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Optimal control of smoking cessation programs for two subclasses of smoker

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Abstract

In this paper, we propose a dynamic system model representing the interaction between smokers in mixed populations of beginners and regular/heavy smokers and incorporate a smoking cessation program. Since not all smokers acquire treatments, we divide each subclass, beginners and smokers, into untreated and treated groups. From the mathematical analysis, we obtained the basic reproduction number, which is the condition for the smoking-free and endemic equilibriums. This study focuses on two intervention programs as control variables to reduce the smoking habit of smokers, namely educational campaigns for the subclass of beginners and counselling with nicotine therapy for the second subclass of regular/heavy smokers. The objective of the control strategy is to minimize the number of individuals in both subclasses of smokers and maximize the number of quitters with minimum cost. The existence of a solution to the optimal control problem is derived using Pontryagin's maximum principle. The numerical simulations are conducted to visualise and confirm the analytical results, which show the effectiveness of the treatments in reducing the number of smokers. Compared to mono-therapy, the combination therapy of educational campaigns and counselling with nicotine replacement is more effective in reducing the number of smokers.

Keywords: Smoking cessation, equilibrium, optimal control, Pontryagin's maximum principle.

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1. Introduction

The effects on health conditions of smoking cigarettes and tobacco remain the leading cause of death and deadly illnesses worldwide. According to the Tobacco Atlas report [21], nearly five trillion cigarettes are consumed each year worldwide; and this contribute to about eight million deaths and nearly US$ 2 trillion in economic losses. Meanwhile, data from the Ministry of Health of the Republic of Indonesia in 2019 shows that the number of non-communicable cases related to tobacco consumption, such as heart disease, stroke, and cancer, is 17.5 million cases at the cost of more than IDR 16.3 trillion. Chronic obstructive pulmonary disease (COPD), mainly affected by tobacco smoking, remains the main problem of mortality and was estimated to rank fifth worldwide in 2020, as reported by the World Health Organization (WHO) [24]. Thus, it is a very critical public health issue to date. Repeated use of nicotine or

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tobacco makes an increased tolerance to nicotine doses, and if nicotine levels fall it will produce withdrawal symptoms. Physicians and other professionals have recommended interventions and strategies to promote tobacco dependence treatment.

Smoking cessation strategies consist of effective programs preventing COPD progression. Some studies have revealed that smoking cessation in COPD patients can preserve or improve lung function in the first year and reduce the natural development of chronic diseases [2, 5, 15, 16]. Cessation programs can be divided into two categories: pharmacological and non-pharmacological interventions. Pharmacological interventions are those in which smokers take prescribed drugs to help with their nicotine withdrawal syndromes. Nicotine replacement therapy (NRT) reduces nicotine withdrawal symptoms to help a smoker stop smoking [15]. Non-pharmacological interventions include cognitive behavioural therapy, motivational interviews, and counselling. These interventions can increase the quit-smoking rates over the self-initiated strategies. Few studies examine the combined impact of multiple interventions targeting initiation and cessation. The right combination of interventions could accelerate the reduction of smoking prevalence [20].

A number of mathematical models have been proposed and theoretical analyses were performed to capture the dynamical behaviour of the smoking epidemic. Analysis of well-posed model, basic reproduction number and stability of equilibrium points are important. These analyses are discussed in Castillo-Garsow et al. [4], where the population is classified into three groups: potential smokers, smokers, and individuals who have ceased smoking permanently. Sharomi and Gumel [18] introduced a model incorporating variability in smoking habits to study the impact of illnesses caused by the smoking habit. The model by Zaman [26] assumed that smokers who quit smoking may become potential smokers again. Effective cessation programs, mentioned earlier, are considered as controls in [26], two types of optimal control strategies were studied: the educational strategy and the campaign treatment strategy. The existence of the optimal control is obtained using optimal control theory. Meanwhile, a dynamic smoking model by Pang et al. [13] addresses the effect of treatment on smoking cessation by incorporating the smoking area and the price of cigarettes as controls to reduce the number of smokers. Yadav et al. [25] studied a model incorporating the implication of educational program and determination. Other models describing the dynamic behaviour of smoking are also discussed in [1, 6, 8, 10, 17, 19].

In this study, we propose a model of the smoker population consisting of treated and untreated classes for both beginner (early) smokers and regular/heavy smokers, respectively. In a developing country, it is not financially sufficient for the population of all smokers immediately to be treated by intervention programs for smoking cessation. Hence, the contribution of this study is the modification of the smoking model [26] where we assume that only a portion of beginners and heavy smokers acquire the cessation program and the remaining portion is considered an untreated sub-population of beginners and smokers. Thus, the population is classified into six sub-populations: potential smokers, untreated and treated beginners, untreated and treated smokers and quitters. We provide the dynamic of each sub-population in constructing the model of ordinary differential equations and prove the positive invariant of the feasible region for the model. A closed form of the basic reproduction number is also obtained using the next-generation matrix method [13, 22].

The model is then extended to focus on treatment interventions for the sub-populations of treated beginners and smokers. Here, we devote two treatment strategies: the educational campaign treatment for the sub-population of beginners and counselling with nicotine replacement therapy for heavy smokers. Optimal control strategies are known to be quite effective in controlling many diseases. A feasible strategy of a bounded time-varying controls is needed to balance the cost and control goal. We prove the existence of optimal control and derive the optimality system, and further apply the Pontryagin's maximum principle [14], also used in [12, 13, 26], to find the pair of optimal control strategies that minimise the cost function.

The rest of this paper is organised as follows. In Section 2, we present the model in detail and derive the closed form of the basic reproduction number used to obtain the equilibrium points. In Section 3, we discuss the existence of optimal controls. The theorem which guarantees the solution to the optimal control problem is discussed in Section 4. Then, Section 5 discusses the numerical simulations conducted to visualise and confirm the analytical results. Finally, a concluding remark is given in Section 6.

2. Mathematical Model of Smoking Cessation with Control

We introduce a smoking cessation model, taking into account the treatment efforts towards beginner and (heavy) smoker individuals to curtail smoking habits. The treatment interventions for smoking cessation include educational campaign treatment for subclass of beginners and counselling with nicotine replacement for subclass of smoker. In the model, populations of beginners and smokers are separated into two classes, namely untreated and treated. The population at time t, $N(t)$, comprises of six sub-populations. These are P(t): non-smokers that are the potential to become smokers; $B_U(t)$: untreated beginners; $B_T(t)$: treated beginners (beginners who acquire the first type treatment/therapy); $S_U(t)$: untreated smokers; $S_T(t)$: treated smoker (smoker individuals who acquire the second type treatment/therapy); and $Q(t)$: sub-population of permanent quitters. In the treated beginner individuals, the educational campaign program, denoted as control variable $\nu_1(t)$, may be given in the form of motivational interviews, psychosocial, cognitive behavioural therapy and counselling. While the treatment of counselling with nicotine replacement (as the second type of treatment) is provided for the smoker population, $S_T(t)$, to reduce the frequency of smoking habit leading to stop smoking. A variable control $\nu_2(t)$ represents this treatment level. Nicotine replacement therapy by giving drugs is used to block the brain's nicotine receptors, reducing the urge to smoke. A schematic diagram of the proposed smoking epidemic is given in the compartment model Figure 1.

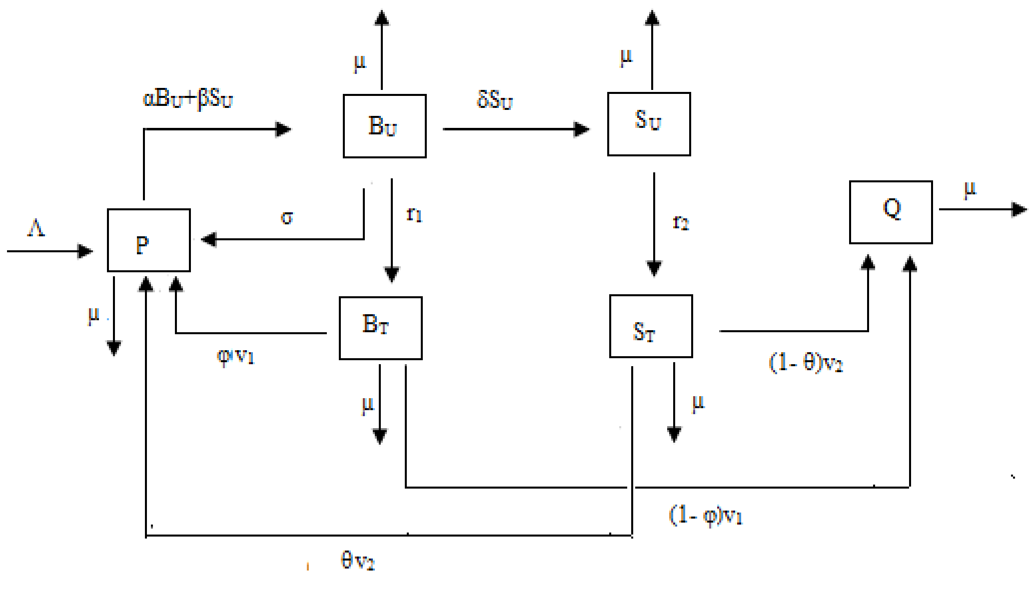


Figure 1: *Flow diagram of smoking epidemic*

We assumed that the non-smokers population (P(t)) increased with a constant rate Λ. The population increases when the untreated beginners begin to stop smoking temporarily due to self-control influence at a constant rate σ. Fractions $\varphi\nu_1(t)$ and $\theta\nu_2(t)$ are the rates of effective treatments in which the classes of the treated beginners and the treated smokers, respectively, return to the potential-smokers class. The remaining $(1-\varphi)\nu_1(t)$ and $(1-\theta)\nu_2(t)$ of the treated beginner and smokers individuals, respectively, become permanent quitters. Non-smokers (potential) population can acquire the attitude toward smoking (as beginners) via effective interaction with both untreated beginners and untreated smokers at constant rates α and β, respectively. The population diminishes due to natural death at a rate μ. The growth rate of potential population P(t) is:

$$\frac{dP}{dt} = \Lambda - (\alpha B_U + \beta S_U)\, P - \mu P + \sigma B_U + \varphi \nu_1(t) B_T + \theta \nu_2(t) S_T. \tag{2.1}$$

Once potential individuals start to smoke, they are called the class of untreated beginners $B_U(t)$. A part of untreated beginners continues smoking at a constant rate δ, when he/she contacts effectively with untreated smokers. Other parts of this population quit smoking temporarily due to self-willingness (at the rate σ). Some beginner individuals respond to following the first type of treatment, and move to the treated beginners class at a rate r_1. Thus, the growth rate of untreated beginner smokers is:

$$\frac{dB_U}{dt} = (\alpha B_U + \beta S_U)\, P - \delta B_U S_U - (\sigma + r_1 + \mu)\, B_U. \tag{2.2}$$

The population of treated beginners $B_T(t)$ increases through untreated beginners (at a rate r_1) when they acquire the first type of treatment and as the source. Due to the efficacy of the first type treatment, a fraction of this population, $\varphi \nu_1(t) B_T$, returns to the potential-smoker class, and another fraction, $(1 - \varphi \nu_1(t)) B_T$, moves to the permanent quitter class. Thus, the change rate of the treated beginners is

$$\frac{dB_T}{dt} = r_1 B_U - \nu_1(t) B_T - \mu B_T. \tag{2.3}$$

The population of untreated smokers is generated by untreated beginners at the rate δ when they begin to smoke permanently through effective contact with untreated smokers. Some smoker individuals respond and follow the second type treatment at a rate r_2, and they move to the treated smoker class. Thus, the growth rate of untreated smokers is:

$$\frac{dS_U}{dt} = \delta B_U S_U - (r_2 + \mu)\, S_U. \tag{2.4}$$

The population of treated smokers is recruited by untreated smokers (at a rate r_2), when the second type of treatment has been acquired. Due to the efficacy of the second type of treatment, a fraction of this population, $\theta \nu_2(t) S_T$, returns to the potential-smoker class, and the remaining fraction of $(1-\theta)\nu_2(t) S_T)$ moves to the permanent quitter class. Thus,

$$\frac{dS_T}{dt} = r_2 S_U - \nu_2(t) S_T - \mu\, S_T. \tag{2.5}$$

The population of quitters is generated by treated beginners and smokers individuals who permanently quit smoking, at rates $(1-\varphi)\nu_1(t) B_T$, and $(1-\theta)\nu_2(t) S_T$, respectively. Therefore,

$$\frac{dQ}{dt} = (1-\varphi)\nu_1(t) B_T + (1-\theta)\, \nu_2(t) S_T - \mu\, Q. \tag{2.6}$$

In the case of control variables $\nu_i(t) = 0, i = 1, 2$, then we have the model without control and the equations (2.1) - (2.6) are reduced into the following system of equations

$$\begin{aligned}
\frac{dP}{dt} &= \Lambda - (\alpha B_U + \beta S_U)\, P - \mu P + \sigma B_U, \\
\frac{dB_U}{dt} &= (\alpha B_U + \beta S_U)\, P - \delta B_U S_U - (\sigma + r_1 + \mu)\, B_U, \\
\frac{dB_T}{dt} &= r_1 B_U - \mu B_T, \\
\frac{dS_U}{dt} &= \delta B_U S_U - (r_2 + \mu)\, S_U,
\end{aligned} \tag{2.7}$$

$$\begin{aligned}\frac{dS_T}{dt} &= r_2 S_U - \mu S_T,\\ \frac{dQ}{dt} &= -\mu Q.\end{aligned}$$

where $N(t) = P(t) + B_U(t) + B_T(t) + S_U(t) + S_T(t) + Q(t)$ and initial conditions:

$$P(0) \geqslant 0, B_U(0) \geqslant 0, B_T(0) \geqslant 0, S_U(0) \geqslant 0, S_T(0) \geqslant 0, Q(0) \geqslant 0. \tag{2.8}$$

In the next section, we establish the existence of a solution for (2.7) in which the system satisfies the properties of non-negativeness and well-posedness.

2.1. *Wellposed model without control*

To guarantee the system of ordinary differential equations (2.7) with initial conditions (2.8) is well-posed mathematically and biologically, we prove the following theorem

Theorem 2.1. *Let* $(P, B_u, B_T, S_u, S_t, Q)$ *be the solution of the smoking model without the controls (2.7) with the given initial condition (2.8). Then the biological feasible closed region*

$$\Omega = \{(P(t), B_U(t), B_T(t), S_U(t), S_T(t), Q(t)) \in \mathbb{R}^6_+ : 0 < P(t) + B_U(t) + B_T(t) + S_U(t) + S_T(t) + Q(t) \leqslant \frac{\Lambda}{\mu}\} \tag{2.9}$$

is positively invariant and well-posed for the system (2.7-2.8).

Proof. Firstly, using the initial conditions, we can easily obtain that all rates are non-negative as follows:

$$\begin{aligned}\frac{dP}{dt}|(P=0, B_U, B_T, S_U, S_T, Q) &= \Lambda + \sigma B_U > 0,\\ \frac{dB_U}{dt}|(P, B_U = 0, B_T, S_U, S_T, Q) &= \beta S_U P \geqslant 0,\\ \frac{dB_T}{dt}|(P, B_U, B_T = 0, S_U, S_T, Q) &= r_1 B_U \geqslant 0,\\ \frac{dS_U}{dt}|(P, B_U, B_T, S_U = 0, S_T, Q) &= 0 \geqslant 0,\\ \frac{dS_T}{dt}|(P, B_U, B_T, S_U, S_T = 0, Q) &= r_2 S_U \geqslant 0,\\ \frac{dQ}{dt}|(P, B_U, B_T, S_U, S_T, Q = 0) &= 0 \geqslant 0.\end{aligned} \tag{2.10}$$

This proves all variables are positive over the boundary planes of the cone $\mathbb{R}^6_+$. Furthermore, since the total population at all time, $t \geqslant 0$, is $N(t) = P(t) + B_U(t) + B_T(t) + S_U(t) + S_T(t) + Q(t)$, by adding the equations in (2.7) we have the total population dynamic

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t), \ \textit{with}\ N(0) = N_0 > 0 \tag{2.11}$$

which yields non-negative value $N(t) = \frac{\Lambda}{\mu} + e^{-\mu t}\left(N_0 - \frac{\Lambda}{\mu}\right) > 0$, for $t \geqslant 0$. It concludes that $N(t) \leqslant \frac{\Lambda}{\mu}$, if $N_0 \leqslant \frac{\Lambda}{\mu}$, and $\lim_{t\to\infty} \sup N(t) = \frac{\Lambda}{\mu}$. Therefore, the model (2.7-2.8) is well-posed, and the feasible region Ω is positively invariant and attracting the system. □

Note that the population size is a variable. The population is constant when $N(0) = \frac{\Lambda}{\mu}$. The system (2.7) has a smoking-free equilibrium which is given by, $E = (P, B_U, B_T, S_U, S_T, Q) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0)$.

2.2. *The basic reproduction number*

The basic reproduction number $\mathfrak{R}_0$ characterises the ability of the smoking epidemic to spread; it is usually assumed that an epidemic occurs if $\mathfrak{R}_0 > 1$. Otherwise, when $\mathfrak{R}_0 < 1$, the smoking habit will disappears without any control treatment strategy. We apply the next-generation-matrix (NGM) method [22] to find $\mathfrak{R}_0$ for system (2.7), where $\mathfrak{R}_0$ is the spectral radius of the NGM matrix and obtain (see also [23])

$$\mathfrak{R}_0 = \frac{\Lambda\,\alpha}{\mu\,(\sigma + r_1 + \mu)}. \tag{2.12}$$

It represents the number of secondary new cases of smoking habits that are affected by a smoker when he/she introduces to a potential population during the period of smoking habits. By letting the right-hand side of (2.7) equal to zero, we also obtain the smoking equilibrium point of the model denoted by $(P^*, B_U^*, B_T^*, S_U^*, S_T^*, Q^*)$ where:

$$\begin{aligned}
P^* &= \frac{\Lambda\,\delta + \mu\,\sigma + r_2\,\sigma}{\beta\,\delta S_U{}^* + \alpha\,\mu + \alpha\,r_2 + \delta\,\mu}\\
B_U^* &= \frac{r_2 + \mu}{\delta}\\
B_T^* &= \frac{r_1\,(r_2 + \mu)}{\delta\,\mu}\\
S_T^* &= \frac{r_2\,S_U{}^*}{\mu},\\
Q^* &= 0,
\end{aligned}$$

and S_U^* is a solution of a quadratic polynomial, $a_2 S_U^2 + a_1 S_U^1 + a_0 = 0$, where

$$\begin{aligned}
a_2 &= -\beta\,\delta^2\,(r_2 + \mu)\\
a_1 &= \delta\,\left(\Lambda\,\delta - \mu^2 - \mu\,r_1 - \mu\,r_2 - r_1\,r_2\right)\beta - \delta\,(r_2 + \mu)\,(\alpha\,\mu + \alpha\,r_2 + \delta\,\mu)\\
a_0 &= (r_2 + \mu)\,((r_2 + \mu)\,(\mu + r_1)\,\alpha + \delta\,\mu\,(\sigma + r_1 + \mu))\,(\mathfrak{R}_0 - 1)\,.
\end{aligned}$$

It shows that this quadratic polynomial of S_U has only one positive root if $\frac{a_0}{a_2} < 0$, and due to a_2 is always negative. It must be $a_0 > 0$ to fulfil the condition of uniqueness of endemic equilibrium. So, the existence and uniqueness of the smoking endemic equilibrium is guaranteed if $\mathfrak{R}_0 > 1$.

3. Existence of solution for the controlled model

Let $P(t), B_U(t), B_T(t), S_U(t), S_T(t)$ and $Q(t)$ be state variables with control $v_1(t), v_2(t)$ in (4.1). We now present the existence of solution for the controlled model (2.1)- (2.6) with initial condition (2.8). The controlled system model can be written in matrix form:

$$\mathbf{X}' = \mathbf{B}\mathbf{X} + \mathbf{F}(\mathbf{X}) \tag{3.1}$$

where $\mathbf{X} = \begin{bmatrix} P(t) & B_U(t) & B_T(t) & S_U(t) & S_T(t) & Q(t) \end{bmatrix}^\top$,

$$\mathbf{B} = \begin{bmatrix}
-\mu & \sigma & \varphi v_1(t) & 0 & \theta v_2(t) & 0\\
0 & -(\sigma + r_1 + \mu) & 0 & 0 & 0 & 0\\
0 & r_1 & -(v_1(t) + \mu) & 0 & 0 & 0\\
0 & 0 & 0 & -(r_2 + \mu) & 0 & 0\\
0 & 0 & 0 & r_2 & -(v_2(t) + \mu) & 0\\
0 & 0 & (1-\varphi)v_1(t) & 0 & (1-\theta)v_2(t) & -\mu
\end{bmatrix},$$

$$\mathbf{F}(\mathbf{X}) = \begin{bmatrix} \Lambda - (\alpha\, B_U + \beta\, S_U)\, P \\ (\alpha\, B_U + \beta\, S_U)\, P - \delta\, B_U\, S_U \\ 0 \\ \delta\, B_U\, S_U \\ 0 \\ 0 \end{bmatrix}$$

and $\mathbf{X}'$ denotes derivative of vector $\mathbf{X}$ with respect to time t. Equation (3.1) is a non-linear system with bounded coefficients. We can rewrite (3.1) as follows:

$$D(\mathbf{X}) = \mathbf{B}\mathbf{X} + \mathbf{F}(\mathbf{X})$$

where $D = \frac{d}{dt}$. For, $\mathbf{X}_i = [P_i,\, B_{U_i},\, B_{T_i},\, S_{U_i},\, S_{T_i},\, Q_i]$, $i = 1, 2$, we have

$$\begin{aligned} \parallel \mathbf{F}(\mathbf{X}_1) - \mathbf{F}(\mathbf{X}_2) \parallel \;&=\; |\alpha\,(-B_{U_1}P_1 + B_{U_2}P_2) + \beta\,(-P_1S_{U_1} + P_2S_{U_2})| + |\delta\;(B_{U_1}S_{U_1} - S_{U_2}B_{U_2})| \\ &\quad + |(-B_{U_2}P_2 + B_{U_1}P_1)\,\alpha + (P_1S_{U_1} - P_2S_{U_2})\,\beta - \delta\;(B_{U_1}S_{U_1} - S_{U_2}B_{U_2})| \\ &\leqslant\; 2\,\alpha\;|B_{U_1}P_1 - B_{U_2}P_2| + 2\,\beta\;|P_1S_{U_1} - P_2S_{U_2}| + 2\,\delta\;|B_{U_1}S_{U_1} - S_{U_2}B_{U_2}| \\ &\leqslant\; |P_1 - P_2|\,(2\,\alpha\;|B_{U_1}| + 2\,\beta\;|S_{U_1}|) + |S_{U_1} - S_{U_2}|\,(2\,\beta\;|P_2| + 2\,\delta\;|B_{U_1}|) + \\ &\quad |B_{U_1} - B_{U_2}|\,(2\,\alpha\;|P_2| + 2\,\delta\;|S_{U_2}|) \\ &\leqslant\; |P_1 - P_2|\left(2\,\alpha\frac{\Lambda}{\mu} + 2\beta\;\frac{\Lambda}{\mu}\right) + |S_{U_1} - S_{U_2}|\left(2\,\beta\frac{\Lambda}{\mu} + 2\delta\;\frac{\Lambda}{\mu}\right) + \\ &\quad |B_{U_1} - B_{U_2}|\left(2\,\alpha\frac{\Lambda}{\mu} + 2\delta\;\frac{\Lambda}{\mu}\right) \\ &\leqslant\; K\,(|P_1 - P_2| + |B_{U_1} - B_{U_2}| + |S_{U_1} - S_{U_2}|) \end{aligned}$$

where

$$K = 2\,\frac{\Lambda}{\mu}\max\left(\alpha + \beta, \alpha + \delta, \beta + \delta\right).$$

Furthermore, we obtain $\parallel D(\mathbf{X_1}) - \mathbf{D}(\mathbf{X_2}) \parallel \leqslant \mathbf{M} \parallel \mathbf{X_1} - \mathbf{X_2} \parallel$, where $M = K + \parallel \mathbf{B} \parallel < \infty$. Thus, operator D is uniformly Lipschitz continuous. Using restrictions on $P(t), B_U(t), B_T(t), S_U(t), S_T(t), Q(t) \geqslant 0$, and the definition of V given in (4.1) we can conclude that there is a solution for the system in (3.1) (see [3]).

4. The Optimal Control Problem

The control function $v_1(t)$ indicates the treatment effectiveness level of the educational campaign in which beginner individuals stop smoking, thus prevent beginners to become heavy smokers. The control variable $v_2(t)$ represents the effectiveness level of counselling and nicotine replacement therapy (NRT) to help smoker individuals quit smoking permanently or temporarily. In terms of optimal control, we take into account the control functions $v_1(t), v_2(t) \in V$ where

$$V = \{(v_1(t), v_2(t)) | v_i(t) \;\text{ is measurable, }\; 0 \leqslant v_i(t) \leqslant v_{i_{max}} \leqslant 1, t \in [0, t_f],\; i = 1, 2\} \tag{4.1}$$

is an admissible control function set. The values of $v_{1_{max}}$ and $v_{2_{max}}$ are the maximum effectiveness level of the first and the second type of treatments, respectively.

The optimal control problem is to find solutions of treatment strategies for the combination of educational campaign $v_1(t)$ and counselling with nicotine replacement therapy $v_2(t)$ in the interval from $t = 0$ through $t = t_f$ which reduce the number of both beginner and smoker individuals at minimum cost. In this case, we seek $\left(v_1^*(t), v_2^*(t)\right)$ that minimise the objective cost functional given by

$$J(v_1, v_2) = \int_0^{t_f} \left[\omega_{B_T} B_T^2 + \omega_{S_T} S_T^2 + \omega_{v_1} v_1^2(t) + \omega_{v_2} v_2^2(t)\right] dt \tag{4.2}$$

subject to the control system (2.1)- (2.6) with initial condition (2.8). We set coefficients ω_{v_1} and ω_{v_2} as positive weight constants associated with controls $v_1(t)$ and $v_2(t)$, respectively. Since higher values of controls $v_1(t)$, $v_2(t)$, elevate the cessation program costs, quadratic functional $\omega_{v_1}v_1^2(t)$ and $\omega_{v_2}v_2^2(t)$ are chosen to represent the costs incurred by the smoking cessation program that reflect the side effect of the first type and second type treatments, respectively. Furthermore, we penalize if the amount of control is too large. Such costs are commonly used as found in [12], [13], [26]. The coefficients $\omega_{B_T}, \omega_{S_T}$, are positive constant to keep balancing in the size of $B_T(t)$ and $S_T(t)$, respectively, that represent costs rising from resource consumptions for treating beginners and smokers due to the negative effects of smoking that we want to also minimise.

4.1. *Existence of optimal control*

Theorem 4.1. *There exists an optimal control $v^*(t) = (v_1^*(t), v_2^*(t))$ such that $J(v^*) = \min_{v(t)\in V} J(v)$ subject to control system (2.1)- (2.6) with initial condition (2.8).*

Proof. We shall present a proof based on the result in [7]. (1) In Section 3 we have showed that for any $v(t) \in V$ and initial variables (2.8) in Ω, the solution of the controlled system exists. (2) Also, we showed the RHS of the state system is bounded by a linear function which determines compactness. (3) The control space V in (4.1) is obviously closed and convex. (4) The integrand in $J(v)$ is also convex in V. (5) We will show there exist constants $k_1 > 0$ and $k_2 > 1$ such that $J(v) \geqslant k_1 \parallel v \parallel^{k_2}$. We have $J(v) \geqslant \frac{1}{2} \parallel v \parallel^2$. So $k_1 = \frac{1}{2}$, $k_2 = 2$. Thus, we have verified the five conditions in [7]. Therefore, an optimal control exists. □

4.2. *Solution of the optimal control problem*

We defined the Hamiltonian (H) associated with the optimisation problem by

$$\begin{aligned} H &= \omega_{B_T}B_T^2 + \omega_{S_T}S_T^2 + \omega_{v_1}v_1^2 + \omega_{v_2}v_2^2 + \lambda_P\frac{dP}{dt} + \lambda_{B_U}\frac{dB_U}{dt} + \lambda_{B_T}\frac{dB_T}{dt} + \\ &\quad \lambda_{S_U}\frac{dS_U}{dt} + \lambda_{S_T}\frac{dS_T}{dt} + \lambda_Q\frac{dQ}{dt}. \end{aligned} \tag{4.3}$$

The Hamiltonian consist of the sum of the integrand of the objective function and the inner product the state equations with the adjoint variables . The conditions in the Pontryagin's maximum principle [14] must hold by the optimal trajectory. Thus, we provide Theorem 4.2 to help find the solution of the optimal control problem.

Theorem 4.2. *Consider optimal control of variables v_1^*, v_1^* and solutions $P^*, B_U^*, B_T^*, S_U^*, S_T^*, Q^*$ of the system (2.1) - (2.6) for minimising $J(v_1, v_2)$ over V. Then there exists adjoint variables $\lambda_P, \lambda_{B_U}, \lambda_{B_T}, \lambda_{S_U}, \lambda_{S_T}, \lambda_Q$ satisfying*

$$\begin{aligned} \frac{d\lambda_P}{dt} &= \lambda_P\left(\alpha B_U + \beta S_U + \mu\right) - \lambda_{B_U}\left(\alpha B_U + \beta S_U\right), \\ \frac{d\lambda_{B_U}}{dt} &= -\lambda_P\left(+\sigma - \alpha P\right) - \lambda_{B_U}\left(\alpha P - \delta S_U - \mu - r_1 - \sigma\right) - \lambda_{B_T}r_1 - \lambda_{S_U}\delta S_U, \\ \frac{d\lambda_{B_T}}{dt} &= -2\,\omega_{B_T}B_T - \lambda_P\varphi v_1 + \lambda_{B_T}\left(\mu + v_1\right) - \lambda_Q\left(1-\varphi\right)v_1, \\ \frac{d\lambda_{S_U}}{dt} &= \lambda_P\beta P - \lambda_{B_U}\left(\beta P - \delta B_U\right) - \lambda_{S_U}\left(\delta B_U - \mu - r_2\right) - \lambda_{S_T}r_2, \\ \frac{d\lambda_{S_T}}{dt} &= -2\,\omega_{S_T}S_T - \lambda_P\theta v_2 + \lambda_{S_T}\left(\mu + v_2\right) - \lambda_Q\left(1-\theta\right)v_2, \\ \frac{d\lambda_Q}{dt} &= \lambda_Q\mu, \end{aligned} \tag{4.4}$$

with transversality conditions

$$\lambda_P(t_f) = \lambda_{B_U}(t_f) = \lambda_{B_T}(t_f) = \lambda_{S_U}(t_f) = \lambda_{S_T}(t_f) = \lambda_Q(t_f) = 0, \tag{4.5}$$

and the controls v_1^* and v_2^* satisfy the optimality condition

$$\begin{aligned} v_1^* &= \frac{1}{2}\frac{B_T^*\left(\lambda_{B_T} - \varphi\lambda_P - (1-\varphi)\lambda_Q\right)}{\omega_{v_1}}, \\ v_2^* &= \frac{1}{2}\frac{S_T^*\left(\lambda_{S_T} - \theta\lambda_P - (1-\theta)\lambda_Q\right)}{\omega_{v_2}}. \end{aligned} \tag{4.6}$$

Proof. Differentiating the Hamiltonian in (4.3) with respect to each state variable, we obtain the differential equation for the corresponding adjoint variables:

$$\begin{aligned} \frac{d\lambda_P}{dt} &= -\frac{\partial H}{\partial P} = \lambda_P\left(\alpha B_U + \beta S_U + \mu\right) - \lambda_{B_U}\left(\alpha B_U + \beta S_U\right), \\ \frac{d\lambda_{B_U}}{dt} &= -\frac{\partial H}{\partial B_U} = -\lambda_P\left(+\sigma - \alpha P\right) - \lambda_{B_U}\left(\alpha P - \delta S_U - \mu - r_1 - \sigma\right) - \lambda_{B_T} r_1 - \lambda_{S_U}\delta S_U, \\ \frac{d\lambda_{B_T}}{dt} &= -\frac{\partial H}{\partial B_T} = -2\omega_{B_T} B_T - \lambda_P\varphi v_1 + \lambda_{B_T}\left(\mu + v_1\right) - \lambda_Q\left(1-\varphi\right)v_1, \\ \frac{d\lambda_{S_U}}{dt} &= -\frac{\partial H}{\partial S_U} = \lambda_P\beta P - \lambda_{B_U}\left(\beta P - \delta B_U\right) - \lambda_{S_U}\left(\delta B_U - \mu - r_2\right) - \lambda_{S_T} r_2, \\ \frac{d\lambda_{S_T}}{dt} &= -\frac{\partial H}{\partial S_T} = -2\omega_{S_T} S_T - \lambda_P\theta v_2 + \lambda_{S_T}\left(\mu + v_2\right) - \lambda_Q\left(1-\theta\right)v_2, \\ \frac{d\lambda_Q}{dt} &= -\frac{\partial H}{\partial Q} = \lambda_Q\mu, \end{aligned} \tag{4.7}$$

with conditions

$$\lambda_P(t_f) = \lambda_{B_U}(t_f) = \lambda_{B_T}(t_f) = \lambda_{S_U}(t_f) = \lambda_{S_T}(t_f) = \lambda_Q(t_f) = 0. \tag{4.8}$$

Using optimality conditions, we differentiate the Hamiltonian function (4.3) with respect to each v_1, v_2 and evaluating at the optimal control variables, we have

$$\begin{aligned} 0 &= \frac{\partial H}{\partial v_1} = 2\omega_{v_1} v_1^* + \lambda_P\varphi B_T^* - \lambda_{B_T} B_T^* + \lambda_Q\left(1-\varphi\right)B_T^*, \\ 0 &= \frac{\partial H}{\partial v_2} = 2\omega_{v_2} v_2^* + \lambda_P\theta S_T^* - \lambda_{S_T} S_T^* + \lambda_Q\left(1-\theta\right)S_T^*. \end{aligned} \tag{4.9}$$

Thus, we obtain

$$\begin{aligned} v_1^* &= \frac{1}{2}\frac{B_T^*\left(\lambda_{B_T} - \varphi\lambda_P - (1-\varphi)\lambda_Q\right)}{\omega_{v_1}} \\ v_2^* &= \frac{1}{2}\frac{S_T^*\left(\lambda_{S_T} - \theta\lambda_P - (1-\theta)\lambda_Q\right)}{\omega_{v_2}}. \end{aligned} \tag{4.10}$$

By using the property of the control space (4.1), we have the following conditions. If $\frac{\partial H}{\partial v_i} < 0$, at t, then $v_i^*(t) = 0$, for $i = 1, 2$, conversely, if $\frac{\partial H}{\partial v_i} > 0$, at t, we take $v_i^*(t) = 1$. Therefore, we can rewrite in compact form the optimal control variables v_1^* and v_2^* by

$$\begin{aligned} v_1^* &= \max\left\{0, \min\left(1, \frac{1}{2}\frac{B_T^*\left(\lambda_{B_T} - \varphi\lambda_P - (1-\varphi)\lambda_Q\right)}{\omega_{v_1}}\right)\right\} \\ v_2^* &= \max\left\{0, \min\left(1, \frac{1}{2}\frac{S_T^*\left(\lambda_{S_T} - \theta\lambda_P - (1-\theta)\lambda_Q\right)}{\omega_{v_2}}\right)\right\}. \end{aligned} \tag{4.11}$$

□

This exhibits the uniqueness of the optimal control of the system (2.1) - (2.6), (4.7), and (4.8) with characterization (4.11).

5. Numerical Simulations

In this section numerical simulations are performed. The parameter values used are listed in Table 1. The mortality rate is assumed by a life expectancy of 70 years; hence $\mu = \frac{1}{(70\times365)} = 4\times10^{-5}$. Firstly, we simulate the smoke-free or non-endemic equilibrium stability result by assuming $\alpha = 0.000014$, thus the basic reproduction number is $\Re_0 = 0.43074 < 1$. The initial condition of the state variables is chosen within the feasible region (2.9) with $P(0) = 5000, Q(0) = 50$, and we assume that $B_U(0) = 60$, $B_T(0) = 60$, $S_U(0) = 800$, and $S_T(0) = 280$, so the total $N(0) = \frac{\Lambda}{\mu} = \frac{0.25}{4\times10^{-5}}$ based on [13]. The dynamic of the state variables is depicted in Figure 2; the time $t_f = 10\times10^4$ is chosen in order to capture the dynamics until the equilibrium state is achieved at $E(P, B_U, B_T, S_U, S_T, Q) = (\frac{\Lambda}{\mu}, 0, 0, 0, 0, 0)$ as $t \to \infty$. This shows that the smoke-free equilibrium is stable (globally asymptotically stable).

Table 1: Table of parameters and descriptions

Parameter	Description	Value	References
Λ	Recruitment rate	0.25	[13]
μ	Natural death rate	4×10^{-5}	[13]
α	The infection rate of smoking $P - B_U$	0.00014	[26]
β	The infection rate of smoking $P - S_U$	8×10^{-4}	[18]
δ	The infection rate of smoking $B_U - S_U$	0.0024	assumed
σ	The rate of the desire to stop smoking	0.0031	[26]
r_1	Constant prevention rate	0.2	assumed
r_2	Constant treatment rate	0.4	assumed
ω_{B_T}	Weight factor for the treated beginner	0.091	[26]
ω_{S_T}	Weight factor for the treated smoker	0.001	[26]
ω_{v_1}	Campaign level of acceptance for beginner	0.02	[26]
ω_{v_2}	Therapy level of acceptance for smoker	0.1	[26]

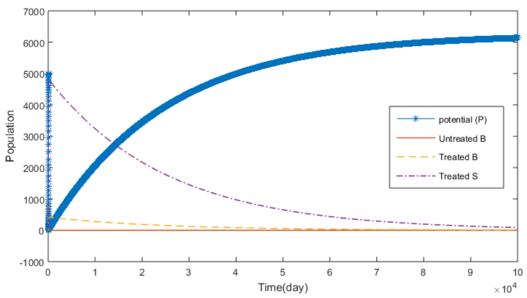


Figure 2: Smoke free, non-endemic equilibrium state, asymptotically stable.

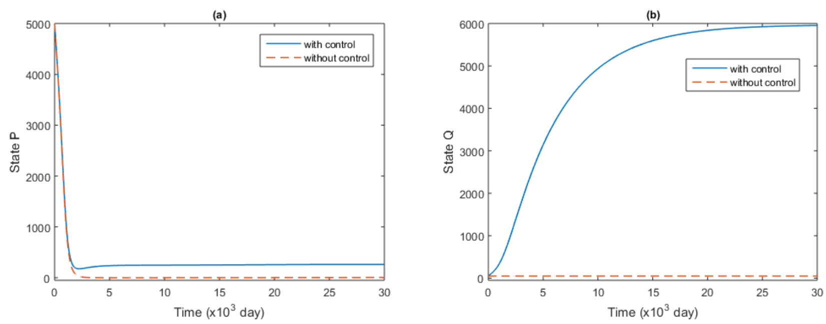


Figure 3: Variations of Potentials (P) smokers and Quitters populations for system with and without controls.

To illustrate the effect of the cessation program, we performed numerical simulations of the optimal control problem by applying the forward-backward sweep Runge-Kutta iterative method [11], [9]. The results are compared with the case when the treatments are ineffective, i.e. $v_1(t) = v_2(t) = 0$. We assumed that the weight factor ωv_2 associated with control $v_2(t)$ is greater than ωv_1 . This assumption is based on the fact that education counselling (campaign) is also included in the counselling with nicotine therapy $v_2(t)$. Using the parameter values in Table 1, the basic reproduction number obtained is $\mathfrak{R}_0 = 4.3074 > 1$; hence an endemic occurs. The simulations are carried out until time $t_f = 30 \times 10^3 days$ in order to visualize the asymptotic global stability of the endemic equilibrium point.

The numerical simulations are conducted in two situations, $\varphi < \theta$ and $\varphi > \theta$. For the first scenario we set: $\varphi = 0.2$, and $\theta = 0.4$. In other words, the number of treated beginners who quit permanently is higher than the number of treated smokers, which means the educational campaign is more effective. Results on changes in the number of potential (P) and permanent quitter (Q) are shown in Figure 3. Clearly the combined optimal control can significantly reduce the number of smokers (S) and beginners (B) as shown in Figure 4. The state variables values at the end are $P^* = 262$, $B^*_U = 17$, $B^*_T = 4$, $S^*_U = 2$, $S^*_T = 10$, $Q^* = 5954$. In this case the cost is $J = 341.239$.

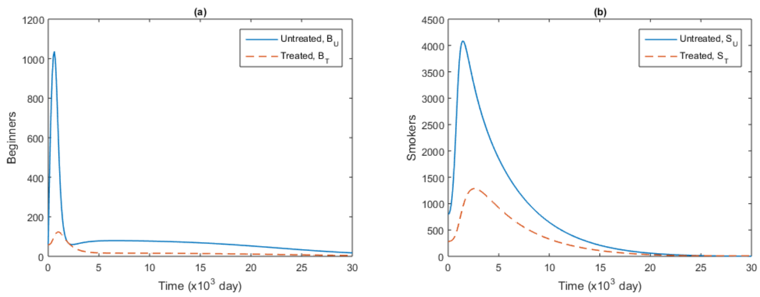


Figure 4: Variations of untreated and treated Beginners (B) and untreated and treated Smoker (S) populations

Figure 5 depicts the total population in the beginners $(B_U + B_T)$ and smokers group, $(S_U + S_T)$. Here the beginner subclass decreases faster than the smoker subclass, since the portion of untreated-smokers who acquire the treatment $(r_2 = 0.5)$ is higher than those from untreated-beginners $(r_1 = 0.2)$. In addition, it clearly shows the effectiveness of the cessation program where the number of individuals both the beginner and smoker groups increases rapidly in the beginning; then, it decreases sharply going to its stable state. While without the cessation programs, the number of smokers rise continuously to 5941 and beginners reached 267. The corresponding control variables, $v_1(t)$ and $v_2(t)$, both take maximum values, are shown in Figure 6.

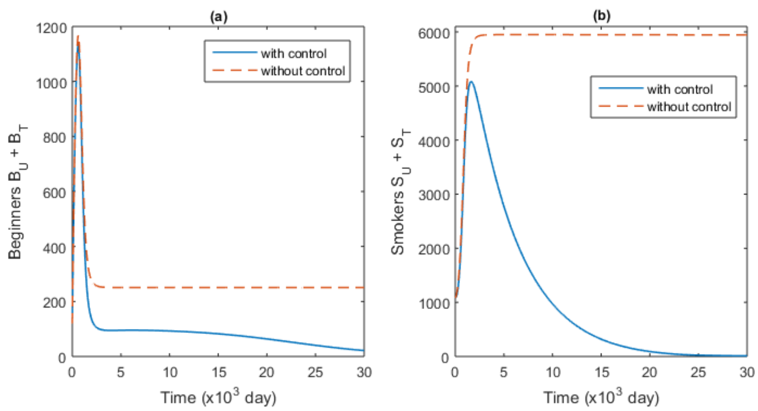


Figure 5: Variations of Total Beginners and Smokers for Case 1.

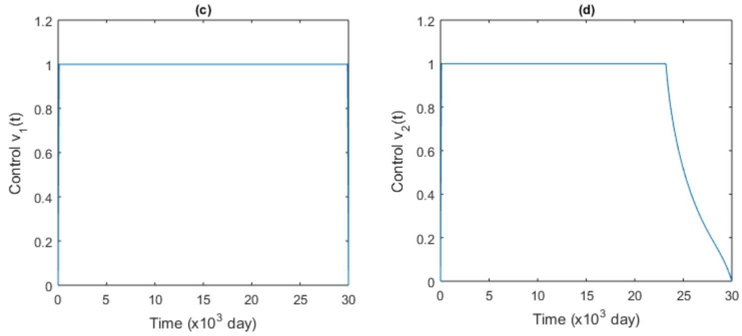


Figure 6: Corresponding Optimal Control. Case 1.

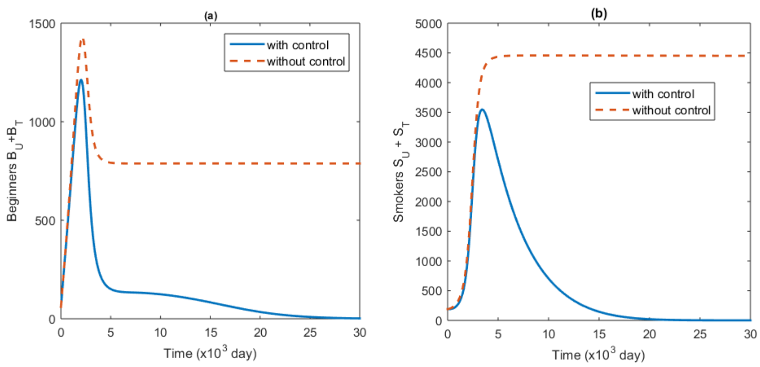
In the second scenario, the portion of treated-beginners returning to a potential smoker is set higher ($\varphi = 0.6$) or those who transfer to permanent quitter are less compared to the effectiveness of the treatment in the treated-smokers group with $\theta = 0.4$. The result shows slightly less achievement, where the total number of beginners and smoker at the end of the program is 28 and 14, respectively. The result from the two cases is summarised in Table 2. It shows that the program is more effective in Case 1 where $\varphi < \theta$.

Table 2: Group size at end of program

Population Group	Case 1 $\varphi = 0.2$ $\theta = 0.4$	Case 2 $\varphi = 0.6$ $\theta = 0.4$	Case 3 $r_1 = 0.4, r_2 = 0.5$ $\varphi = 0.2, \theta = 0.4$
Potential (P)	262	317	375
Beginners (B)	22	28	2
Smokers (S)	12	14	8
Quitters (Q)	5954	5891	5865

We further observe the Case 1 when we increase the portion of beginners who acquire treatment to $r_1 = 0.4$, and $\mathfrak{R}_0$ is now reduced to 2.1705. The result is given in column 4 in Table 2, namely Case 3. We obtained better results; the number of beginners and smokers was reduced to 2 and 8, respectively. However, this incurs more cost with $J = 455.969$. Illustration of the numerical results of Case 3 are shown in Figure 7.

Figure 7 shows the total population of smokers first increases rapidly, then decays significantly compared to when the treatments are ineffective or without control. Hence, we observe the control strategy's efficiency in successfully eliminating the number of beginners and smokers. We obtained that both interventions must be at the highest possible effort from the beginning of the program.



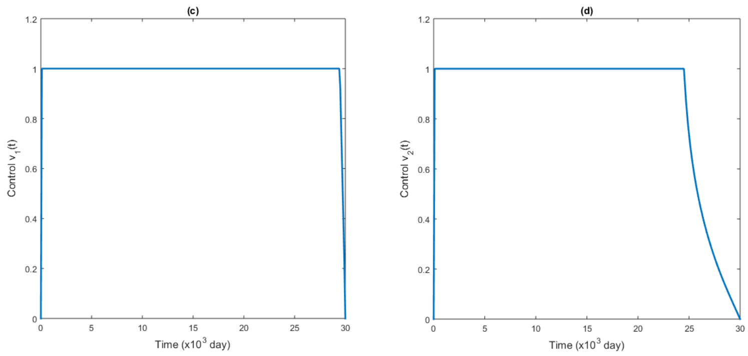


Figure 7: Case 3. Top: Total B and total S, with control and without control. Bottom: the corresponding optimal controls.

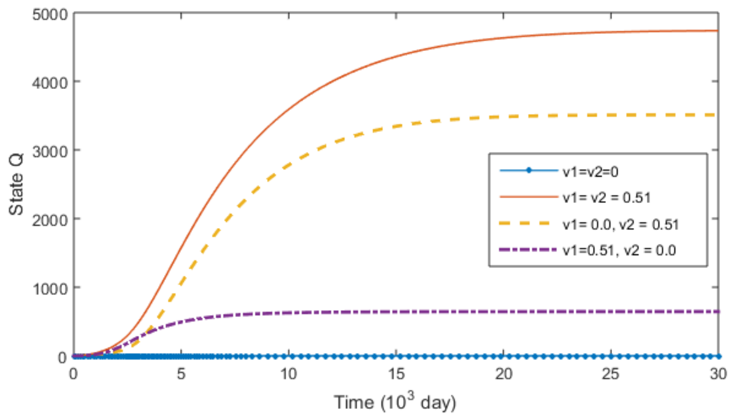


Figure 8: The effect of mono-therapy and combination treatments on the number of quitters.

To visualise which combination of treatments is most effective, we compare the mono-therapy of educational campaign treatment, the therapy of counselling with nicotine replacement, and the combination treatment of educational campaign and counselling with nicotine replacement. For this, we define the overall efficacy which equals to $\bar{\epsilon} = 1 - (1 - \nu_1)(1 - \nu_2)$ with $\bar{\epsilon} = 0.51$. The result exhibited in Figure 8 confirms that the therapy of counselling with nicotine replacement is effective compared to educational campaign. However, combined therapies of educational campaign and nicotine replacement are the most effective for smoking cessation programs.

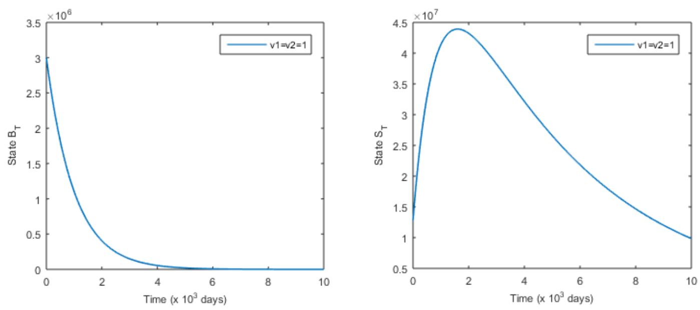


Figure 9: Decreasing number of B_T and S_T ($\nu_1 = \nu_2 = 1$).

In Figure 9, we provide an illustration of decreasing $B_T(t)$ and $S_T(t)$ where the initial value of the variables are from the Ministry of Health, Indonesia, 2019. Namely, $P(0) = 120 \times 10^6$, the number of smokers is 70.4 million, of which 9.1% ($\approx$ 6.4 million) are beginners under the age of 18 years. We assume $B_U(0) = 3.4 \times 10^6$, then $B_T(0) = 3 \times 10^6$, $S_U(0) = 51.2 \times 10^6$, $S_T(0) = 12.8 \times 10^6$ and $Q(0) = 9.6 \times 10^6$. We apply control treatments at maximum level $\nu_1 = \nu_2 = 1$ for approximately 27.4 years. S_T increases in the beginning due to transfer from S_U, then declines to 10 million or 78%. The smoking behaviour of beginners are eliminated within 6000 days.

6. Conclusion

This work proposed a dynamic model for smoking control strategies. We analysed the responses of two subclasses, beginners and smokers, in acquiring therapy programs that combine educational campaigns and counselling with nicotine replacement therapy. In offering the programs, only some beginners and smokers were willing to participate in the treatment. We considered those individuals from beginner and smoker groups that did not acquire the treatments. The existence of equilibrium for the smoking individuals and the optimal control to reach the equilibrium were obtained. Numerical results show that the combined cessation program is sufficient to achieve the minimum of both beginner and smoker populations and a maximum number of non-smokers and quitters in a community. We concluded that the control program following this strategy could effectively reduce the population of both treated beginners and treated smokers. By putting maximum effectiveness in treating the beginner and smoker groups continuously for a long period, we obtained high efficiency in reducing the total number of smokers and thus increase the number of quitters. Conversely, if control is not applied, the number of smokers will not decrease.

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