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Spatiotemporal dynamics of a delayed IS-LM model with interest rate and general investment function

Maha Elkarmouchi[a,*], Sara Lasfar[a], Khalid Hattaf[a,b], Noura Yousfi[a]

[a]Laboratory of Analysis, Modeling and Simulation (LAMS), Faculty of Sciences Ben M'Sick, Hassan II University of Casablanca, P.O. Box 7955 Sidi Othman, Casablanca, Morocco.

[b]Equipe de Recherche en Modélisation et Enseignement des Mathématiques (ERMEM), Centre Régional des Métiers de l'Education et de la Formation (CRMEF), 20340 Derb Ghalef, Casablanca, Morocco.

Abstract

In this paper, we develop a mathematical model that describes the spatiotemporal dynamics of business cycle under the goods and services market as well as the money market. We first prove that the developed model is mathematically and economically well-posed. The conditions for the existence of economic equilibrium are rigorously established. Moreover, the stability analysis and the existence of Hopf bifurcation are carefully investigated. Finally, our theoretical results are illustrated with some numerical simulations.

Keywords: IS-LM model, reaction-diffusion equations, time delay, asymptotic stability, Hopf bifurcation.

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1. Introduction

Recently, IS-LM models are used to describe the dynamics of business cycle by taking into account the interest rate. The acronym IS stands for "investment-savings" and LM for "liquidity preference-money supply". The first formulation of this type of models was introduced by Hicks [12] based on Keynesian theory. In this formulation, the curve of the equilibrium interest rates and incomes in the commodity market is usually labeled by IS, and the curve of the equilibrium interest rates and incomes in the money market is usually labeled by LM.

In the literature, several IS-LM macroeconomic models have been developed based on the idea of Hicks [12] in order to better understand the dynamics of business cycle. For instance, Torre [21] proposed a standard IS-LM model governed by two ordinary differential equations (ODEs) in order to study the existence of limit cycles in the set of complete Keynesian systems by means of bifurcation theory. Gabisch and Lorenz [3] considered an augmented IS-LM business cycle model with three ODEs. Based on the ideas given in [3, 14], Cai [2] proposed a delayed IS-LM model by assuming that the investment, the saving and the demand for money functions depend linearly on their arguments. The delay in [2] denotes

*Corresponding author
Email address: mahaelkarmouchi123@gmail.com (Maha Elkarmouchi)

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the time lag between the decision of investment and his implementation. Abta et al. [1] extended the model of Cai [2] to a nonlinear case of investment function of the form given in [17]. A recent IS-LM model was introduced in [20] in order to improve and generalize the above models.

All the mathematical models cited above ignored the diffusion of macroeconomic aggregates. However, in this paper, we propose an IS-LM business cycle model involving time and space. This model is expressed by the following nonlinear system of reaction-diffusion equations:

$$\begin{cases} \frac{\partial Y(t,x)}{\partial t} = d_1\Delta Y(t,x) + \alpha[I(Y(t,x),K(t,x),R(t,x)) - s_1Y(t,x) - s_2R(t,x)], \\ \frac{\partial K(t,x)}{\partial t} = d_2\Delta K(t,x) + I(Y(t-\tau,x),K(t,x),R(t,x)) - \delta K(t,x), \\ \frac{\partial R(t,x)}{\partial t} = d_3\Delta R(t,x) + \beta[L(Y(t,x),R(t,x)) - \overline{M}], \end{cases} \tag{1.1}$$

where $Y(t,x)$, $K(t,x)$, and $R(t,x)$ respectively represent the gross product, the capital stock and the interest rate at time t, and location x. Δ is the Laplacian operator as well as d_1, d_2, and d_3 are the diffusion coefficients of Y, K, and R, respectively. The parameter α is the adjustment coefficient in the goods market while β is the coefficient of adjustment in the money market. The demand for money or liquidity preference function is labeled by $L(Y,R)$ while the investment is presented by $I(Y,K,R)$. The constant money supply is denoted by $\overline{M}$. The delay τ is the time lag between the decision of investment and his implementation. The positive constants s_1 and s_2 are the propensities to save. Finally, δ is the depreciation rate of the capital stock. In addition, we consider model (1.1) with the initial conditions:

$$Y(t,x) = \phi_1(t,x),\ K(t,x) = \phi_2(t,x),\ R(t,x) = \phi_3(t,x),\ (t,x) \in [-\tau,0] \times \overline{\Omega}, \tag{1.2}$$

and Neumann boundary conditions:

$$\frac{\partial Y}{\partial \nu} = \frac{\partial K}{\partial \nu} = \frac{\partial R}{\partial \nu} = 0, \quad \text{on } (0,+\infty) \times \partial\Omega, \tag{1.3}$$

where Ω is the market capacity and $\frac{\partial}{\partial \nu}$ indicates the outward normal derivative on the smooth boundary $\partial\Omega$.

It is important to note that our IS-LM model formulated by system (1.1) improves and generalizes various economic models existing in the literature. For example, when we neglect the diffusion effect, we get the model introduced in [20] that includes the temporal models proposed in [1, 2]. Further, the spatiotemporal model presented by Hu et al. [13] is a special case of our model (1.1) when the interest rate is absent and the investment function has a particular form.

The organization of the present paper is as follows. Section 2 deals with the existence of economic equilibrium and the properties of solutions including the existence, uniqueness and uniform boundedness. Section 3 establishes the local stability of the economic equilibrium and the existence of Hopf bifurcation. Section 4 presents some numerical simulations to illustrate our main theoretical results. Finally, the paper ends with a conclusion presented in Section 5.

2. Existence of solutions and economic equilibrium

To establish the existence of the solutions of problem (1.1)-(1.3), we introduce some notations. Let $X = C\left(\overline{\Omega}, \mathbb{R}^3\right)$ be the Banach space of continuous functions from $\overline{\Omega}$ into $\mathbb{R}^3$ and $\mathcal{C} = C\left([-\tau,0], X\right)$ be the Banach space of continuous functions of $[-\tau,0]$ into X with standard uniform topology. For simplicity, we identify an element $\phi \in \mathcal{C}$ as a function from $[-\tau,0] \times \overline{\Omega}$ into $\mathbb{R}^3$ defined by $\phi(s,x) = \phi(s)(x)$. For any continuous function $\omega(.) : [-\tau,b) \to X$ for $b > 0$, we set $\omega_t \in \mathcal{C}$ by $\omega_t(s) = \omega(t+s)$ for $s \in [-\tau,0]$.

As in [9, 19], we assume that the general investment function $I(Y,K,R)$ satisfies the following hypothesis:

(H_1) there exist two constants $A > 0$ and $\bar{q} \geqslant 0$ such that $|I(Y,K,R) + \bar{q}K| \leqslant A$ for all $Y,\ K,\ R \in \mathbb{R}$.

As in [20], we assume that the liquidity preference function $L(Y, R)$ is of the form $L(Y, R) = \mathcal{L}(Y) - \gamma R$, where γ measures the variation of demand of liquidity related to interest rate and $\mathcal{L}(Y)$ satisfies the following hypothesis:

(H_2) There exists a constant $B > 0$ such that $|\mathcal{L}(Y)| \leqslant B$ for all $Y \in \mathbb{R}$.

Based on these hypotheses, we have the following result.

Theorem 2.1. *If* (H_1) *and* (H_2) *hold, then for any given initial* $\phi \in \mathcal{C}$ *there exists a unique solution of problem* (1.1)-(1.3) *defined on* $[0, +\infty)$. *Furthermore, this solution is uniformly bounded.*

Proof. For each $\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in \mathcal{C}$ and $x \in \overline{\Omega}$, we define $F = (F_1, F_2, F_3) : \mathcal{C} \to X$ by

$$F_1(\varphi)(x) = \alpha[I(\varphi_1(0, x), \varphi_2(0, x), \varphi_3(0, x)) - s_1\varphi_1(0, x) - s_2\varphi_3(0, x)],$$
$$F_2(\varphi)(x) = I(\varphi_1(-\tau, x), \varphi_2(0, x), \varphi_3(0, x)) - \delta\varphi_2(0, x),$$
$$F_3(\varphi)(x) = \beta[\mathcal{L}(\varphi_1(0, x)) - \gamma\varphi_3(0, x) - \overline{M}].$$

Then problem (1.1)-(1.3) can be rewritten as the following abstract functional differential equation

$$\begin{cases} u'(t) = Eu(t) + F(u_t), \ t > 0, \\ u(0) = \phi \in \mathcal{C}, \end{cases} \tag{2.1}$$

where $u = (Y, K, R)^T$ and $Eu = (d_1\Delta Y, d_2\Delta K, d_3\Delta R)^T$. It is obvious that F is locally Lipschitz in $\mathcal{C}$, and as in [10], we conclude that problem (2.1) has a unique local solution on $[0, T_{max})$, where T_{max} is the maximal existence time for solution of system (2.1).

From the second equation of (1.1) and (H_1), we get

$$\begin{cases} \frac{\partial K}{\partial t} - d_2\Delta K \leqslant A - (\delta + \bar{q})K, \\ \frac{\partial K}{\partial \nu} = 0, \\ K(0, x) = \phi_2(0, x), x \in \bar{\Omega}, \end{cases}$$

According to Lemma 1 presented in [6], we have

$$K(t, x) \leqslant \max\left\{\frac{A}{\delta + \bar{q}}, \max_{x\in\overline{\Omega}} \phi_2(0, x)\right\}, \ \forall (x, t) \in \overline{\Omega} \times [0, T_{max}).$$

This implies that K is bounded. From the third equation of (1.1) and (H_2), we obtain

$$\begin{cases} \frac{\partial R}{\partial t} - d_3\Delta R \leqslant \beta(B - \overline{M}) - \beta\gamma R, \\ \frac{\partial R}{\partial \nu} = 0, \\ R(0, x) = \phi_3(0, x), x \in \overline{\Omega}. \end{cases}$$

Similarly to above, we get

$$R(t, x) \leqslant \max\left\{\frac{B - \overline{M}}{\gamma}, \max_{x\in\overline{\Omega}} \phi_3(0, x)\right\}, \ \forall (x, t) \in \overline{\Omega} \times [0, T_{max}).$$

This implies that R is bounded. According to the first equation of the system (1.1), we have

$$\begin{cases} \frac{\partial Y}{\partial t} - d_1\Delta Y \leqslant \rho - \alpha s_1 Y, \\ \frac{\partial Y}{\partial \nu} = 0, \\ Y(0, x) = \phi_1(0, x), x \in \overline{\Omega}, \end{cases}$$

where $\rho = \alpha(A + \bar{q}n_1 + s_2n_2)$ with $n_1 = \max\{\frac{A}{\delta+\bar{q}}, \max_{x\in\overline{\Omega}} \phi_2(0, x)\}$ and $n_2 = \max\{\frac{B-\overline{M}}{\gamma}, \max_{x\in\overline{\Omega}} \phi_3(0, x)\}$. Hence,

$$Y(t, x) \leqslant \max\left\{\frac{\rho}{\alpha s_1}, \max_{x\in\overline{\Omega}} \phi_1(0, x)\right\}, \ \forall (x, t) \in \overline{\Omega} \times [0, T_{max}),$$

which implies that Y is bounded. It follows from the standard theory for semilinear parabolic systems [11] that $T_{max} = +\infty$. This completes the proof. □

To study the existence of economical equilibrium of (1.1), we need the following hypotheses:

(H_3) $\gamma I\left(0, \frac{s_2(\mathcal{L}(0)-\overline{M})}{\gamma\delta}, \frac{\mathcal{L}(0)-\overline{M}}{\gamma}\right) - s_2\left(\mathcal{L}(0)-\overline{M}\right) > 0$;

(H_4) $\gamma\frac{\partial I}{\partial Y} + \left(\frac{\gamma s_1}{\delta} + \frac{s_2}{\delta}\mathcal{L}'(Y)\right)\frac{\partial I}{\partial K} + \mathcal{L}'(Y)\frac{\partial I}{\partial R} < \gamma s_1 + s_2\mathcal{L}'(Y), \forall (Y, K, R) \in \mathbb{R}^3$.

Theorem 2.2. *If* (H_1)-(H_4) *hold, then the system* (1.1) *has a unique economic equilibrium defined by* $E^*\left(Y^*, \frac{\gamma s_1 Y^* + s_2(\mathcal{L}(Y^*)-\overline{M})}{\gamma\delta}, \frac{\mathcal{L}(Y^*)-\overline{M}}{\gamma}\right)$ *such that* Y^* *is the positive solution of the following equation*

$$\gamma I\left(Y, \frac{\gamma s_1 Y + s_2(\mathcal{L}(Y)-\overline{M})}{\gamma\delta}, \frac{\mathcal{L}(Y)-\overline{M}}{\gamma}\right) - \gamma s_1 Y - s_2(\mathcal{L}(Y)-\overline{M}) = 0.$$

Proof. Any equilibrium of (1.1) is a solution of the following equations:

$$I(Y, K, R) - s_1 Y - s_2 R = 0, \tag{2.2}$$
$$I(Y, K, R) - \delta K = 0, \tag{2.3}$$
$$\mathcal{L}(Y) - \gamma R - \overline{M} = 0. \tag{2.4}$$

From (2.2)-(2.4), we have

$$R = \frac{\mathcal{L}(Y)-\overline{M}}{\gamma} \quad \text{and} \quad K = \frac{\gamma s_1 Y + s_2(\mathcal{L}(Y)-\overline{M})}{\gamma\delta}. \tag{2.5}$$

By replacing (2.5) in (2.2), we get

$$\gamma I\left(Y, \frac{\gamma s_1 Y + s_2(\mathcal{L}(Y)-\bar{M})}{\gamma\delta}, \frac{\mathcal{L}(Y)-\overline{M}}{\gamma}\right) - \gamma s_1 Y - s_2\left(\mathcal{L}(Y)-\overline{M}\right) = 0.$$

Therefore, we consider a function ψ defined on interval $[0, +\infty)$ as follows

$$\psi(Y) = \gamma I\left(Y, \frac{\gamma s_1 Y + s_2(\mathcal{L}(Y)-\bar{M})}{\gamma\delta}, \frac{\mathcal{L}(Y)-\overline{M}}{\gamma}\right) - \gamma s_1 Y - s_2\left(\mathcal{L}(Y)-\overline{M}\right).$$

From (H_1)-(H_4), we obtain $\psi(0) > 0$, $\lim_{Y\to+\infty}\psi(Y) = -\infty$, and

$$\psi'(Y) = \gamma\frac{\partial I}{\partial Y} + \left[\frac{\gamma s_1}{\delta} + \frac{s_2}{\delta}\mathcal{L}'(Y)\right]\frac{\partial I}{\partial K} + \mathcal{L}'(Y)\frac{\partial I}{\partial R} - \gamma s_1 - s_2\mathcal{L}'(Y) < 0.$$

Consequently, there exists a unique $Y^* \in (0, +\infty)$ such that Y^* is the positive solution of the equation $\psi(Y) = 0$. This completes the proof. □

3. Stability analysis and Hopf bifurcation

In this section, we focus on local stability of the economic equilibrium $E^*(Y^*, K^*, R^*)$ and the existence of Hopf bifurcation analysis. Let $y = Y - Y^*$, $k = K - K^*$, and $r = R - R^*$. By substituting these new variables into (1.1) and linearizing, we get following system:

$$\begin{cases} \frac{\partial y}{\partial t} = d_1\Delta y(t,x) + \alpha[(a-s_1)y(t,x) + bk(t,x) + (c-s_2)r(t,x)], \\ \frac{\partial k}{\partial t} = d_2\Delta k(t,x) + ay(t-\tau,x) + (b-\delta)k(t,x) + cr(t,x), \\ \frac{\partial r}{\partial t} = d_3\Delta r(t,x) + \beta[l_1 y(t,x) - \gamma r(t,x)], \\ \frac{\partial y}{\partial \nu} = \frac{\partial k}{\partial \nu} = \frac{\partial r}{\partial \nu} = 0, \quad t > 0, x \in \partial\Omega, \end{cases} \tag{3.1}$$

where $a = \frac{\partial I}{\partial Y}(Y^*, K^*, R^*)$, $b = \frac{\partial I}{\partial K}(Y^*, K^*, R^*)$, $c = \frac{\partial I}{\partial R}(Y^*, K^*, R^*)$, and $l_1 = \mathcal{L}'(Y^*)$.

Let $\zeta = C([-\tau, 0], \mathbb{X})$ be the Banach space of continuous functions of $[-\tau, 0]$ into $\mathbb{X}$, where $\mathbb{X}$ is defined by

$$\mathbb{X} = \left\{ y, k, r \in W^{2,2}(\Omega) : \frac{\partial y(t,x)}{\partial \nu} = \frac{\partial k(t,x)}{\partial \nu} = \frac{\partial r(t,x)}{\partial \nu} = 0, x \in \partial\Omega \right\}$$

with the inner product $\langle ., . \rangle$. Hence, system (3.1) can be rewritten as an abstract differential equation in the phase space ζ as follows

$$U'(t) = \mathcal{D}\Delta U + L(U_t), \tag{3.2}$$

where $U = (y, k, r)^T$, $\mathcal{D} = \text{diag}(d_1, d_2, d_3)$, and $L : \zeta \to \mathbb{X}$ defined by

$$L(\phi) = \mathcal{A}\phi(0) + \mathcal{B}\phi(-\tau),$$

with

$$\mathcal{A} = \begin{pmatrix} \alpha(a - s_1) & \alpha b & \alpha(c - s_2) \\ 0 & b - \delta & c \\ \beta l_1 & 0 & -\beta\gamma \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the characteristic of system (3.2) is as follows

$$\lambda y - \mathcal{D}\Delta y - L\left(e^{\lambda \cdot} y\right) = 0, \ y \in \text{dom}(\Delta) \backslash \{0\}. \tag{3.3}$$

Let $-k^2$ $(k \in \mathbb{N} = \{0, 1, 2, \ldots\})$ be the eigenvalue of the operator Δ under the Neumann boundary conditions on $\mathbb{X}$ and the corresponding eigenvectors take the form:

$$\beta_k^1 = (\gamma_k, 0, 0)^T, \beta_k^2 = (0, \gamma_k, 0)^T, \beta_k^3 = (0, 0, \gamma_k)^T, \gamma_k = \cos(kx), \quad k = 0, 1, 2, \ldots,$$

and $\left\{\beta_k^1, \beta_k^2, \beta_k^3\right\}_{k=0}^{+\infty}$ construct a basis of the phase space $\mathbb{X}$. Hence, we can expand in the form of Fourier on the phase space $\mathbb{X}$, which is as follows:

$$y = \sum_{k=0}^{\infty} Y_k^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \end{pmatrix}, \quad Y_k = \begin{pmatrix} \langle y, \beta_k^1 \rangle \\ \langle y, \beta_k^2 \rangle \\ \langle y, \beta_k^3 \rangle \end{pmatrix}. \tag{3.4}$$

Then by calculation, we get

$$L\left(\phi^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \end{pmatrix}\right) = L(\phi)^T \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \end{pmatrix}, \quad k \in \mathbb{N}. \tag{3.5}$$

Substituting (3.5) and (3.4) into (3.3), we can have

$$\sum_{k=0}^{\infty} Y_k^T \left[(\lambda I_3 + \mathcal{D}k^2) - \begin{pmatrix} \alpha(a - s_1) & \alpha b & \alpha(c - s_2) \\ ae^{-\lambda\tau} & b - \delta & c \\ \beta l_1 & 0 & -\beta\gamma \end{pmatrix} \right] \begin{pmatrix} \beta_k^1 \\ \beta_k^2 \\ \beta_k^3 \end{pmatrix} = 0.$$

The characteristic equation of (3.5) is as follows:

$$\begin{vmatrix} \lambda + d_1k^2 - \alpha(a - s_1) & -\alpha b & -\alpha(c - s_2) \\ -ae^{-\lambda\tau} & \lambda + d_2k^2 - b + \delta & -c \\ -\beta l_1 & 0 & \lambda + d_3k^2 + \beta\gamma \end{vmatrix} = 0,$$

which leads to

$$\lambda^3 + p_{2,k}\lambda^2 + p_{1,k}\lambda + p_{0,k} + (q_1\lambda + q_{0,k})e^{-\lambda\tau} = 0, \tag{3.6}$$

where

$$\begin{aligned} p_{0,k} &= [d_1k^2 - \alpha(a-s_1)](d_2k^2 - b + \delta)(d_3k^2 + \beta\gamma), \\ p_{1,k} &= -\beta l_1 \alpha(c - s_2) + (d_3k^2 + \beta\gamma)(d_2k^2 - b + \delta) \\ &\quad + (d_3k^2 + \beta\gamma)[d_1k^2 - \alpha(a - s1)] + [d_1k^2 - \alpha(a - s_1)](d_2k^2 - b + \delta), \\ p_{2,k} &= d_1k^2 - \alpha(a - s_1) + d_2k^2 - b + \delta + d_3k^2 + \beta\gamma, \\ q_{0,k} &= -(\beta\gamma + d_3k^2)\alpha b a, \\ q_1 &= -\alpha a b. \end{aligned}$$

Clearly, for $k \in \mathbb{N}$, $\lambda = 0$ is not a solution of Eq. (3.6). When $\tau = 0$, Eq. (3.6) reduces to

$$\lambda^3 + p_{2,k}\lambda^2 + (p_{1,k} + q_1)\lambda + p_{0,k} + q_{0,k} = 0. \tag{3.7}$$

If $a < s_1$, then it not hard to see that the coefficients of the equation (3.7) satisfy:

$$p_{2,k} > 0 \quad \text{and} \quad p_{2,k}(p_{1,k} + q_1) - (p_{0,k} + q_{0,k}) > 0.$$

By the stability criterion of Routh-Hurwitz, we deduce that all the roots of (3.7) have negative real parts and we have the following result.

Lemma 3.1. *If $a < s_1$, then the economic equilibrium E^* is locally asymptotically stable in the absence of delay.*

When $\tau \neq 0$, let $\lambda = i\omega$ $(\omega > 0)$ be a purely imaginary root of the equation (3.6). Then

$$-i\omega^3 - p_{2,k}\omega^2 + p_{1,k}i\omega + p_{0,k} + (q_1 i\omega + q_{0,k})e^{-i\omega\tau} = 0.$$

Hence,

$$\begin{cases} p_{2,k}\omega^2 - p_{0,k} = q_1\omega\sin(\omega\tau) + p_{0,k}\cos(\omega\tau), \\ \omega^3 - p_{1,k}\omega = q_1\omega\cos(\omega\tau) - q_{0,k}\sin(\omega\tau), \end{cases} \tag{3.8}$$

which implies that

$$\omega^6 + (p_{2,k}^2 - 2p_{1,k})\omega^4 + (p_{1,k}^2 - 2p_{0,k}p_{2,k} - q_1^2)\omega^2 + p_{0,k}^2 - q_{0,k}^2 = 0. \tag{3.9}$$

Let $z = \omega^2$. Thus, the equation (3.9) becomes

$$h(z) = z^3 + c_{2,k}z^2 + c_{1,k}z + c_{0,k} = 0, \tag{3.10}$$

where

$$c_{2,k} = p_{2,k}^2 - 2p_{1,k}, \quad c_{1,k} = p_{1,k}^2 - 2p_{0,k}p_{2,k} - q_1^2, \quad \text{and} \quad c_{0,k} = p_{0,k}^2 - q_{0,k}^2.$$

By calculations, we obtain

$$\begin{aligned} c_{0,k} &= [d_1k^2 - \alpha(a - s_1)]^2(d_2k^2 - b + \delta)^2(d_3k^2 + \beta\gamma)^2 - [(d_3k^2 + \beta\gamma)\alpha a b]^2, \\ c_{1,k} &= (-\beta l_1\alpha(c - s_2))^2 + 2[-\beta l_1\alpha(c - s_2)][(d_3k^2 + \beta\gamma)(d_2k^2 - b + \delta) \\ &\quad + (d_3k^2 + \beta\gamma)(d_1k^2 - \alpha(a - s_1)) + (d_2k^2 - b + \delta)(d_1k^2 - \alpha(a - s_1))] \\ &\quad + (d_3k^2 + \beta\gamma)^2(d_2k^2 - b + \delta)^2 + (d_3k^2 + \beta\gamma)^2[d_1k^2 - \alpha(a - s_1)]^2 \\ &\quad + [d_1k^2 - \alpha(a - s_1)]^2(d_2k^2 - b + \delta)^2 - (\alpha a b)^2, \\ c_{2,k} &= [d_1k^2 - \alpha(a - s_1)]^2 + (d_2k^2 - b + \delta)^2 + (d_3k^2 + \beta\gamma)^2 + \beta l_1\alpha(c - s_2). \end{aligned}$$

When $k = 0$, it is easy to show that

$$c_{0,0} = [-\beta\gamma\alpha(a - s_1)(-b + \delta)]^2 - (\beta\gamma\alpha a b)^2.$$

If the following condition:

(A_1) $|\beta\gamma\alpha a b| > [-\beta\gamma\alpha(a - s_1)(-b + \delta)]$

holds, then $c_{0,0} < 0$. In this case, Eq. (3.10) has a unique positive root z_0, and thus Eq. (3.9) has a unique positive root $\omega_0 = \sqrt{z_0}$. By (3.8), we get

$$\tau_0^{(j)} = \frac{1}{\omega_0} \arcsin \frac{(p_{2,0}q_1 - q_{0,0})\,\omega_0^3 + (p_{1,0}q_{0,0} - p_{0,0}q_1)\,\omega}{q_{0,0}^2 + q_1^2\omega^2} + \frac{2j\pi}{\omega_0}, \quad j = 0, 1, 2, \dots,$$

at which Eq. (3.6) with $k = 0$ has a pair of purely imaginary roots of the form $\pm i\omega_0$ and all roots of Eq. (3.6), except $\pm i\omega_0$, have no zero real parts. Then, by the general theory on characteristic equations of delay differential equations from [18, Theorem 4.1], we see that if $a < s_1$ and (A_1) hold, E^* remains stable for $\tau < \tau_0$, where $\tau_0 = \tau_0^{(0)}$. Let $\lambda(\tau) = \nu(\tau) + i\omega(\tau)$ be a root of Eq. (3.6) satisfying $\nu(\tau_0) = 0,\ \omega(\tau_0) = \omega_0$. We now verify that

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\tau}\right|_{\tau=\tau_0} > 0.$$

This will prove that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. In addition, the conditions for the existence of a Hopf bifurcation [5] are then satisfied yielding a periodic solution. To this end, differentiating Eq. (3.6) with respect τ, we derive that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{3\lambda^2 + 2p_{2,k}\lambda + p_{1,k}}{\lambda\,(\lambda^3 + p_{2,k}\lambda^2 + p_{1,k}\lambda + p_{0,k})} + \frac{q_1}{\lambda\,(q_1\lambda + q_{0,k})} - \frac{\tau}{\lambda}.$$

By direct calculations one obtains that

$$\begin{aligned}
\operatorname{sign}\left\{\frac{d(\operatorname{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} &= \operatorname{sign}\left\{\operatorname{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0} \\
&= \operatorname{sign}\left\{\frac{3\omega_0^4 + 2c_{2,k}\omega_0^2 + c_{1,k}}{q_1^2\omega_0^2 + q_{0,k}^2}\right\} = \operatorname{sign}\left\{\frac{h'(z_0)}{q_1^2\omega_0^2 + q_{0,k}^2}\right\},
\end{aligned}$$

where $z_0 = \omega_0^2$. Hence, the transversal condition holds and a Hopf bifurcation occurs at $\omega = \omega_0$, $\tau = \tau_0$. Further, if $i\omega_1 (\omega_1 > 0)$ is a solution of (3.6) with $k \geqslant 1$, by calculation, we get that

$$\omega_1^6 + (p_{2,k}^2 - 2p_{1,k})\omega_1^4 + (p_{1,k}^2 - 2p_{0,k}p_{2,k} - q_1^2)\omega_1^2 + p_{0,k}^2 - q_{0,k}^2 = 0.$$

Then, we get

$$\begin{aligned}
c_{0,k} &\geqslant (d_1 - \alpha(a - s_1))^2(d_2 - b + \delta)^2(d_3 + \beta\gamma)^2 - [(d_3 + \beta\gamma)\alpha ab]^2, \\
c_{1,k} &\geqslant (-\beta l_1\alpha(c - s_2))^2 + 2[-\beta l_1\alpha(c - s_2)][(d_3 + \beta\gamma)(d_2 - b + \delta) \\
&\quad + (d_3 + \beta\gamma)(d_1 - \alpha(a - s_1)) + (d_2 - b + \delta)(d_1 - \alpha(a - s_1))] \\
&\quad + (d_3 + \beta\gamma)^2(d_2 - b + \delta)^2 + (d_3 + \beta\gamma)^2(d_1 - \alpha(a - s_1))^2 \\
&\quad + (d_1 - \alpha(a - s_1))^2(d_2 - b + \delta)^2 - (\alpha ab)^2, \\
c_{2,k} &\geqslant (d_1 - \alpha(a - s_1))^2 + (d_2 - b + \delta)^2 + (d_3 + \beta\gamma)^2 + \beta l_1\alpha(c - s_2).
\end{aligned}$$

Clearly, if the following conditions:

(A_2) $(d_1 - \alpha(a - s_1))(d_2 - b + \delta)(d_3 + \beta\gamma) > |(d_3 + \beta\gamma)\alpha ab|$;

(A_3)

$$\begin{aligned}
&(-\beta l_1\alpha(c - s_2))^2 + 2[-\beta l_1\alpha(c - s_2)][(d_3 + \beta\gamma)(d_2 - b + \delta) + (d_3 + \beta\gamma)(d_1 - \alpha(a - s_1)) \\
&\quad + (d_2 - b + \delta)(d_1 - \alpha(a - s_1))] + (d_3 + \beta\gamma)^2(d_2 - b + \delta)^2 + (d_3 + \beta\gamma)^2(d_1 - \alpha(a - s_1))^2 \\
&\quad + (d_1 - \alpha(a - s_1))^2(d_2 - b + \delta)^2 > (\alpha ab)^2;
\end{aligned}$$

(A_4) $(d_1 - \alpha(a - s_1))^2 + (d_2 - b + \delta)^2 + (d_3 + \beta\gamma)^2 > -\beta l_1\alpha(c - s_2)$,

hold, then Eq. (3.6) with $k \geqslant 1$ has no purely imaginary roots. In conclusion, we have the following results.

Theorem 3.2. *If $a < s_1$ and (A_1)-(A_4) hold, then the economic equilibrium E^* is locally asymptotically stable forall*

$\tau \in [0, \tau_0)$ *and becomes unstable when* $\tau > \tau_0$. *In addition, the system* (1.1) *undergoes a Hopf bifurcation at* E^* *when* $\tau = \tau_0^{(j)}$, *for* $j \in \mathbb{N}$.

4. Numerical simulations

In this section, we present some numerical simulations to illustrate our theoretical results. Here, we consider $I(Y, K, R) = I(Y) + \frac{q_1 K}{\sqrt{1+\epsilon K^2}} + q_2 R$, where $q_1, q_2 < 0$, $\epsilon \geqslant 0$ and $I(Y)$ is the Kaldor-type investment function defined by $\frac{e^Y}{1+e^Y}$. The liquidity preference function is chosen as $L(Y, R) = s_3 Y - s_4 R$, where $s_3, s_4 \geqslant 0$. We use the following parameter values: $\alpha = 3$, $\delta = 0.2$, $q_1 = -0.3$, $q_2 = -0.2$, $\epsilon = 0.01$, $\overline{M} = 0.05$, $s_1 = 0.2$, $s_2 = 0.1$, $s_3 = 0.3$, $s_4 = 0.2$, $\beta = 0.2$, $d_1 = d_2 = d_3 = 0.1$. In this case, our model has an economic equilibrium $E^*(0.65775, 1.02607, 0.73663)$. By a simple computation, we can obtain $\tau_0 \approx 1.8666$. First, we choose $\tau = 1.5 < \tau_0$. According to the result of Theorem 3.2, we know that the economic equilibrium E^* is locally asymptotically stable for all $\tau \in [0, \tau_0)$, Figures 1 and 2 demonstrate this result. Next, we choose $\tau = \tau_0$. By Theorem 3.2, the system (1.1) undergoes a Hopf bifurcation at the economic equilibrium E^*, Figures 3 and 4 illustrate this result. Finally, we choose $\tau = 1.92 > \tau_0$. It follows from Theorem 3.2 that E^* is unstable, which is illustrated in Figures 5 and 6.

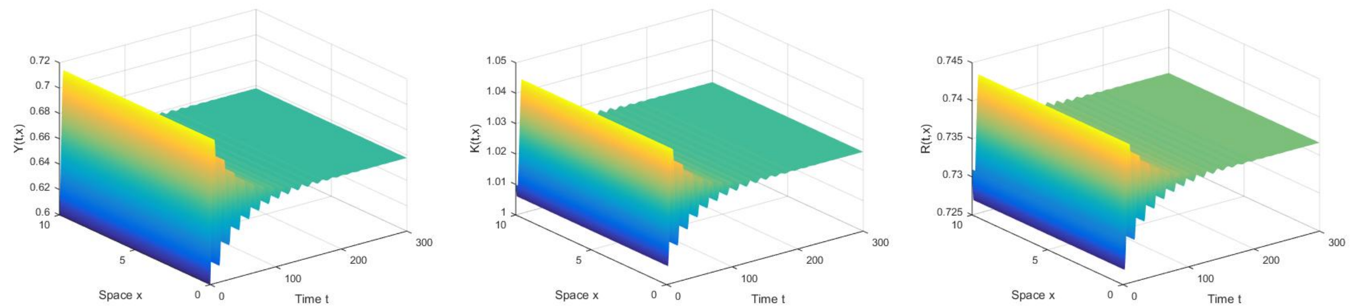

Figure 1: Spatiotemporel dynamics of system (1.1), when $\tau = 1.5$.

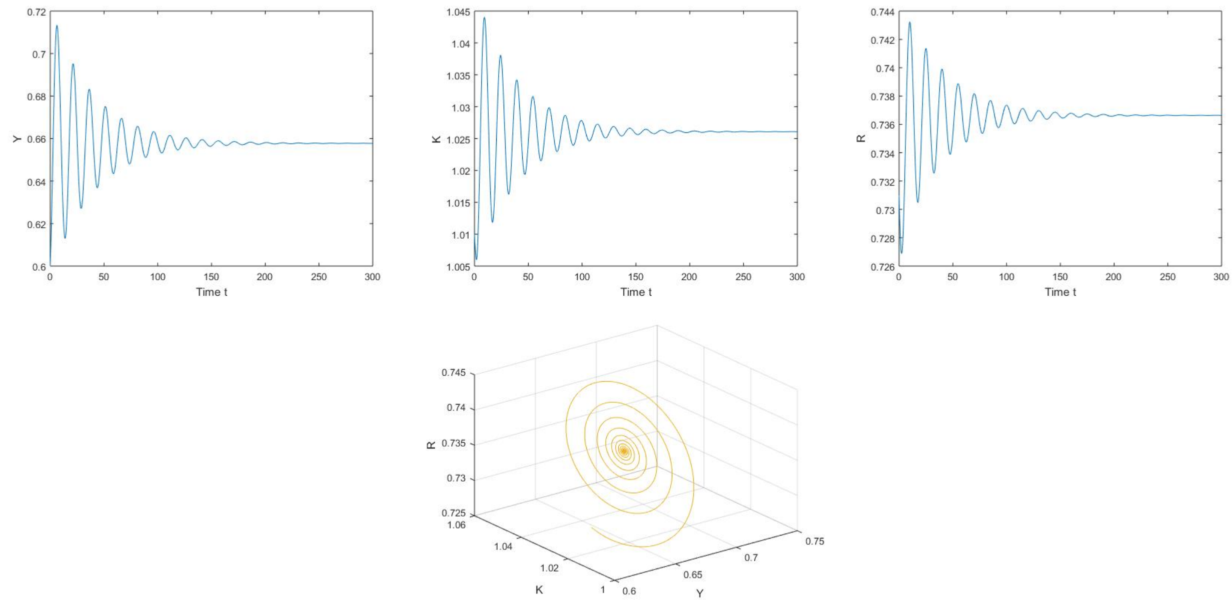

Figure 2: The economic equilibrium E^* is stable, when $\tau = 1.5$ and $x = 5$.

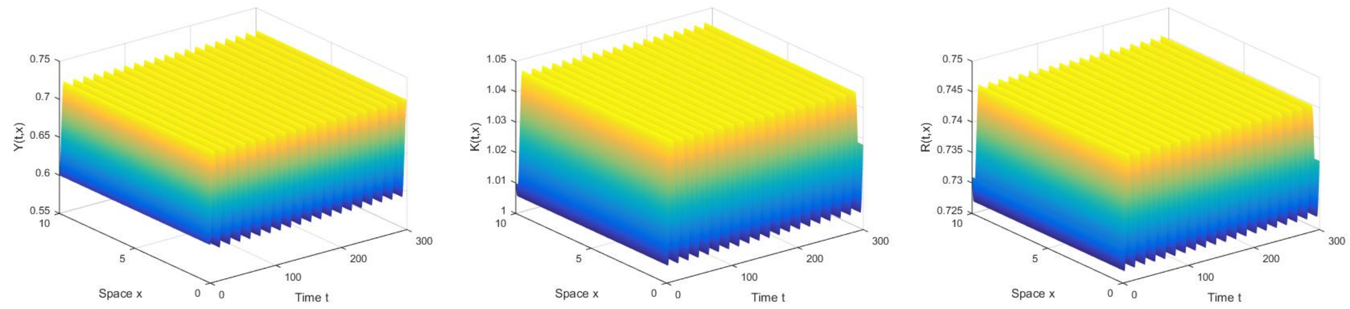

Figure 3: Spatiotemporel dynamics of system (1.1), when $\tau = \tau_0$.

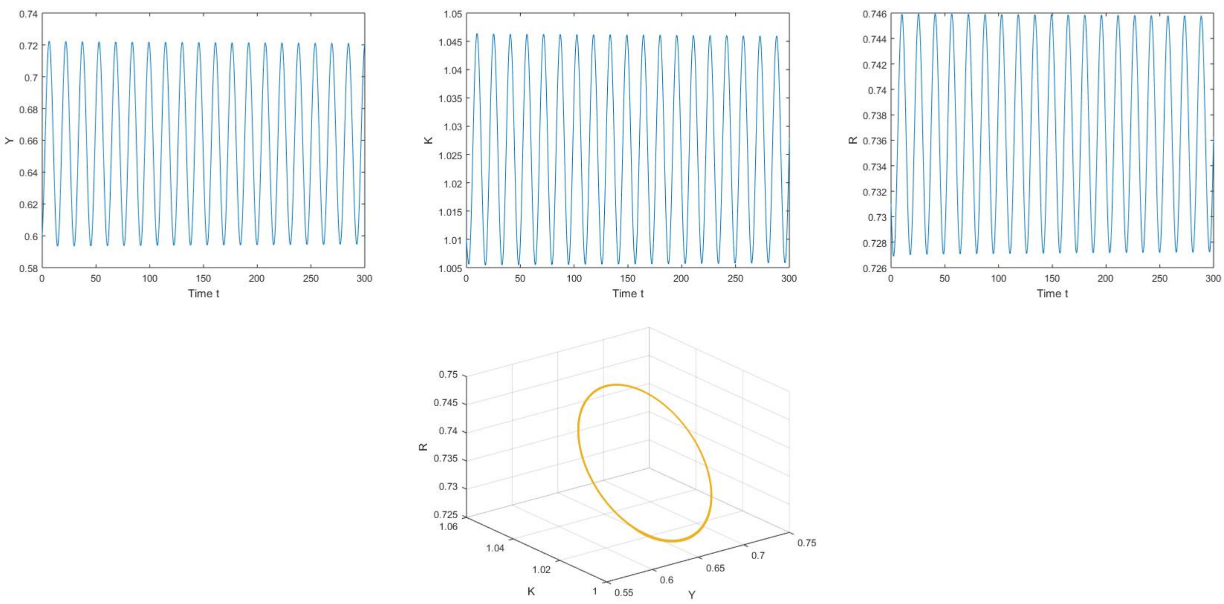

Figure 4: System (1.1) undergoes a Hopf bifurcation at the economic equilibrium E^*, when $\tau = \tau_0$ and $x = 5$.

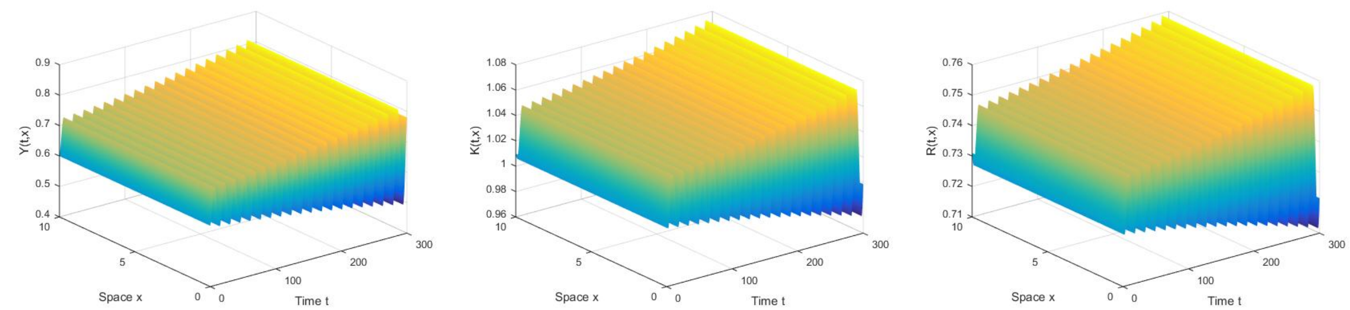

Figure 5: Spatiotemporel dynamics of system (1.1), when $\tau = 1.92$.

5. Conclusion

In this paper, we have proposed a delayed IS-LM model with diffusion effect, interest rate and general investment function under homogeneous Neumann boundary conditions. We have studied the existence, the uniqueness, and the uniform boundedness of solutions as well as the existence of economic equilibrium. Using the Routh-Hurwitz criterion and the analysis of the roots of the characteristic equation of

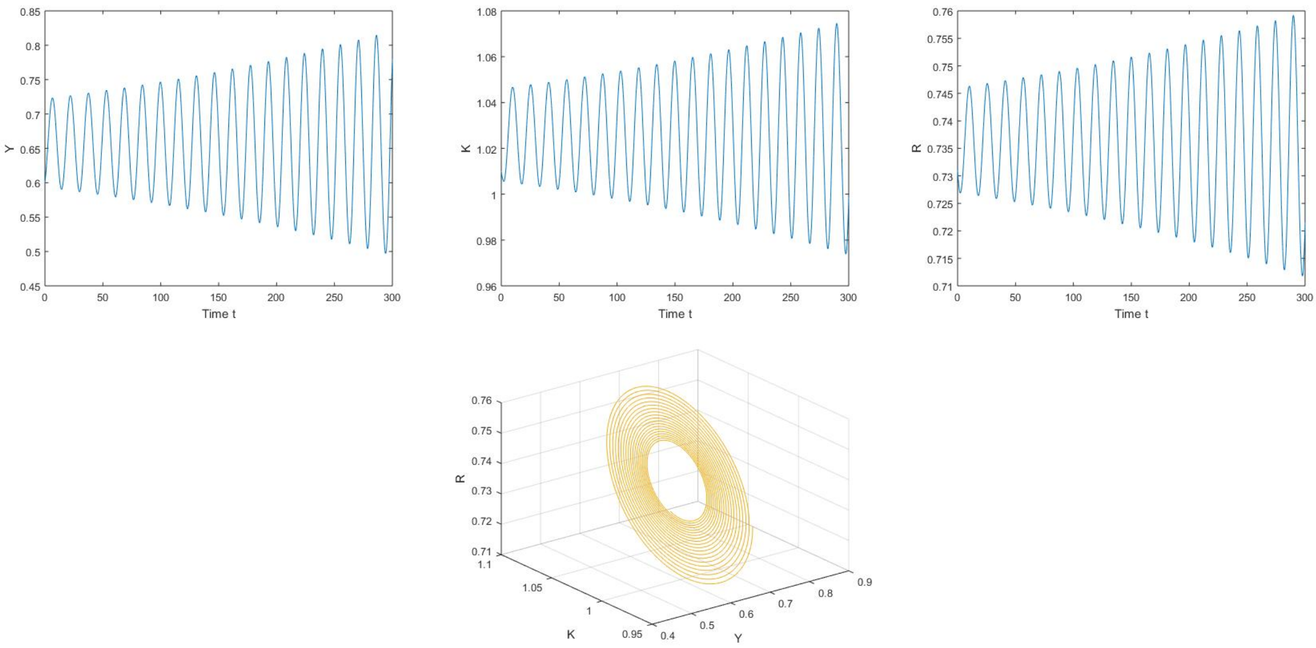

Figure 6: The economic equilibrium E^* is unstable, when $\tau = 1.92$ and $x = 5$.

the linearized system, we have established the local asymptotic stability of the economic equilibrium and the existence of Hopf bifurcation. Furthermore, the business cycle models and the IS-LM macroeconomic models presented in [1, 2, 13, 20] are improved and extended by considering other economic factors.

In economics, memory refers to the ability of economic agents to remember past events and use them to make economic decisions in the present. In addition, most economic systems have long-term memories. Therefore, the impact of memory effect on the dynamics of our model by using the new generalized Hattaf fractional (GHF) derivative [7, 8] and other fractal-fractional derivatives as used in [4, 15, 16], will be the main purpose of our future works.

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