamma}(K))$ *holds;*
(d) $f_{pu}:\left(M,\lambda^{\times},B\right)\longrightarrow(N,\gamma,D)$ *is soft continuous.*

Proof.

(a) $\longrightarrow$ (b): Let $H\in SS(M,B)$ and let $d_n\widetilde{\in}f_{pu}\left(Cl^{g}_{\lambda}(H)\right)$. Choose $b_m\widetilde{\in}Cl^{g}_{\lambda}(H)$ such that $d_n=f_{pu}(b_m)$. To see that $d_n\widetilde{\in}Cl_{\gamma}(f_{pu}(H))$, let $G\in\gamma$ such that $d_n=f_{pu}(b_m)\widetilde{\in}G$. Then by (a), there exists $K\in GO(M,\lambda,B)$ such that $b_m\widetilde{\in}K$ and $f_{pu}(K)\widetilde{\subseteq}G$. Since $b_m\widetilde{\in}Cl^{g}_{\lambda}(H)$ and $b_m\widetilde{\in}K\in GO(M,\lambda,B)$, then by Theorem 2.2, $H\widetilde{\cap}K\neq 0_B$. Choose $e_r\widetilde{\in}H\widetilde{\cap}K$. Then $f_{pu}(e_r)\widetilde{\in}f_{pu}(H)$ and $f_{pu}(e_r)\widetilde{\in}f_{pu}(K)\widetilde{\subseteq}G$. Therefore, $f_{pu}(H)\widetilde{\cap}G\neq 0_D$. Hence, $d_n\widetilde{\in}Cl_{\gamma}(f_{pu}(H))$.

(b) $\longrightarrow$ (a): Let $b_m\in SP(M,B)$ and let $G\in\gamma$ such that $f_{pu}(b_m)\widetilde{\in}G$. Let $H=f^{-1}_{pu}(1_D-G)$. Then by (b),

$$f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\widetilde{\subseteq}Cl_{\gamma}\left(f_{pu}\left(f^{-1}_{pu}(1_D-G)\right)\right)\widetilde{\subseteq}Cl_{\gamma}(1_D-G)=1_D-G.$$

and so,

$$Cl^{g}_{\lambda}(H)\widetilde{\subseteq}f^{-1}_{pu}\left(f_{pu}\left(Cl^{g}_{\lambda}(H)\right)\right)\widetilde{\subseteq}f^{-1}_{pu}(1_D-G)=H.$$

This shows that $Cl^{g}_{\lambda}(H)=H$.

Since $b_m\widetilde{\notin}H$, then $b_m\widetilde{\notin}Cl^{g}_{\lambda}(H)$. Then, by Theorem 2.2, there exists $K\in GO(M,\lambda,B)$ such that $b_m\widetilde{\in}K$ and $K\widetilde{\cap}H=0_B$. Thus, $K\widetilde{\subseteq}1_B-H$ and hence,

$$f_{pu}(K)\widetilde{\subseteq}f_{pu}(1_B-H)=f_{pu}\left(1_B-f^{-1}_{pu}(1_D-G)\right)=f_{pu}\left(1_B-\left(1_B-f^{-1}_{pu}(G)\right)\right)=f_{pu}\left(f^{-1}_{pu}(G)\right)\widetilde{\subseteq}G.$$

(b) $\longrightarrow$ (c): Let $K \in SS(N, D)$. Then by (b),

$$f_{pu}\left(Cl^g_\lambda\left(f^{-1}_{pu}(K)\right)\right) \widetilde{\subseteq} Cl_\gamma\left(f_{pu}\left(f^{-1}_{pu}(K)\right)\right) \widetilde{\subseteq} Cl_\gamma(K).$$

Thus,

$$Cl^g_\lambda\left(f^{-1}_{pu}(K)\right) \widetilde{\subseteq} f^{-1}_{pu}\left(f_{pu}\left(Cl^g_\lambda\left(f^{-1}_{pu}(K)\right)\right)\right) \widetilde{\subseteq} f^{-1}_{pu}(Cl_\gamma(K)).$$

(c) $\longrightarrow$ (b): Let $H \in SS(M, B)$. Then by (c) and Theorem 2.7 (c),

$$Cl^g_\lambda(H) \widetilde{\subseteq} Cl^g_\lambda\left(f^{-1}_{pu}(f_{pu}(H))\right) \widetilde{\subseteq} f^{-1}_{pu}(Cl_\gamma(f_{pu}(H))).$$

Thus,

$$f_{pu}\left(Cl^g_\lambda(H)\right) \widetilde{\subseteq} f_{pu}\left(f^{-1}_{pu}(Cl_\gamma(f_{pu}(H)))\right) \widetilde{\subseteq} Cl_\gamma(f_{pu}(H)).$$

(b) $\longrightarrow$ (d): Let $H \in SS(M, B)$. Then by (b), $f_{pu}\left(Cl^g_\lambda(H)\right) \widetilde{\subseteq} Cl_\gamma(f_{pu}(H))$. Since by Theorem 2.17, $Cl^g_\lambda(H) = Cl_{\lambda^\times}(H)$, then we have $f_{pu}(Cl_{\lambda^\times}(H)) \widetilde{\subseteq} Cl_\gamma(f_{pu}(H))$. This shows that $f_{pu} : (M, \lambda^\times, B) \longrightarrow (N, \gamma, D)$ is soft continuous.

(d) $\longrightarrow$ (b): Let $H \in SS(M, B)$. Then by (d), $f_{pu}(Cl_{\lambda^\times}(H)) \widetilde{\subseteq} Cl_\gamma(f_{pu}(H))$. Since by Theorem 2.17, $Cl^g_\lambda(H) = Cl_{\lambda^\times}(H)$, then we have $f_{pu}\left(Cl^g_\lambda(H)\right) \widetilde{\subseteq} Cl_\gamma(f_{pu}(H))$. This ends the proof. □

Theorem 3.7. *Let $f_{pu} : (M, \lambda, B) \longrightarrow (N, \gamma, D)$ be a soft function such that $GC(M, \lambda, B)$ is closed under soft intersection. Then the following are equivalent:*

(a) *f_{pu} is soft g-continuous;*
(b) *for each $b_m \in SP(M, B)$ and each $G \in \gamma$ such that $f_{pu}(b_m) \widetilde{\in} G$, there exists $K \in GO(M, \lambda, B)$ such that $b_m \widetilde{\in} K$ and $f_{pu}(K) \widetilde{\subseteq} G$;*
(c) *for every $H \in SS(M, B)$, $f_{pu}\left(Cl^g_\lambda(H)\right) \widetilde{\subseteq} Cl_\gamma(f_{pu}(H))$ holds;*
(d) *for every $K \in SS(N, D)$, $Cl^g_\lambda\left(f^{-1}_{pu}(K)\right) \widetilde{\subseteq} f^{-1}_{pu}(Cl_\gamma(K))$ holds;*
(e) *$f_{pu} : (M, \lambda^\times, B) \longrightarrow (N, \gamma, D)$ is soft continuous.*

Proof. Follows form Theorems 3.5 and 3.6. □

4. Conclusion

Many aspects of our daily lives are uncertain. One of the concepts proposed to cope with uncertainty is the "soft set" theory. Soft topology, a unique mathematical framework established by topologists utilizing soft sets, is the subject of this research.

For a given soft topological space (M, λ, B), we have introduced a new soft closure operator via soft g-closed sets. We provided several properties of this operator, and using these properties we show that it generated a new soft topology $\lambda^\times$ which is finer than λ. Several properties of this new topology are presented, with a focus on soft separation axioms. In addition, new properties and characterizations of the well-known concept of "soft g-continuity" are provided. Finally, the correspondences between the novel soft topological concepts and their general topological analogs are examined.

The following topics could be considered for future studies: 1) defining a type of soft compactness via soft g-open sets; 2) defining a type of soft connectedness via soft g-open sets; 3) introducing soft g-border, soft g-frontier, and soft g-exterior via soft g-open sets; 4) defining some classes of generalized soft continuity via soft g-open sets.

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