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Combination complex synchronization among three incommensurate fractional-order chaotic systems

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Abstract

The problem of combination complex synchronization among three incommensurate fractional-order chaotic systems is considered. Based on the stability theory of incommensurate fractional-order systems and the feedback control technique, some robust criteria on combination complex synchronization are presented. Notably, the proposed combination complex synchronization can establish a link between the incommensurate fractional-order complex chaos and real chaos. Moreover, three numerical simulations are provided, which agree well with the theoretical analysis.

Keywords: Combination complex synchronization, chaotic complex system, fractional-order system, feedback control technique.
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1. Introduction

Since the pioneering work of Pecora and Carroll [33], chaos synchronization, as a very hot research topic in nonlinear science, has been extensively investigated for its potential applications in many disciplines, especially in some areas closely related to our real life, such as secure communications, neural dynamics, mechanical engineering, neural dynamics, mechanical engineering, and

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image encryption [1, 7, 11, 32, 40]. So a substantial amount of synchronization phenomena have been reported, for instance, complete synchronization [24], anti-synchronization [16], phase and anti-phase synchronization [3, 34], lag synchronization [19], projective synchronization [30], etc. With in-depth study of fractional-order calculus, synchronization of fractional-order chaotic systems starts to attract increasing attention among researchers. Most of synchronization schemes available for integer-order chaotic systems have been verified to be effective in synchronizing fractional-order chaotic systems, such as [2, 4, 8, 10, 35, 39] to name a few.

In most of the above-mentioned works, many researchers focus on the usual drive-response synchronization based on one drive system and one response system. Recently, Luo et al. [27] presented a novel form of synchronization named combination synchronization, in which three classical chaotic systems were made to synchronize simultaneously via systematically designed nonlinear controllers. The implication of combination synchronization for secure communications is such that a signal can be split into two, each loaded and transmitted between two drive systems or at different intervals. Further developments in this direction are reported in [36–38], for example, combination complex synchronization, combination-combination synchronization, compound synchronization, and so forth.

However, to the best of our knowledge, most of the studies about the synchronization of muti-chaotic systems mainly focus on the integer-order chaotic systems, not involving fractional-order systems with complex variables. Complex variables increase the contents and security of the transmitted information so that complex chaotic systems can be widely studied for applications in secure communication [22, 23, 29]. In addition, compared with the integer-order chaotic systems, the fractional-order chaotic systems can have higher nonlinearity and spreading power spectrum. Consequently, more and more authors have paid attention to study fractional-order complex chaotic systems in recent years. Luo and Wang proposed the fractional-order complex Lorenz system, complex Chen system and studied their synchronization in [25, 26]. Liu et al. [21] introduced the fractional-order complex T system and further achieved its function projective synchronization. The fractional-order complex Lü system was presented and its anti-synchronization was realized in [12]. Liu [20] achieved complex modified hybrid projective synchronization between fractional-order complex chaos and real hyper-chaos by designing nonlinear controllers. Mahmoud et al. [28] presented the generalization of combination-combination synchronization among n-dimensional fractional-order dynamical systems. As introduced in [13], Jiang et al. studied combination complex synchronization among three different dimensional fractional-order chaotic systems with commensurate orders, where the state variables of two drive systems and one response system synchronize up to two complex scaling matrices. Nevertheless, in the aforementioned literature [12, 13, 20, 21, 25, 26, 28], authors are all concerned with the synchronization for the commensurate fractional-order complex chaotic systems. As a matter of fact, there exist many incommensurate fractional-order systems in practical applications. These above-mentioned synchronization schemes will not be approached under the effects of incommensurate orders. In a recent paper [14], complex modified projective synchronization was discussed between two incommensurate fractional-order chaotic systems. As yet, there are few results regarding the study of synchronization for fractional-order chaotic complex systems with incommensurate orders.

Inspired by the above discussions, in this paper, we have made an endeavor to study and analyze combination complex synchronization among three incommensurate fractional-order chaotic nonlinear systems. On the basis of the stability theory of incommensurate fractional-order systems, and by employing the feedback control technique, the corresponding nonlinear controllers are designed. By virtue of two complex scaling matrices, we can realize combination complex synchronization between the incommensurate fractional-order real chaos and complex chaos. As a generalization of several synchronization schemes, combination complex synchronization for incommensurate fractional-order complex chaotic systems including combination synchronization, complex generalized projective synchronization, complex projective synchronization, projective synchronization, just to enumerate a few examples. As a result, our work will extend previous results.

The structure of this paper is as follows. In the next section, some preliminaries are presented. Combination complex synchronization of the incommensurate fractional-order chaotic systems is discussed in detail in Sections 3-5. Finally, conclusions are outlined in Section 6.

2. Preliminaries

Fractional calculus can be developed through various definitions of derivatives such as Riemann-Liouville, Grünwald, and Caputo differential operators. In this paper, we use the Caputo definition [15] to describe the fractional-order systems and our computation scheme is based on the so-called "Adams-Bashforth-Moulton scheme" [6] which is suitable for theoretical analysis.

Definition 2.1. The Caputo fractional derivative of order $\alpha \in \mathbb{R}$ is defined as follows:

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad t > a, $$

where $n = \min\{k \in \mathbb{N}/k > \alpha\}$, $\Gamma(\cdot)$ is the Gamma function, and ${}_aD_t^\alpha$ is generally called α-order Caputo differential operator. In the sequel, the notation d^α/dt^α is chosen as ${}_aD_t^\alpha$ and we mainly consider the order $0 < \alpha < 1$.

In order to obtain our desired results, it is necessary to introduce the main stability properties of the linear fractional-order system. Considering the following linear fractional-order system:

$$ \frac{d^\alpha}{dt^\alpha} x = \Phi x, \qquad x(0) = x_0, \tag{2.1} $$

where $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_n]$ denotes the fractional orders, $d^\alpha/dt^\alpha = [d^{\alpha_1}/dt^{\alpha_1}, d^{\alpha_2}/dt^{\alpha_2}, \cdots, d^{\alpha_n}/dt^{\alpha_n}]$, the state vector $x \in \mathbb{R}^n$, the matrix $\Phi \in \mathbb{R}^{n\times n}$.

When $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, we have the following stability of the fractional-order system (2.1) which was introduced in [31].

Lemma 2.2. *When $\alpha_1 = \alpha_2 = \cdots = \alpha_n$, the fractional-order system* (2.1) *is asymptotically stable if $|\arg(\lambda_l(\Phi))| > \alpha\pi/2$, where $\arg(\lambda_l(\Phi))$ denotes the argument of the eigenvalue λ_l of Φ. In this case, the components of the state decay towards 0 like $t^{-\alpha}$.*

When $\alpha_1, \alpha_2, \cdots, \alpha_n$ are rational positive numbers, Deng et al. [5] explored the following stability of fractional-order system (2.1).

Lemma 2.3. *Assume that α_i's are rational numbers between 0 and 1, for $i = 1, 2, \cdots, n$. Let $\gamma = 1/m$, where m is the least common multiple of the denominators m_i of α_i's, $\alpha_i = k_i/m_i$, $k_i, m_i \in \mathbb{N}$, $i = 1, 2, \cdots, n$. Then system* (2.1) *is asymptotically stable if all roots λ of the equation $\det(\mathrm{diag}(\lambda^{m\alpha_1}, \lambda^{m\alpha_2}, \cdots, \lambda^{m\alpha_n}) - \Phi) = 0$ satisfy $|\arg(\lambda)| > \gamma\pi/2$.*

3. Combination complex synchronization between two fractional-order real chaotic drive systems and one fractional-order complex chaotic response system

3.1. Mathematical model and problem descriptions

Consider a fractional-order real chaotic system as the first drive system

$$\frac{d^\alpha}{dt^\alpha}\omega = p(\omega), \tag{3.1}$$

the second real chaotic drive system is given as:

$$\frac{d^\alpha}{dt^\alpha}\upsilon = q(\upsilon), \tag{3.2}$$

while a fractional-order complex chaotic response system is configured as

$$\frac{d^\alpha}{dt^\alpha}z = h(z) + U, \tag{3.3}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ and $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)^T$ are real state vectors of systems (3.1) and (3.2), respectively, $z = z^r + jz^i \in \mathbb{C}^{n\times 1}$ is a complex state vector of system (3.3), $p = (p_1, p_2, \cdots, p_n)^T$ and $q = (q_1, q_2, \cdots, q_n)^T$ are vectors of nonlinear real functions, $h = (h_1, h_2, \cdots, h_n)^T$ is a nonlinear complex function, and U is a controller to be designed, $U = (U_1, U_2, \cdots, U_n)^T = U^r + jU^i$, $U^r = (u_1, u_3, \cdots, u_{2n-1})^T$, $U^i = (u_2, u_4, \cdots, u_{2n})^T$. Superscripts r and i stand for the real and imaginary parts of a state complex vector.

Definition 3.1. Two drive systems (3.1) and (3.2) are said to exhibit combination complex synchronization with the response system (3.3) if there exist two complex matrices $A = A^r + jA^i \in \mathbb{C}^{n\times n}$ and $B = B^r + jB^i \in \mathbb{C}^{n\times n}$, such that

$$\lim_{t\to\infty} ||e(t)|| = \lim_{t\to\infty} ||z(t) - A\omega(t) - B\upsilon(t)|| = 0,$$

where $||.||$ is the Euclidean norm, $e = e^r + je^i$ is called the error vector, $e^r = (e_1, e_3, \ldots, e_{2n-1})^T = z^r - A^r\omega - B^r\upsilon$, $e^i = (e_2, e_4, \ldots, e_{2n})^T = z^i - A^i\omega - B^i\upsilon$.

For the sake of convenience, we suppose $A = A^r + jA^i =$diag$(\alpha_1 + j\alpha_2, \alpha_3 + j\alpha_4, \cdots, \alpha_{2n-1} + j\alpha_{2n})$ and $B = B^r + jB^i =$diag$(\beta_1 + j\beta_2, \beta_3 + j\beta_4, \cdots, \beta_{2n-1} + j\beta_{2n})$ in our synchronization scheme. If there exists $z_l \in R$, then $\alpha_{2l} = \beta_{2l} = 0$ are chosen to avoid increasing a new imaginary part in the response system ($l = 1, 2, \cdots, n$).

Remark 3.2. Most of the classical fractional-order real chaotic systems can be written as the form of system (3.1), such as the fractional-order Lorenz system [9], the fractional-order Chen system [17], the fractional-order Rössler system [18], and so forth. Several fractional-order complex chaotic systems can also be described by (3.3), such as the fractional-order complex Lorenz system [25], the fractional-order complex Chen system [26], the fractional-order complex T system [21], the fractional-order complex Lü system [12], and so on.

Remark 3.3. In Definition 3.1, A and B are often called the scaling matrices. Moreover, two drive systems in the scheme of combination complex synchronization can be completely identical or different.

Remark 3.4. The combination complex synchronization may be considered as a generalization of several synchronization schemes in the literature, for instance, if the scaling matrices $A^i = O$ and $B^i = O$, then combination synchronization can be carried out. If $A = O$ or $B = O$, then we can achieve complex projective synchronization. If $A = O$ and $B = O$, then synchronization problem will degenerate into the chaos control problem.

The main aim of this paper is to design proper controllers to approach combination complex synchronization among three incommensurate fractional-order chaotic systems. Next, we consider that the incommensurate fractional-order real Chen system and real Rössler system drive fractional-order complex Lü system.

3.2. *Synchronization of fractional-order real Chen system, real Rössler system, and complex Lü system*

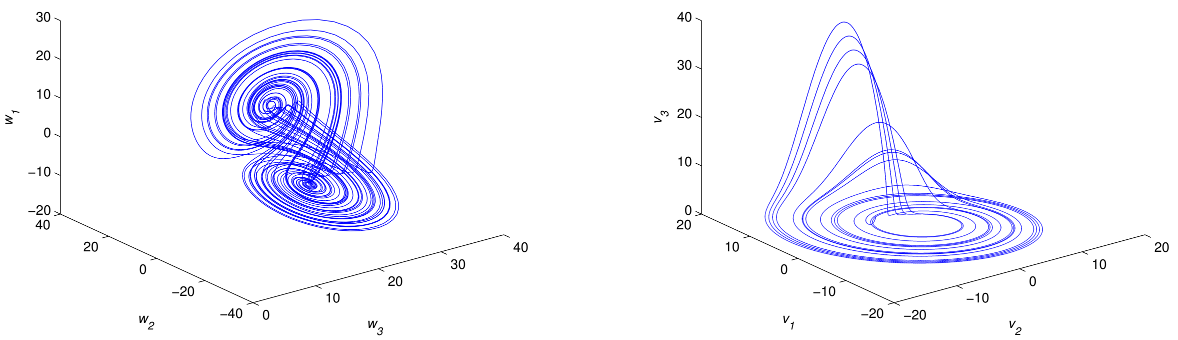
The first drive system is the incommensurate fractional-order real Chen system described by

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}\omega_1 = d_1(\omega_2 - \omega_1), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}\omega_2 = (d_2 - d_1)\omega_1 - \omega_1\omega_3 + d_2\omega_2, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}\omega_3 = \omega_1\omega_2 - d_3\omega_3, \end{cases} \tag{3.4}$$

and the incommensurate fractional-order real Rössler system as the second drive system is depicted as follows:

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}\upsilon_1 = -(\upsilon_2 + \upsilon_3), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}\upsilon_2 = \upsilon_1 + \delta_1\upsilon_2, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}\upsilon_3 = \delta_2 + \upsilon_3(\upsilon_1 - \delta_3), \end{cases} \tag{3.5}$$

where ω_i and υ_i are real state variables, d_i and δ_i are real parameters ($i = 1, 2, 3$). When $(\alpha_1, \alpha_2, \alpha_3) = (0.91, 0.94, 0.96)$, $(d_1, d_2, d_3) = (35, 28, 3)$, and $(\delta_1, \delta_2, \delta_3) = (0.4, 0.2, 10)$, systems (3.4) and (3.5) can generate chaotic attractors as displayed in Fig. 1.



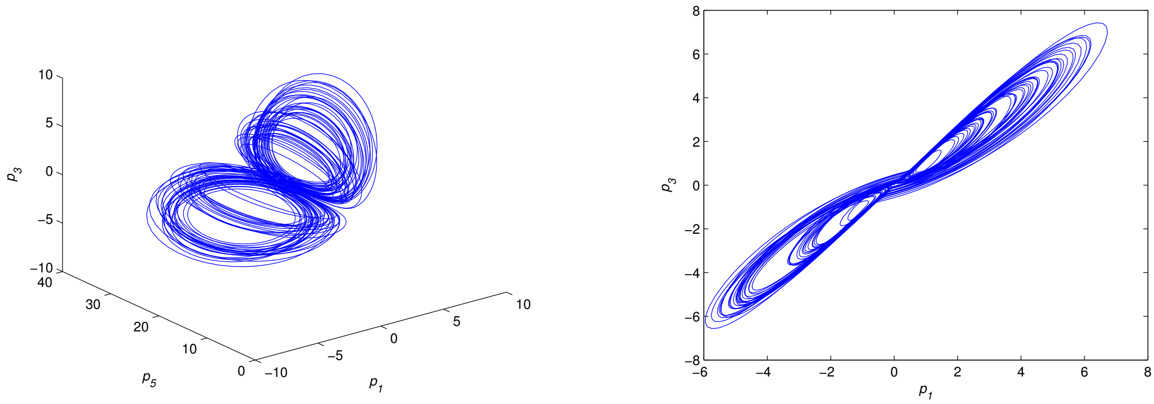
(a) Attractor of the first drive system (3.4)　　(b) Attractor of the second drive system (3.5)

Fig. 1: Chaotic attractors of two drive systems (3.4) and (3.5).

The response system is the incommensurate fractional-order complex Lü system:

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}z_1 = c_1(z_2 - z_1) + U_1, \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}z_2 = c_2z_2 - z_1z_3 + U_2, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}z_3 = \frac{1}{2}(\bar{z}_1z_2 + z_1\bar{z}_2) - c_3z_3 + U_3, \end{cases} \tag{3.6}$$

where $z_1 = p_1 + jp_2$, $z_2 = p_3 + jp_4$ are complex variables and $z_3 = p_5$ is a real variable, c_i are real parameters $(i = 1, 2, 3)$, $U_1 = u_1 + ju_2$, $U_2 = u_3 + ju_4$ are complex functions and $U_3 = u_5$ is a real control function. The parameter values of system (3.6) are set as $c_1 = 42$, $c_2 = 22$, $c_3 = 5$ to ensure the chaotic motion with incommensurate fractional orders $\alpha_1 = 0.91$, $\alpha_2 = 0.94$, $\alpha_3 = 0.96$, see Fig. 2.



(a) (b)

Fig. 2: Chaotic attractor and phase portrait of the response system (3.6).

For two given scaling matrices $A = \mathrm{diag}(\alpha_1 + j\alpha_2, \alpha_3 + j\alpha_4, \alpha_5)$ and $B = \mathrm{diag}(\beta_1 + j\beta_2, \beta_3 + j\beta_4, \beta_5)$, we define the synchronization error as follows:

$$\begin{cases} e_1 = p_1 - \alpha_1\omega_1 - \beta_1\upsilon_1, \\ e_2 = p_2 - \alpha_2\omega_1 - \beta_2\upsilon_1, \\ e_3 = p_3 - \alpha_3\omega_2 - \beta_3\upsilon_2, \\ e_4 = p_4 - \alpha_4\omega_2 - \beta_4\upsilon_2, \\ e_5 = p_5 - \alpha_5\omega_3 - \beta_5\upsilon_3. \end{cases}$$

Combining systems (3.4), (3.5), with (3.6), one can get

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}e_1 = c_1e_3 - c_1e_1 + (c_1\alpha_3 - d_1\alpha_1)\omega_2 + (d_1 - c_1)\alpha_1\omega_1 + (\beta_1 + \beta_3c_1)\upsilon_2 \\ \qquad\qquad + \beta_1(\upsilon_3 - c_1\upsilon_1) + u_1, \\ \frac{d^{\alpha_1}}{dt^{\alpha_1}}e_2 = c_1e_4 - c_1e_2 + (c_1\alpha_4 - d_1\alpha_2)\omega_2 + (d_1 - c_1)\alpha_2\omega_1 + (\beta_2 + \beta_4c_1)\upsilon_2 \\ \qquad\qquad + \beta_2(\upsilon_3 - c_1\upsilon_1) + u_2, \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}e_3 = c_2e_3 + (c_2 - d_2)\alpha_3\omega_2 - (d_2 - d_1)\alpha_3\omega_1 - \beta_3\upsilon_1 + (c_2 - \delta_1)\beta_3\upsilon_2 - p_1p_5 \\ \qquad\qquad + \alpha_3\omega_1\omega_3 + u_3, \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}e_4 = c_2e_4 + (c_2 - d_2)\alpha_4\omega_2 - (d_2 - d_1)\alpha_4\omega_1 - \beta_4\upsilon_1 + (c_2 - \delta_1)\beta_4\upsilon_2 - p_2p_5 \\ \qquad\qquad + \alpha_4\omega_1\omega_3 + u_4, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}e_5 = -c_3e_5 + (d_3 - c_3)\alpha_5\omega_3 + (\delta_3 - c_3 - \upsilon_1)\beta_5\upsilon_3 - \beta_5\delta_2 - \alpha_5\omega_1\omega_2 + p_1p_3 \\ \qquad\qquad + p_2p_4 + u_5. \end{cases}$$

Then, we define active control functions $u_i(t)$ $(i = 1, 2, \cdots, 5)$ as

$$\begin{cases} u_1(t) = (d_1\alpha_1 - c_1\alpha_3)\omega_2 - (d_1 - c_1)\alpha_1\omega_1 - (\beta_1 + \beta_3c_1)\upsilon_2 - \beta_1(\upsilon_3 - c_1\upsilon_1) + \varphi_1(t), \\ u_2(t) = (d_1\alpha_2 - c_1\alpha_4)\omega_2 - (d_1 - c_1)\alpha_2\omega_1 - (\beta_2 + \beta_4c_1)\upsilon_2 - \beta_2(\upsilon_3 - c_1\upsilon_1) + \varphi_2(t), \\ u_3(t) = (d_2 - c_2)\alpha_3\omega_2 + (d_2 - d_1)\alpha_3\omega_1 + \beta_3\upsilon_1 - (c_2 - \delta_1)\beta_3\upsilon_2 + p_1p_5 - \alpha_3\omega_1\omega_3 + \varphi_3(t), \\ u_4(t) = (d_2 - c_2)\alpha_4\omega_2 + (d_2 - d_1)\alpha_4\omega_1 + \beta_4\upsilon_1 - (c_2 - \delta_1)\beta_4\upsilon_2 + p_2p_5 - \alpha_4\omega_1\omega_3 + \varphi_4(t), \\ u_5(t) = (c_3 - d_3)\alpha_5\omega_3 - (\delta_3 - c_3 - \upsilon_1)\beta_5\upsilon_3 + \beta_5\delta_2 + \alpha_5\omega_1\omega_2 - p_1p_3 - p_2p_4 + \varphi_5(t), \end{cases}$$

which provides

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}e_1 = c_1e_3 - c_1e_1 + \varphi_1(t), \\ \frac{d^{\alpha_1}}{dt^{\alpha_1}}e_2 = c_1e_4 - c_1e_2 + \varphi_2(t), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}e_3 = c_2e_3 + \varphi_3(t), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}e_4 = c_2e_4 + \varphi_4(t), \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}e_5 = -c_3e_5 + \varphi_5(t). \end{cases} \tag{3.7}$$

The above synchronization error system is a linear system with active control inputs $\varphi_i(t)$ $(i = 1, 2, \cdots, 5)$. Next we design an appropriate feedback control to stabilize the synchronization system. There are many possible choices for the control inputs $\varphi_i(t)$ $(i = 1, 2, \cdots, 5)$. In view of the fact that the term $\varphi_i(t)$ are linear functions, we set

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \end{pmatrix} = L \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{pmatrix},$$

where $L = (l_{ij})_{5\times 5}$ is a 5×5 real matrix. So system (3.7) can be rewritten as

$$\frac{d^\alpha}{dt^\alpha}e = \Phi e,$$

where

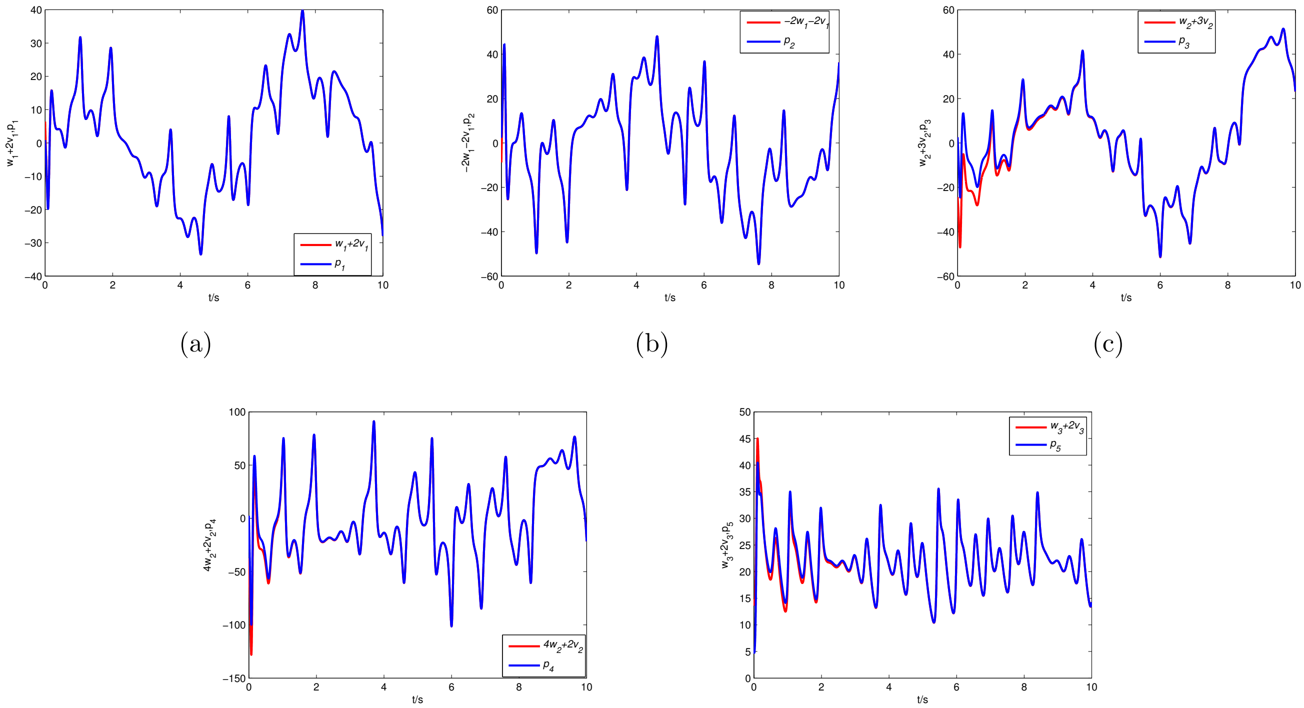
$$\Phi = \begin{pmatrix} l_{11} - c_1 & l_{12} & l_{13} + c_1 & l_{14} & l_{15} \\ l_{21} & l_{22} - c_1 & l_{23} & l_{24} + c_1 & l_{25} \\ l_{31} & l_{32} & l_{33} + c_2 & l_{34} & l_{35} \\ l_{41} & l_{42} & l_{43} & l_{44} + c_2 & l_{45} \\ l_{51} & l_{52} & l_{53} & l_{54} & l_{55} - c_3 \end{pmatrix}.$$

To make the error system stable, the matrix L should be selected in such a way that the feedback system satisfies conditions of Lemma 2.3. There is not a unique choice for such matrix L, a good choice can be as follows: $l_{13} = l_{24} = -42$, $l_{22} = 2$, $l_{33} = -24$, $l_{44} = -26$, $l_{55} = 2$, $l_{11} = l_{12} = l_{14} = l_{15} = l_{23} = l_{25} = l_{34} = l_{35} = l_{45} = 0$, and other elements of L can be chosen arbitrarily. Taking $(\alpha_1, \alpha_2, \alpha_3) = (0.91, 0.94, 0.96)$, $m = 100$, and $\gamma = 1/100$, we obtain

$$(\lambda^{91} + 42)(\lambda^{91} + 40)(\lambda^{94} + 2)(\lambda^{94} + 4)(\lambda^{96} + 3) = 0. \tag{3.8}$$

By a simple calculation, we conclude that all roots of (3.8) lie in the region $|\arg(\lambda)| > \gamma\pi/2$. It follows from Lemma 2.3 that the error vector $e(t)$ asymptotically converges to zero as $t \to \infty$.

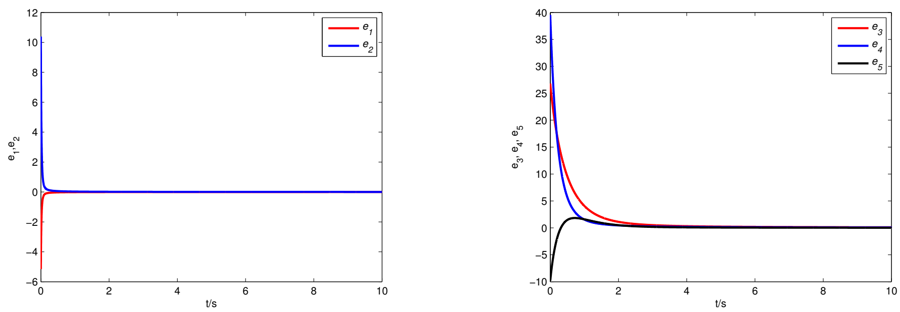
In the numerical simulations, the initial values of systems (3.4), (3.5), and (3.6) are chosen as $(\omega_1, \omega_2, \omega_3)^T = (3, -6, 9)^T$, $(\upsilon_1, \upsilon_2, \upsilon_3)^T = (2, -6, 3)^T$, and $(z_1, z_2, z_3)^T = (1 + 2j, 3 + 4j, 5)^T$, respectively. Taking two scaling matrices as $A = \mathrm{diag}(1-2j, 1+4j, 1)$ and $B = \mathrm{diag}(2-2j, 3+2j, 2)$, we obtain the initial errors as $(-6 + 12j, 27 + 40j, -10)^T$. Further, selecting $l_{31} = l_{42} = l_{53} = 1$, $l_{21} = l_{32} = l_{41} = l_{43} = l_{51} = l_{52} = l_{54} = 0$, we have the corresponding numerical results as displayed in Figs. 3-4. Fig. 3 depicts the state variables of two drive systems and the response system. From Fig. 4, it can be observed that the errors of synchronization converge asymptotically to zero. Therefore, the fractional-order real Chen system, real Rössler system, and complex Lü system with incommensurate order achieve combination complex synchronization.



(a) (b) (c)

(d) (e)

Fig. 3: State variables of drive systems (3.4), (3.5), and response system (3.6).



(a) (b)

Fig. 4: The time evolution of synchronization errors between drive systems (3.4), (3.5), and response system (3.6).

4. Combination complex synchronization between two fractional-order complex chaotic drive systems and one fractional-order real response system

4.1. Mathematical model and problem descriptions

Two fractional-order complex chaotic systems take the following form:

$$\frac{d^\alpha}{dt^\alpha}x = f(x) \tag{4.1}$$

and

$$\frac{d^\alpha}{dt^\alpha}y = g(y), \tag{4.2}$$

while a fractional-order real chaotic response system is assumed as

$$\frac{d^{\alpha}}{dt^{\alpha}}\upsilon = q(\upsilon) + U, \tag{4.3}$$

where $x = x^r + jx^i \in \mathbb{C}^{n\times 1}$ and $y = y^r + jy^i \in \mathbb{C}^{n\times 1}$ are complex state vectors of systems (4.1) and (4.2), respectively, $\upsilon = (\upsilon_1, \upsilon_2, \ldots, \upsilon_n)^T$ is a real state vector of system (4.3), $f = (f_1, f_2, \cdots, f_n)^T$ and $g = (g_1, g_2, \cdots, g_n)^T$ are vectors of nonlinear complex functions, $q = (q_1, q_2, \cdots, q_n)^T$ is a nonlinear real function.

As $\upsilon(t)$ is real, a real U is chosen to ensure combination complex synchronization of real parts and avoid increasing the imaginary parts of response system. And in this case, the error vector with two given scaling matrices $A = A^r + jA^i \in \mathbb{C}^{n\times n}$ and $B = B^r + jB^i \in \mathbb{C}^{n\times n}$ is defined as

$$e = \upsilon - A^r x^r + A^i x^i - B^r y^r + B^i y^i.$$

Therefore, two complex drive systems (4.1), (4.2), and one real response system (4.3) are combination complex synchronization of real parts.

In what follows, our aim is to approach combination complex synchronization among the fractional-order complex Lorenz system, complex Chen system, and real Rössler system with incommensurate orders.

4.2. Synchronization of fractional-order complex Lorenz system, complex Chen system, and real Rössler system

Consider the fractional-order complex Lorenz system as the first drive system

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}x_1 = a_1(x_2 - x_1), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}x_2 = a_2x_1 - x_2 - x_1x_3, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}x_3 = \frac{1}{2}(\bar{x}_1x_2 + x_1\bar{x}_2) - a_3x_3, \end{cases} \tag{4.4}$$

and the fractional-order complex Chen system as the second drive system is depicted as follows:

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}y_1 = b_1(y_2 - y_1), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}y_2 = (b_2 - b_1)y_1 + b_2y_2 - y_1y_3, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}y_3 = \frac{1}{2}(\bar{y}_1y_2 + y_1\bar{y}_2) - b_3y_3, \end{cases} \tag{4.5}$$

where $x_1 = m_1 + jm_2$, $x_2 = m_3 + jm_4$, $y_1 = s_1 + js_2$, and $y_2 = s_3 + js_4$ are complex variables, $x_3 = m_5$ and $y_3 = s_5$ are real variables, a_i and b_i are real parameters $(i = 1, 2, 3)$. When $(\alpha_1, \alpha_2, \alpha_3) = (0.91, 0.96, 0.99)$, $(a_1, a_2, a_3) = (10, 180, 1)$, $(b_1, b_2, b_3) = (35, 28, 3)$, systems (4.4) and (4.5) behave chaotically as shown in Fig. 5.

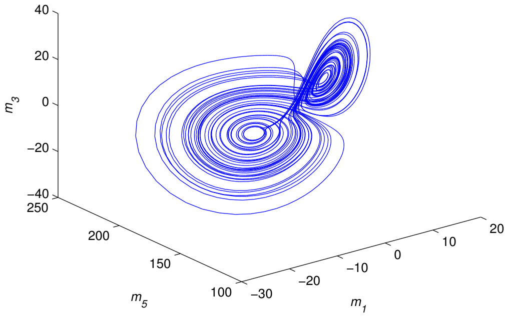
The response system is the fractional-order real Rössler system:

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}\upsilon_1 = -(\upsilon_2 + \upsilon_3) + U_1, \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}\upsilon_2 = \upsilon_1 + \delta_1\upsilon_2 + U_2, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}\upsilon_3 = \delta_2 + \upsilon_3(\upsilon_1 - \delta_3) + U_3, \end{cases} \tag{4.6}$$

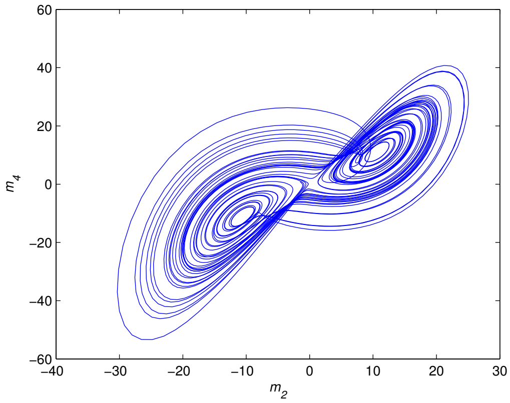
where υ_i are real variables, δ_i are real parameters, and U_i are real control functions $(i = 1, 2, 3)$.

For two given scaling matrices $A = \mathrm{diag}(\alpha_1 + j\alpha_2, \alpha_3 + j\alpha_4, \alpha_5)$ and $B = \mathrm{diag}(\beta_1 + j\beta_2, \beta_3 + j\beta_4, \beta_5)$, the synchronization error can be presented in the form of

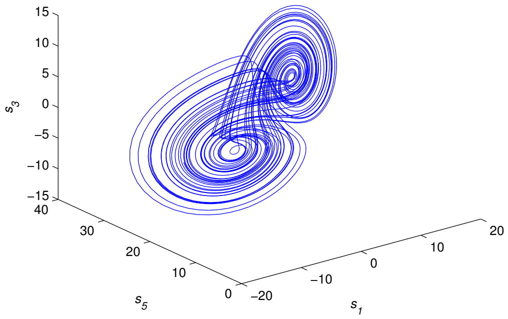
$$\begin{cases} e_1 = \upsilon_1 - \alpha_1m_1 + \alpha_2m_2 - \beta_1s_1 + \beta_2s_2, \\ e_2 = \upsilon_2 - \alpha_3m_3 + \alpha_4m_4 - \beta_3s_3 + \beta_4s_4, \\ e_3 = \upsilon_3 - \alpha_5m_5 - \beta_5s_5. \end{cases}$$



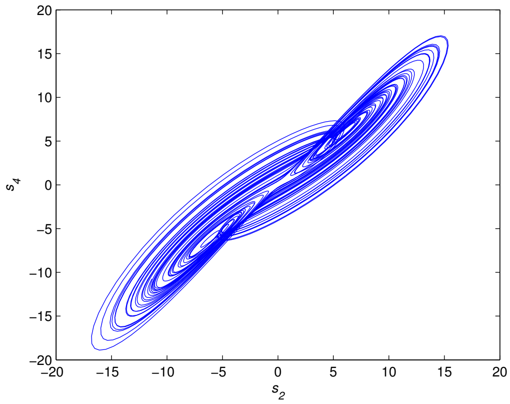
(a) Attractor of the first drive system (4.4)



(b) Phase portrait of the first drive system (4.4)



(c) Attractor of the second drive system (4.5)



(d) Phase portrait of the second drive system (4.5)

Fig. 5: Chaotic attractor and phase portrait of two drive systems (4.4) and (4.5).

By virtue of systems (4.4), (4.5), and (4.6), the error dynamical system becomes:

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}}e_1 = -e_2 - e_3 - (\alpha_3 + a_1\alpha_1)m_3 + (\alpha_4 + a_1\alpha_2)m_4 + a_1(\alpha_1 m_1 - \alpha_2 m_2) - (\beta_3 + b_1\beta_1)s_3 \\ \qquad +(\beta_4 + b_1\beta_2)s_4 + b_1(\beta_1 s_1 - \beta_2 s_2) - \alpha_5 m_5 - \beta_5 s_5 + U_1, \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}}e_2 = e_1 + \delta_1 e_2 + (\alpha_1 - a_2\alpha_3)m_1 - (\alpha_2 - a_2\alpha_4)m_2 + (\delta_1 + 1)(\alpha_3 m_3 - \alpha_4 m_4) + \alpha_3 m_1 m_5 \\ \qquad -\alpha_4 m_2 m_5 + [\beta_1 - (b_2 - b_1)\beta_3]s_1 - [\beta_2 - (b_2 - b_1)\beta_4]s_2 + (\delta_1 - b_2)(\beta_3 s_3 - \beta_4 s_4) \\ \qquad +\beta_3 s_1 s_5 - \beta_4 s_2 s_5 + U_2, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}}e_3 = -\delta_3 e_3 + (a_3 - \delta_3)\alpha_5 m_5 + (b_3 - \delta_3)\beta_5 s_5 + \delta_2 + \upsilon_1\upsilon_3 - \alpha_5(m_1 m_3 + m_2 m_4) \\ \qquad -\beta_5(s_1 s_3 + s_2 s_4) + U_3. \end{cases}$$

Hence, control functions $U_i(t)$ $(i = 1, 2, 3)$ are designed as

$$\begin{cases} U_1(t) = (\alpha_3 + a_1\alpha_1)m_3 - (\alpha_4 + a_1\alpha_2)m_4 - a_1(\alpha_1 m_1 - \alpha_2 m_2) + (\beta_3 + b_1\beta_1)s_3 \\ \qquad -(\beta_4 + b_1\beta_2)s_4 - b_1(\beta_1 s_1 - \beta_2 s_2) + \alpha_5 m_5 + \beta_5 s_5 + \varphi_1(t), \\ U_2(t) = (a_2\alpha_3 - \alpha_1)m_1 + (\alpha_2 - a_2\alpha_4)m_2 - (\delta_1 + 1)(\alpha_3 m_3 - \alpha_4 m_4) - \alpha_3 m_1 m_5 + \alpha_4 m_2 m_5 \\ \qquad -[\beta_1 - (b_2 - b_1)\beta_3]s_1 + [\beta_2 - (b_2 - b_1)\beta_4]s_2 - (\delta_1 - b_2)(\beta_3 s_3 - \beta_4 s_4) - \beta_3 s_1 s_5 \\ \qquad +\beta_4 s_2 s_5 + \varphi_2(t), \\ U_3(t) = (\delta_3 - a_3)\alpha_5 m_5 - (b_3 - \delta_3)\beta_5 s_5 - \delta_2 - \upsilon_1\upsilon_3 + \alpha_5(m_1 m_3 + m_2 m_4) \\ \qquad +\beta_5(s_1 s_3 + s_2 s_4) + \varphi_3(t), \end{cases}$$

which yields

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}} e_1 = -e_2 - e_3 + \varphi_1(t), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}} e_2 = e_1 + \delta_1 e_2 + \varphi_2(t), \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}} e_3 = -\delta_3 e_3 + \varphi_3(t). \end{cases}$$

On the basis of the fact that the term $\varphi_i(t)$ are linear functions of the error terms $e_i(t)$ $(i = 1, 2, 3)$, we choose

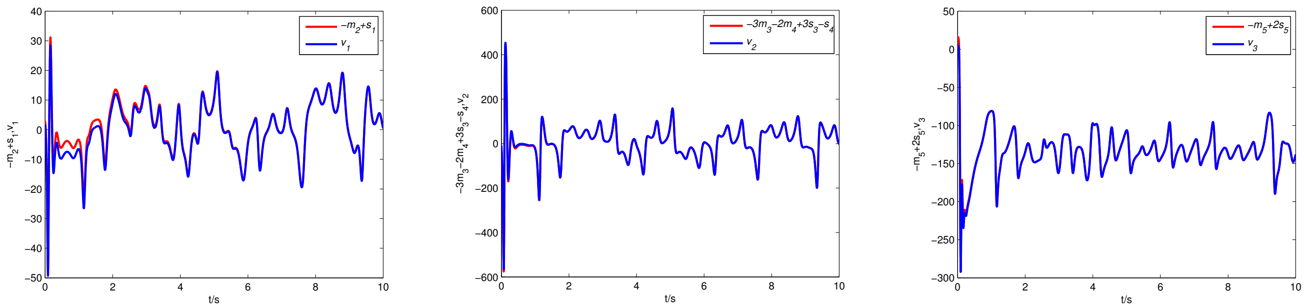
$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \end{pmatrix} = L \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{pmatrix},$$

where $L = (l_{ij})_{3\times3}$ is a 3×3 real matrix. Next, we choose that $l_{11} = l_{21} = -1$, $l_{22} = -2.4$, $l_{33} = 5$, $l_{31} = l_{32} = 0$, and l_{13}, l_{23} can be chosen arbitrarily. Since $(\alpha_1, \alpha_2, \alpha_3) = (0.91, 0.96, 0.99)$, $m = 100$, and $\gamma = 1/100$, we obtain the characteristic equation

$$(\lambda^{91} + 1)(\lambda^{96} + 2)(\lambda^{99} + 5) = 0. \tag{4.7}$$

It is not difficult to verify that all roots of (4.7) satisfy $|\arg(\lambda)| > \gamma\pi/2$. According to Lemma 2.3, the error vector $e(t)$ asymptotically converges to zero as $t \to \infty$.

The initial values of systems (4.4), (4.5), and (4.6) are chosen as $(x_1, x_2, x_3)^T = (2+3j, 5+6j, 9)^T$, $(y_1, y_2, y_3)^T = (6 + 9j, 5 + 7j, 12)^T$, and $(v_1, v_2, v_3)^T = (2, -6, 3)^T$, respectively. Taking two scaling matrices as $A = \mathrm{diag}(j, -3 + 2j, -1)$ and $B = \mathrm{diag}(1, 3 + j, 2)$, we obtain the initial errors as $(-1, 13, -12)^T$. Further, selecting $l_{13} = 1$, $l_{12} = l_{23} = 0$, we have simulation results as displayed in Figs. 6-7. The synchronization process of systems (4.4), (4.5), and (4.6) is described in Fig. 6, where the red line presents the states of two drive systems and the blue line shows the states of response system. Fig. 7 illustrates the errors of synchronization converge asymptotically to zero, i.e., the incommensurate fractional-order complex Lorenz system, complex Chen system, and real Rössler system can achieve combination complex synchronization of real parts.



(a) (b) (c)

Fig. 6: State variables of drive systems (4.4), (4.5), and response system (4.6).

5. Combination complex synchronization of incommensurate fractional-order complex chaotic systems

5.1. Mathematical model and problem descriptions

Now, we consider combination complex synchronization among fractional-order complex chaotic drive systems (4.1), (4.2), and response system (3.3). For two given complex transformation matrices

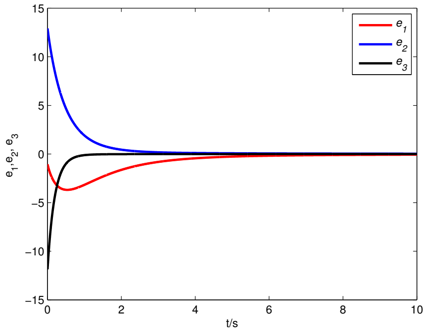


Fig. 7: The error dynamics of synchronization among systems (4.4), (4.5), and (4.6).

$A = A^r + jA^i$ =diag$(\alpha_1 + j\alpha_2, \alpha_3 + j\alpha_4, \cdots, \alpha_{2n-1} + j\alpha_{2n})$ and $B = B^r + jB^i$ =diag$(\beta_1 + j\beta_2, \beta_3 + j\beta_4, \cdots, \beta_{2n-1} + j\beta_{2n})$, the error of combination complex synchronization is defined as:

$$e(t) = e^r(t) + je^i(t) = z(t) - Ax(t) - By(t),$$

i.e.,

$$\begin{cases} e^r = z^r - A^r x^r + A^i x^i - B^r y^r + B^i y^i, \\ e^i = z^i - A^r x^i - A^i x^r - B^r y^i - B^i y^r. \end{cases}$$

In the following, we design a controller to realize combination complex synchronization among the fractional-order complex Lorenz system, complex Lü system, and complex Chen system with incommensurate orders.

5.2. Synchronization of the fractional-order complex Lorenz system, complex Lü system, and complex Chen system

The first drive system is fractional-order complex Lorenz system described by (4.4) and the fractional-order complex Chen system (4.5) is taken as the second drive system, while the response system is the fractional-order complex Lü system described by (3.6).

In our synchronization scheme, we assume $A = \mathrm{diag}(\alpha_1 + j\alpha_2, \alpha_3 + j\alpha_4, \alpha_5)$ and $B = \mathrm{diag}(\beta_1 + j\beta_2, \beta_3 + j\beta_4, \beta_5)$. The synchronization error is set as:

$$\begin{cases} e_1 = p_1 - \alpha_1 m_1 + \alpha_2 m_2 - \beta_1 s_1 + \beta_2 s_2, \\ e_2 = p_2 - \alpha_1 m_2 - \alpha_2 m_1 - \beta_1 s_2 - \beta_2 s_1, \\ e_3 = p_3 - \alpha_3 m_3 + \alpha_4 m_4 - \beta_3 s_3 + \beta_4 s_4, \\ e_4 = p_4 - \alpha_3 m_4 - \alpha_4 m_3 - \beta_3 s_4 - \beta_4 s_3, \\ e_5 = p_5 - \alpha_5 m_5 - \beta_5 s_5. \end{cases}$$

Taking account into systems (4.4), (4.5), and (3.6), we get the following error dynamical system:

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}} e_1 = c_1 e_3 - c_1 e_1 + (c_1\alpha_3 - a_1\alpha_1) m_3 + (a_1\alpha_2 - c_1\alpha_4) m_4 + (a_1 - c_1)(\alpha_1 m_1 - \alpha_2 m_2) \\ \qquad +(b_1 - c_1)(\beta_1 s_1 - \beta_2 s_2) + (c_1\beta_3 - b_1\beta_1) s_3 + (b_1\beta_2 - c_1\beta_4) s_4 + u_1, \\ \frac{d^{\alpha_1}}{dt^{\alpha_1}} e_2 = c_1 e_4 - c_1 e_2 + (c_1\alpha_3 - a_1\alpha_1) m_4 - (a_1\alpha_2 - c_1\alpha_4) m_3 + (a_1 - c_1)(\alpha_2 m_1 + \alpha_1 m_2) \\ \qquad +(b_1 - c_1)(\beta_2 s_1 + \beta_1 s_2) + (c_1\beta_4 - b_1\beta_2) s_3 - (b_1\beta_1 - c_1\beta_3) s_4 + u_2, \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}} e_3 = c_2 e_3 + (c_2 + 1)(\alpha_3 m_3 - \alpha_4 m_4) + a_2(\alpha_4 m_2 - \alpha_3 m_1) + (c_2 - b_2)(\beta_3 s_3 - \beta_4 s_4) \\ \qquad -(b_2 - b_1)(\beta_3 s_1 - \beta_4 s_2) - p_1 p_5 + \alpha_3 m_1 m_5 - \alpha_4 m_2 m_5 + \beta_3 s_1 s_5 - \beta_4 s_2 s_5 + u_3, \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}} e_4 = c_2 e_4 + (c_2 + 1)(\alpha_3 m_4 + \alpha_4 m_3) - a_2(\alpha_4 m_1 + \alpha_3 m_2) + (c_2 - b_2)(\beta_3 s_4 + \beta_4 s_3) \\ \qquad -(b_2 - b_1)(\beta_3 s_2 + \beta_4 s_1) - p_2 p_5 + \alpha_3 m_2 m_5 + \alpha_4 m_1 m_5 + \beta_3 s_2 s_5 + \beta_4 s_1 s_5 + u_4, \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}} e_5 = -c_3 e_5 + (a_3 - c_3)\alpha_5 m_5 + (b_3 - c_3)\beta_5 s_5 + p_1 p_3 + p_2 p_4 - \alpha_5(m_1 m_3 + m_2 m_4) \\ \qquad -\beta_5(s_1 s_3 + s_2 s_4) + u_5. \end{cases}$$

Thus, we define the active control inputs $u_i(t)$ $(i = 1, 2, \cdots, 5)$ as

$$\begin{cases} u_1(t) = (a_1\alpha_1 - c_1\alpha_3)m_3 - (a_1\alpha_2 - c_1\alpha_4)m_4 - (a_1 - c_1)(\alpha_1 m_1 - \alpha_2 m_2) - (b_1 - c_1)\beta_1 s_1 \\ \qquad +(b_1 - c_1)\beta_2 s_2 - (c_1\beta_3 - b_1\beta_1)s_3 - (b_1\beta_2 - c_1\beta_4)s_4 + \varphi_1(t), \\ u_2(t) = (a_1\alpha_1 - c_1\alpha_3)m_4 + (a_1\alpha_2 - c_1\alpha_4)m_3 - (a_1 - c_1)(\alpha_2 m_1 + \alpha_1 m_2) - (b_1 - c_1)\beta_2 s_1 \\ \qquad -(b_1 - c_1)\beta_1 s_2 - (c_1\beta_4 - b_1\beta_2)s_3 + (b_1\beta_1 - c_1\beta_3)s_4 + \varphi_2(t), \\ u_3(t) = (c_2 + 1)(\alpha_4 m_4 - \alpha_3 m_3) - a_2(\alpha_4 m_2 - \alpha_3 m_1) - (c_2 - b_2)(\beta_3 s_3 - \beta_4 s_4) + p_1 p_5 \\ \qquad +(b_2 - b_1)(\beta_3 s_1 - \beta_4 s_2) - \alpha_3 m_1 m_5 + \alpha_4 m_2 m_5 - \beta_3 s_1 s_5 + \beta_4 s_2 s_5 + \varphi_3(t), \\ u_4(t) = -(c_2 + 1)(\alpha_3 m_4 + \alpha_4 m_3) + a_2(\alpha_4 m_1 + \alpha_3 m_2) - (c_2 - b_2)(\beta_3 s_4 + \beta_4 s_3) + p_2 p_5 \\ \qquad +(b_2 - b_1)(\beta_3 s_2 + \beta_4 s_1) - \alpha_3 m_2 m_5 - \alpha_4 m_1 m_5 - \beta_3 s_2 s_5 - \beta_4 s_1 s_5 + \varphi_4(t), \\ u_5(t) = (c_3 - a_3)\alpha_5 m_5 - (b_3 - c_3)\beta_5 s_5 - p_1 p_3 - p_2 p_4 + \alpha_5(m_1 m_3 + m_2 m_4) \\ \qquad +\beta_5(s_1 s_3 + s_2 s_4) + \varphi_5(t), \end{cases}$$

which leads to

$$\begin{cases} \frac{d^{\alpha_1}}{dt^{\alpha_1}} e_1 = c_1 e_3 - c_1 e_1 + \varphi_1(t), \\ \frac{d^{\alpha_1}}{dt^{\alpha_1}} e_2 = c_1 e_4 - c_1 e_2 + \varphi_2(t), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}} e_3 = c_2 e_3 + \varphi_3(t), \\ \frac{d^{\alpha_2}}{dt^{\alpha_2}} e_4 = c_2 e_4 + \varphi_4(t), \\ \frac{d^{\alpha_3}}{dt^{\alpha_3}} e_5 = -c_3 e_5 + \varphi_5(t). \end{cases}$$

Considering the fact that the term $\varphi_i(t)$ are linear functions of the error terms $e_i(t)$ $(i = 1, 2, \cdots, 5)$, we choose

$$\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \\ \varphi_3(t) \\ \varphi_4(t) \\ \varphi_5(t) \end{pmatrix} = L \begin{pmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \\ e_4(t) \\ e_5(t) \end{pmatrix},$$

where $L = (l_{ij})_{5\times 5}$ is a 5×5 real matrix. L can be set as follows: $l_{11} = 40$, $l_{22} = 38$, $l_{33} = -23$, $l_{44} = -25$, $l_{55} = 3$, $l_{21} = l_{31} = l_{32} = l_{41} = l_{42} = l_{43} = l_{51} = l_{52} = l_{53} = l_{54} = 0$, and other elements can be chosen arbitrarily. Taking $(\alpha_1, \alpha_2, \alpha_3) = (0.92, 0.95, 0.98)$, $m = 100$, and $\gamma = 1/100$, we have the characteristic equation

$$(\lambda^{92} + 2)(\lambda^{92} + 4)(\lambda^{95} + 1)(\lambda^{95} + 3)(\lambda^{98} + 2) = 0. \tag{5.1}$$

After some calculations, we can show that all roots of (5.1) satisfy $|\arg(\lambda)| > \gamma\pi/2$. By virtue of Lemma 2.3, we can obtain that the error vector $e(t)$ asymptotically converges to zero as $t \to \infty$. Therefore, combination complex synchronization between the incommensurate fractional-order chaotic complex systems (4.4), (4.5), and (3.6) is achieved.

In the numerical simulations, the initial values of systems (4.4), (4.5), and (3.6) are chosen as $(x_1, x_2, x_3)^T = (2+3j, 5+6j, 9)^T$, $(y_1, y_2, y_3)^T = (6+9j, 5+7j, 12)^T$, $(z_1, z_2, z_3)^T = (1+2j, 3+4j, 5)^T$, and two scaling matrices are taken as $A = \mathrm{diag}(3 + j, -1 + 2j, -2)$ and $B = \mathrm{diag}(2 + j, 2 - j, 1)$. Thus, the initial errors are $(-5 - 33j, 3 - 9j, 11)^T$. Further, taking $l_{13} = -40$, $l_{24} = -38$, $l_{35} = 1$, $l_{45} = -1$, and $l_{12} = l_{14} = l_{15} = l_{23} = l_{25} = l_{34} = 0$, we obtain simulation results as displayed in Figs. 8-9. The synchronization process of systems (4.4), (4.5), and (3.6) is described in Fig. 8, where the solid line presents the states of two drive systems and the dashed line shows the states of response system. Fig. 9 shows all error states converge asymptotically to zero. As expected, we approach combination complex synchronization among the incommensurate fractional-order complex Lorenz system, complex Chen system, and complex Lü system.

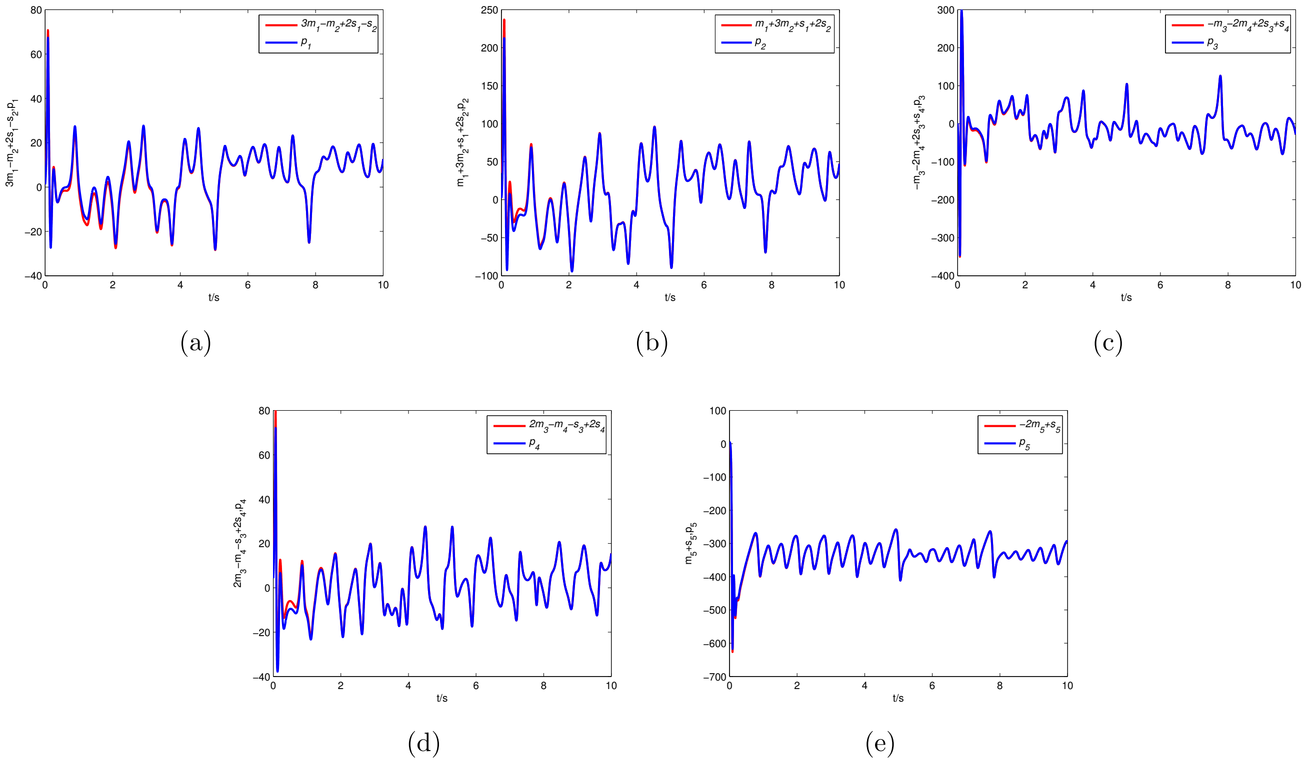


Fig. 8: State variables of drive systems (4.4), (4.5), and response system (3.6).

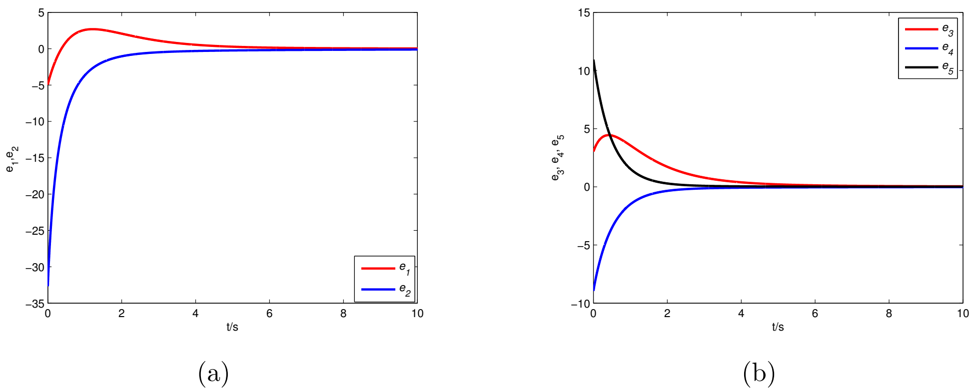


Fig. 9: The error dynamics of synchronization between drive systems (4.4), (4.5), and response system (3.6).

6. Conclusions

In this paper, we introduce, analyze, and validate a novel form of chaos synchronization that can involve two drive systems and one response system, namely combination complex synchronization. On the basis of the stability theory of incommensurate fractional-order systems and the feedback control approach, we design controllers to realize combination complex synchronization among three fractional-order chaotic systems with incommensurate orders. Three groups of examples are considered and their numerical simulations demonstrate the validity and feasibility of the proposed scheme. Additionally, this synchronization is applicable to all fractional-order chaotic systems, including those that can exhibit hyperchaotic behavior. The proposed synchronization scheme directs the attention of secure communication to fractional-order chaotic complex systems with incommensurate orders, which may increase the number of state variables to further enhance the security of private communications. Therefore, it is believed that the proposed scheme will play an important role in practical applications.

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