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Mathematical model and stability analysis of university-PhD-postdoc-industry migration and unemployment in south Africa

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Abstract

Unemployment poses to be a threat in many countries which has been a long age battle with the human race. Our study considers Africa (South Africa to be precise) and focuses on the academic and industry sectors, with PhD graduates and postdoctoral research fellows as key participants. The movement of PhD graduates and postdocs between academia and industry, and its impact (increase or decrease) in unemployment within the sectors was investigated. For the population of University, PhDs, Postdocs, and industry compartments, a model was developed. The primary objectives investigates the model by analyzing the stability at the University-PhD-Postdoc-Industry free equilibrium, University-PhD-Postdoc-Industry equilibrium, determine the recruitment number R_{UE} and understand how to mitigate migrations within the University and Industry compartments. Our findings help to understand the cause and effects of migration, and aims to manage the migration of PhD graduates and Postdocs moving into the Industry while the university suffers research capacities.

Keywords: ODE, stability analysis, recruitment number, mathematical model, Lyapunov function.

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1. Introduction

Modelling of real world problems can be solved using resulting equations and the process is called Mathematical modelling. Researchers have used the idea of mathematical modelling to solve problems in different fields such as infectious disease in Covid-19 [1, 5, 6], Malaria [20], Hepatitis B virus [21]. Not only in infectious disease but also in ecology relating to Lotka-Volterra model such as Holling-Tanner model [2–4]. The same ideology is employed in the investigation of unemployment within the university and industry sector on the study of PhD graduates and Postdoctoral research fellows.

Unemployment is a significant problem in many nations; the level of unemployment varies depending on the country's socioeconomic standing. Using South Africa as an example, the unemployment rate was 34.6 percent as of the third quarter of 2022, making it the highest country in the world. In [15], a study conducted in 2015, investigating why south Africa's PhD holders struggles to find employment (which

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can be in different sectors of the economy). This study observed the growing gap between the number of Doctor of Philosophy (PhD) graduates and the number who find employment, while sometimes some employers see PhD holders as over qualified to be given certain jobs. This is due to a number of variables, including unemployment in the economy, which prompts PhD holders to downplay or conceal their credentials. It's interesting to note that, according to Hako of News24 [15], every year, approximately 1800 PhD students graduate from South African universities, and when their employability is assessed, it is discovered that half of them had trouble obtaining job or gaining employment.

Unemployment rate in countries is a socio-economic problem which has place more request on demand and supply in the labor market. Increase in population grows exponentially in the world of today, the unemployed statuscope has become saturated and youth with various level of education add to the saturation. This saturation has lead to difficulty in maintaining adequate sustainability of welfare packages or resources that is to be shared in the community in different spheres [13]; example such as prison (was overcrowded at a time [9]). Because of this, the younger generation is threatened by the unemployment rate. Given the current situation, the strain on nutrition, social safety, housing, and other necessities of life puts a thorn on the government's financial capacity and the accessibility of facilities. The older workers who have been in the service for a long time (in government and non-government parastatals) are retiring as their post becomes competitive and hundreds of people apply for it, meaning that it's the ratio of 1 position to $1,500$ applicants. It is critical to recognize that this issue has an impact on several economic sectors. As a result of unemployment, we have focused our analysis on migration within the sectors of academia and industry. Due to unemployment and other factors, there is a progressive decline in interest in STEM fields, which is soon becoming a concern in most countries. Statistics show that the number of students interested in STEM is statistically declining each year [12, 16, 25]. As a result of landing faculty posts, which provide a hurdle to their career ambition, PhD students are evidently becoming less interested in pursuing academic careers in STEM fields [23]. Early-career researchers, Postdocs and PhD graduates are gradually and eventually choosing industry over academia as they pursue their careers [17] as a result of their desire for passion (prestige) and financial security. The movement of PhD graduates in search of job outside academia its increasing on daily basis [8]. More than half of STEM PhD holders work in non-academic professions since STEM academic employment are becoming more difficult to get, and in certain situations, the pay is poor compared to the input and activity [7, 18, 24] required to maintain job security and funds attractions to the university. From research, it shows that 67% of PhD students desire a career in academia (especially research position) or teaching position but only 30% stay in academia three years on or move to the industry as other means of survival [26]. From the investigate work done by Bethan Cornell, unlikely reason some PhD students or graduates will not pursue academic research as a career is predicated on 20% as a result of Lack of work (life balance) and 13% said the salary will be too low [11]. To give an overview of the situation, [19] explored the problem by stating R_0 as the mean number of new PhD students a tenure-track professor will graduate during his or her academic career. Its imperative to say if $R_0 = 1$, then it means there is a PhD graduate that will replace the professor, if $R_0 < 1$, the position will die out and if $R_0 > 1$, then there would be too many PhD graduates applying for one position. From literature, it depicts there are many parameters that suggest or influence the decision of the PhD graduates/early career researchers migrating to the industry. The impact of Covid-19 also affected several universities in terms of job positions by canceling or postponing the position [27]. The pandemic does only affect the senior academic researchers but the impact (i.e., canceling faculty and teaching positions, postdoctoral research funding and other funds for research) is massively felt by the early career researchers. This study's focal point comes from the perspective of both academia and industry. Following is how the remaining portions of the paper are structured. Section 2 talks about how the model is formulated, the assumptions, and the schematic diagram, while Section 3 gives details information on the positivity and boundedness of the model. The stability analysis of the model and the recruitment number is given in Section 4 while in Section 5, we investigate the nature of the equilibrium points. In section 6, we presented the numerical analysis of increasing a parameter and its effects on other compartments while Section 7 provides some perspective and recommendations to the research.

2. Model formulation

Definition 2.1. In this section, we propose the unemployment model that significantly assumes the doctoral graduates, postdocs that migrates from academia or university for the purpose of research into the industry and this is associated with the following assumptions:

- *ALL* applicants for PhD admissions are qualified and competent for PhD placement.
- The rate of the university population, in accordance to the population of the PhD graduates, postdoc placement predicated upon PhD graduation, and transitioning to the industry is denoted by $U(t), P_h(t), P_D(t)$, and $I(t)$ respectively.
- Some of the *NOT* accepted applicants move else where.
- Those accepted graduate after three years of research, proceed to either postdoc, industry, or be retained at the University.

The dynamics of the model are governed by the following nonlinear system of ordinary differential equations:

$$\frac{dU(t)}{dt} = \Lambda - \beta \cdot P_h(t) \cdot U(t) + \alpha_4 \cdot I(t) + \alpha_5 \cdot P_h(t) + \alpha_6 P_D(t) - \mu \cdot U(t), \tag{2.1}$$

$$\frac{dP_h(t)}{dt} = \beta \cdot P_h(t) \cdot U(t) - (\alpha_1 + \alpha_2 + \alpha_5 + \mu) P_h(t), \tag{2.2}$$

$$\frac{dP_D(t)}{dt} = \alpha_1 \cdot P_h(t) - (\mu + \alpha_3 + \alpha_6) P_D(t), \tag{2.3}$$

$$\frac{dI(t)}{dt} = \alpha_2 \cdot P_h(t) + \alpha_3 \cdot P_D(t) - (\alpha_4 + \mu) I(t). \tag{2.4}$$

The flowchart as seen in Figure 1 shows the four compartments and migration of parameters in each compartments and assumptions.

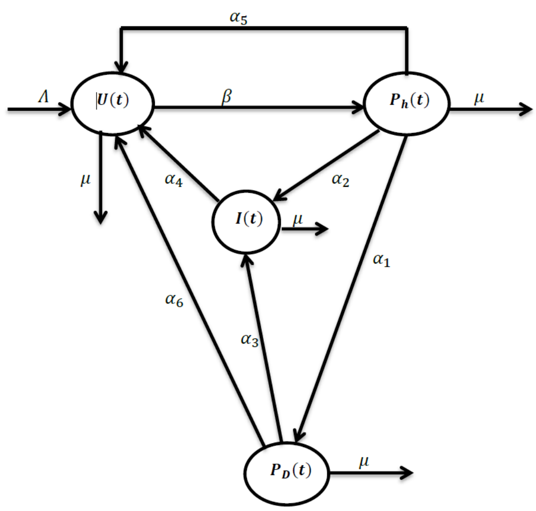

Figure 1

Table 1: Definition of parameters.

Parameters	Description
Λ	Rate at which PhD students are admitted into the university
β	Progression rates of students to PhD class
μ	Rate at which individuals leaves the academia and industry for other endeavors
α_1	Progression rate of individual with a PhD to Postdoctoral class
α_2	Progression rate of individual with a PhD directly to the industry
α_3	Progression rate of individual with a Postdoctoral position to the industry
α_4	Progression rate of individual from the industry to the academia
α_5	rate at which individual with a PhD secure employment in the university
α_6	rate at which individual with a postdoctoral experience secure employment in the university

Table 2: Definition of variables.

Variables	Description
$U(t)$	University Population
$P_h(t)$	Population of PhD graduates
$P_D(t)$	Population of PhD graduates with postdoc placements
$I(t)$	Industry Population

2.1. Variables and parameter description

With respect to equations (2.1)-(2.4), parameters $\Lambda, \beta, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \mu$, are positive constants and can be described in Table 1 while variables definition are seen in Table 2.

We assume that the initial condition of the mathematical model in equations (2.1)-(2.4) are positive as follows:

$$U(0) = U_0 \geqslant 0; \quad P_h(0) = P_{h_0} \geqslant 0; \quad P_D(0) = P_{D_0} \geqslant 0; \quad I_0 = I_0 \geqslant 0.$$

Since the model of Equations (2.1)-(2.4) is patterned after the population of the variables and the parameters are assumed to be positive. The invariant region can be represented as:

$$\Psi_\sigma = \left\{ (U, P_h, P_D, I) \in \mathbb{R}^4_+ : U(t) + P_h(t) + P_D(t) + I(t) \leqslant \frac{\Lambda}{\mu} \right\}, \tag{2.5}$$

w.r.t equation (2.5), the solution to the mathematical model are feasible $\forall t > 0$, provided they remain invariant in the region of Ψ_σ.

3. Positivity and boundedness of the solutions

3.1. Positivity of the solutions

The model describes the university population of PhD students and Postdocs candidates, hence it is imperative to show that the system is well-posed and meaningful. The variables must remain non-negative to discuss the positivity of the model with equations (2.1)-(2.4) using Theorem (3.1). To prove this positivity of equation (2.1), we need to show that equations (2.2)-(2.4) are positive.

Theorem 3.1. *Let* $U(0) > 0$, $P_h(0) > 0$, $P_D(0) > 0$, *and* $I_0 > 0$, *then the solution* $U(t), P_h, P_D, I_0$ *of equations* (2.1)-(2.4) *is always non-negative.*

Proof. Taking equation (2.2), the PhDs population of the model, we can show that

$$\frac{dP_h(t)}{dt} = \beta \cdot P_h(t) \cdot U(t) - (\alpha_1 + \alpha_2 + \alpha_5 + \mu)P_h(t), \tag{3.1}$$

taking initial value to be $t = 0, P_0(t) = P_{h_0} > 0$, introducing separation of variables and integrating both sides of the equation (3.1) from $t \in [0, t]$, we obtain a solution in equation (3.2)

$$P_h(t) > P_{h_0} e^{\{\beta \int_0^t U(\xi) d\xi - (\alpha_1 + \alpha_2 + \alpha_5 + \mu)t\}} > 0. \tag{3.2}$$

From the result of equation (3.2), it is seen that $P_h(t)$ is positive, initial value of P_{h_0} and exponential function is always positive, hence the solution for the model is positive $\forall t > 0$. We apply the same procedure for the other equations can proven to be positive engaging the same idea.

Considering the equation (3.3), the postdocs population and taking $t = 0, P_D(t) = P_{h_0} > 0$ into consideration, we have:

$$\frac{dP_D(t)}{dt} = \alpha_1 \cdot P_h(t) - (\mu + \alpha_3 + \alpha_6) P_D(t) > -(\mu + \alpha_3 + \alpha_6) P_D(t), \tag{3.3}$$

taking $t = 0,\ P_D(t) = P_{h_0} > 0$, due to Grönwall's inequality[14], it implies that

$$P_D(t) > P_{D_0} e^{\{\int_0^t -(\mu + \alpha_3 + \alpha_6)t\}} > 0. \tag{3.4}$$

The industry population and $t = 0, I_t = I_0 > 0$, we have

$$\frac{dI(t)}{dt} = \alpha_2 \cdot P_h(t) + \alpha_3 \cdot P_D(t) - (\alpha_4 + \mu) I(t) > -(\alpha_4 + \mu) I(t).$$

Because $P_{D_0} > 0$, $e^{\{\int_0^t -(\mu + \alpha_3 + \alpha_6)t\}} > 0$, $t > 0$, hence $P_D(t) > 0$. Applying the same principle of Grönwall's inequality[14], we obtain

$$I(t) > I_0 e^{\{-(\alpha_4 + \mu)t\}} > 0. \tag{3.5}$$

Because $I_0 > 0$, $e^{\{-(\alpha_4 + \mu)t\}} > 0$, $t > 0$, hence $I(t) > 0$. To show the positivity of the university population, it's necessary to show the positivity of the PhD's, postdocs and industry population which have been shown in equations (3.2), (3.4), and (3.5) that $U(t) > 0$, $P_h(t) > 0$ and $P_D(t) > 0$. Considering the university population in equation (2.1),

$$\frac{dU(t)}{dt} = \Lambda - \beta \cdot P_h(t) \cdot U(t) + \alpha_4 \cdot I(t) + \alpha_5 \cdot P_h(t) + \alpha_6 P_D(t) - \mu \cdot U(t),$$

then

$$> \Lambda - [\beta \cdot P_h(t) - \mu] U(t).$$

Introducing the Grönwall's inequality, we have

$$U(t) > U_0 e^{[-\beta \int_0^t P_h(\zeta) d\zeta - \mu t]} + \Lambda \int_0^t e^{[-\beta \int_\zeta^t P_h(\sigma) d\sigma - \mu(t - \zeta)]} d\zeta > 0.$$

Recall that the exponent and initial values of U_0 are positive and the parameters are positive constants are stated respectively, hence the solution is always positive. □

3.2. *Boundedness of the solutions*

Theorem 3.2. *The non-negative solutions by Theorem (3.1) are bounded.*

Proof. From the model, the total population rate is given as

$$N(t) = U(t) + P_h(t) + P_D(t) + I(t)$$

such that

$$N'(t) = U'(t) + P'_h(t) + P'_D(t) + I'(t).$$

By simplification, we obtain

$$\frac{dN(t)}{dt} = \Lambda - [U(t) + P_h(t) + P_D(t) + I(t)]\,\mu.$$

Recall that $N(t) = U(t) + P_h(t) + P_D(t) + I(t)$, then

$$\frac{dN(t)}{dt} = \Lambda - \mu \cdot N(t). \tag{3.6}$$

□

With respect to equation (3.6), the solution can be found as represented in equation (3.7)

$$N(t) = \frac{\Lambda}{\mu} - \left[\frac{\Lambda - \mu N_0}{\mu}\right] e^{-\mu t}. \tag{3.7}$$

We can see that $N(t)$ tends to $\frac{\Lambda}{\mu}$, as $t \to \infty$ and the representation of $\frac{\Lambda}{\mu}$ denote the threshold population level. This indicates that the population grows and converges asymptotically.

4. Analysis of PhD-Postdoc Unemployment

Thins section is on the analysis of the model formulation in equations (2.1)-(2.4). The model does not have a close form, hence we investigate it using the stability theorems to understand the behaviour of the equilibrium points with respect to unemployment in the academia. To compute this, we set the right hand side of the equations which is the derivation to zero as representative in equation (4.1):

$$\frac{dU}{dt} = \frac{dP_h}{dt} = \frac{dP_D}{dt} = \frac{dI}{dt} = 0. \tag{4.1}$$

From Equation (4.1), we derive two equilibrium points that is associated with equations (2.1)-(2.4). We engage the idea of epidemiology by obtaining the unemployment free equilibrium point (UPPIFE) and the unemployment equilibrium point (UPPIE). The UPPIFE point occurs in the absence of PhD graduates, postdocs population, and industry. It's also sufficient enough to investigates state of UPPIEE, i.e., the relationship between the PhD graduates, postdocs population, and industry population, respectively.

4.1. U-PhD-Postdoc-Industry free equilibrium point (UPPIFE)

The PPIFE states the model in equations (2.1)-(2.4) as explained, and its denoted by Ξ^0 in equation (4.2) as follows:

$$\Xi^0 = (U_0, P_{h_0}, P_{D_0}, I_0) = \left(\frac{\Lambda}{\mu}, 0, 0, 0\right) \tag{4.2}$$

To understand the existence and stability of the equilibrium, its imperative we obtain the recruitment number from the knowledge of epidemiology of infectious disease which will be denoted by R_{UE} in the research. In this research, the R_{UE} of equations (2.1)-(2.4) is the number of secondary cases of unemployment with increase from one individual who graduated from the university in the absence of PhD and Postdoc placement. However, the model is considered in the class of accepted PhD students and unemployed graduates who either got a postdoc placement or move to the industry to be equivalent to the classes of infected and susceptible people in equations (2.1)-(2.4). To obtain the recruitment number, this procedure is obtained using the next generation matrix [22] and represented by

$$R_{UE} = \left(FV^{-1}\right). \tag{4.3}$$

The vectors for $\mathcal{F}$ and $\mathcal{V}$ are represented in equation (4.4):

$$\mathcal{F} = \begin{bmatrix} \beta(P_h)U \\ 0 \end{bmatrix} \text{ and } \mathcal{V} = \begin{bmatrix} (\alpha_1+\alpha_2+\alpha_5+\mu)P_h \\ \alpha_1 \cdot P_h + (\alpha_3+\mu+\alpha_6)P_D \end{bmatrix}.$$

And using the equation (4.3), we compute the Jacobian matrix from $\mathcal{F}$ and $\mathcal{V}$ to F and V, to obtain equation (4.4):

$$F = \begin{bmatrix} \beta U & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} \alpha_1+\alpha_2+\alpha_5+\mu & 0 \\ -\alpha_1 & \alpha_3+\mu+\alpha_6 \end{bmatrix}. \tag{4.4}$$

Evaluate the Jacobian matrices in equation (4.4) at $(\frac{\Lambda}{\mu}, 0, 0, 0)$, we obtain equation (4.5):

$$F = \begin{bmatrix} \frac{\beta\Lambda}{\mu} & 0 \\ 0 & 0 \end{bmatrix} \text{ and } V = \begin{bmatrix} \alpha_1+\alpha_2+\alpha_5+\mu & 0 \\ -\alpha_1 & \alpha_3+\mu+\alpha_6 \end{bmatrix}, \tag{4.5}$$

such that

$$V^{-1} = \begin{bmatrix} \frac{1}{\alpha_1+\alpha_2+\alpha_5+\mu} & 0 \\ \frac{\alpha_1}{(\alpha_1+\alpha_2+\alpha_5+\mu)(\alpha_3+\mu+\alpha_6)} & \frac{1}{\alpha_3+\mu+\alpha_6} \end{bmatrix}.$$

The recruitment number R_{UE} is the spectral radius of the next generation matrix in equation (4.6),

$$R_{UE} = \left(FV^{-1}\right),$$

therefore,

$$R_{UE} = \begin{bmatrix} \frac{\beta\Lambda}{\mu(\alpha_1+\alpha_2+\alpha_5+\mu)} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.6}$$

To obtain the spectral radius, the eigenvalues λ are calculated from equation (4.6) to give

$$\lambda = \frac{\beta\Lambda}{\mu(\alpha_1+\alpha_2+\alpha_5+\mu)}, 0.$$

Therefore, the dominant eigenvalue λ is represented in equation (4.7)

$$R_{UE} = \frac{\beta\Lambda}{\mu(\alpha_1+\alpha_2+\alpha_5+\mu)}. \tag{4.7}$$

4.2. *Local stability of (UPPIFE), Ξ^0*

This section investigates the the stability of of the U-PhD-Postdoc-Industry free equilibrium, and to carry our procedure, we compute the Jacobian matrix at the UPPIFE. The obtained eigenvalues and its sign, is used to determine the stability.

Theorem 4.1. *If $R_{UE} < 1$, then the UPPIFE ($\Xi^0 = \frac{\Lambda}{\mu}, 0, 0, 0$) is locally asymptotically stable, otherwise, if $R_{UE} > 1$, then it's said to be unstable.*

Proof. Consider the Jacobian matrix of system of equation (2.1)-(2.4) in equation (4.8):

$$J = \begin{bmatrix} -\beta P_h - \mu & -U\beta + \alpha_5 & \alpha_6 & \alpha_4 \\ \beta P_h & U\beta - \mu - \alpha_1 - \alpha_2 - \alpha_5 & 0 & 0 \\ 0 & \alpha_1 & -\mu - \alpha_3 - \alpha_6 & 0 \\ 0 & \alpha_2 & \alpha_3 & -\mu - \alpha_4 \end{bmatrix}. \tag{4.8}$$

Evaluation J at (UPPIFE), Ξ^0 stating in the absence of P_h, P_D, I, we obtain equation (4.9):

$$J\Xi^0 = \begin{bmatrix} -\mu & -\frac{\Lambda\beta}{\mu} + \alpha_5 & \alpha_6 & \alpha_4 \\ 0 & \frac{\Lambda\beta}{\mu} - \mu - \alpha_1 - \alpha_2 - \alpha_5 & 0 & 0 \\ 0 & \alpha_1 & -\mu - \alpha_3 - \alpha_6 & 0 \\ 0 & \alpha_2 & \alpha_3 & -\mu - \alpha_4 \end{bmatrix}. \tag{4.9}$$

The eigenvalues to the evaluated Jacobian matrix in equation (4.9) is obtained in equation (4.10) as follows:

$$\lambda_1 = -\mu, \qquad \lambda_2 = -(\mu + \alpha_4), \qquad \lambda_3 = -(\mu + \alpha_3 + \alpha_6), \qquad \lambda_4 = \frac{\Lambda\beta - \mu^2 - \alpha_1\mu - \alpha_2\mu - \alpha_5\mu}{\mu}. \tag{4.10}$$

By simplification to λ_4, we can deduce as follows:

$$\begin{aligned} \lambda_4 = \frac{\Lambda\beta - \mu^2 - \alpha_1\mu - \alpha_2\mu - \alpha_5\mu}{\mu} &= \frac{\Lambda\beta - \mu(\mu + \alpha_1 + \alpha_2 + \alpha_5)}{\mu} \\ &= \frac{\mu(\mu + \alpha_1 + \alpha_2 + \alpha_5)}{\mu}\left(\frac{\Lambda\beta}{\mu(\mu + \alpha_1 + \alpha_2 + \alpha_5)} - 1\right). \end{aligned} \tag{4.11}$$

With respect to the equation (4.11), λ_4 can be obtained as

$$\lambda_4 = (\mu + \alpha_1 + \alpha_2 + \alpha_5)(R_{UE} - 1) < 1 \text{ if } R_{UE} < 1.$$

Thus local asymptotic stability implies since all the eigenvalues are real and negative whenever $R_{UE} < 1$. □

4.3. Global stability of the (UPPIFE), Ξ^0

Define the following Lyapunov function

$$L(P_h) = P_h. \tag{4.12}$$

By differentiating equation (4.12) w.r.t.t using the solution of the model system of equation (2.1)-(2.4), we obtain:

$$L'(P_h) = P_h' = \beta \cdot P_h \cdot U - (\alpha_1 + \alpha_2 + \alpha_5 + \mu)P_h = [\beta \cdot U - (\alpha_1 + \alpha_2 + \alpha_5 + \mu)]\, P_h. \tag{4.13}$$

Recall that, the U-PhD-Postdoc-Industry free equilibrium, $U = \frac{\Lambda}{\mu}$. From equation (4.13), substituting U, we obtain equation

$$\begin{aligned} = \left[\beta \cdot (\frac{\Lambda}{\mu}) - (\alpha_1 + \alpha_2 + \alpha_5 + \mu)\right] P_h &= (\alpha_1 + \alpha_2 + \alpha_5 + \mu)\left[\frac{\beta\Lambda}{\mu(\alpha_1 + \alpha_2 + \alpha_5 + \mu)} - 1\right] P_h \\ &= (\alpha_1 + \alpha_2 + \alpha_5 + \mu)(R_{UE} - 1)P_h \leqslant 0 \text{ if } R_{UE} \leqslant 1. \end{aligned}$$

The implication of the result is that the U-PhD-Postdoc-Industry free equilibrium is globally asymptotically stable if $R_{UE} \leqslant 1$, otherwise it's unstable.

5. U-PhD-Postdoc-Industry Equilibrium (UPPIEE)

Let $\Xi_{(UPPIEE)} = \left(U^*, P_h^*, P_D^*, I^*\right)$ represents the region where there is unemployment in the academia sector or population. In this section, the compartments considered are not equal to zero.
Let $\Xi^* = U^*, P_h^*, P_D^*, I$ be represented in equation (5.1) as

$$a_1 = \frac{\mu + \alpha_1 + \alpha_2 + \alpha_5}{\beta},$$

$$\begin{aligned} a_2 &= \frac{(\mu\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_2\alpha_6)\cdot\mu(\mu+\alpha_1+\alpha_2+\alpha_5)}{\beta\mu(\mu^2+(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6)\mu+(\alpha_1+\alpha_2+\alpha_4)\alpha_3+(\alpha_1+\alpha_6)\alpha_4+\alpha_2\alpha_6)},\\ a_3 &= \frac{\alpha_1\cdot(\mu+\alpha_4)\cdot\mu(\mu+\alpha_1+\alpha_2+\alpha_5)}{\beta\mu(\mu^2+(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6)\mu+(\alpha_1+\alpha_2+\alpha_4)\alpha_3+(\alpha_1+\alpha_6)\alpha_4+\alpha_2\alpha_6)},\\ a_4 &= \frac{(\mu+\alpha_3+\alpha_6)\cdot(\mu+\alpha_4)\cdot\mu(\mu+\alpha_1+\alpha_2+\alpha_5)}{\beta\mu(\mu^2+(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_6)\mu+(\alpha_1+\alpha_2+\alpha_4)\alpha_3+(\alpha_1+\alpha_6)\alpha_4+\alpha_2\alpha_6)}. \end{aligned} \tag{5.1}$$

For easier computation and representation, we represent the compartments as seen in equation (5.1) as $a_i : i = 1, \dots, 4$ in equation (5.2). The UPPIEE co-exist equilibrium points is expressed in terms of the UPPIFE and represented as follows equation (5.2):

$$U^* = a_1, \qquad P_h^* = a_2\left(R_{UE}-1\right), \qquad P_D^* = a_3\left(R_{UE}-1\right), \qquad I^* = a_4\left(R_{UE}-1\right) \tag{5.2}$$

The UPPIFE is obtained as equilibrium points by taking all the derivatives to zero.

5.1. Local stability of (UPPIE), Ξ^*

We investigate the local stability of the of the U-Phd-Postdoc-Industry equilibrium point. We define $\Xi^* = (a_1, a_2\cdot(R_{UE}-1), a_3\cdot(R_{UE}-1), a_4\cdot(R_{UE}-1))$ and evaluate using Jacobian matrix as represented in equation (5.4)

$$J_{\Xi^*} = \begin{bmatrix} -\beta a_4(R_{UE}-1)-\mu & -\beta a_1+\alpha_5 & \alpha_6 & \alpha_4\\ \beta a_4(R_{UE}-1) & \beta a_1-\mu-\alpha_1-\alpha_2-\alpha_5 & 0 & 0\\ 0 & \alpha_1 & -\mu-\alpha_3-\alpha_6 & 0\\ 0 & \alpha_2 & \alpha_3 & -\mu-\alpha_4 \end{bmatrix}. \tag{5.3}$$

From equation (5.4), we introduce the upper triangular matrix to obtain equation (5.4):

$$J_{\Xi^*} = \begin{bmatrix} -\beta a_4(R_{UE}-1)-\mu & -\beta a_1+\alpha_5 & \alpha_6 & \alpha_4\\ 0 & \beta a_1-\mu-\alpha_1-\alpha_2-\alpha_5 & 0 & 0\\ 0 & 0 & -\mu-\alpha_3-\alpha_6 & 0\\ 0 & 0 & 0 & -\mu-\alpha_4 \end{bmatrix}. \tag{5.4}$$

From equilibrium points in equation (5.2), the eigenvalues obtained from the equation (5.4) are as follows:

$$\lambda_1^* = -(\mu+\alpha_4), \qquad \lambda_2^* = -(\mu+\alpha_3+\alpha_6), \qquad \lambda_3^* = 0, \qquad \lambda_4^* = -\beta a_4(R_{UE}-1)-\mu < 0 \quad \text{if } R_{UE} > 1.$$

Thus, local asymptotic stability may not imply in this case since $\lambda_3 = 0$ indicates a saddle node though $R_{UE} > 1$. This means that more information is required to completely determine the local dynamics of the University-PhD-Postdoc-Industry equilibrium, indeed $\lambda_3 = 0$ could suggest the existence of bifurcation. This phenomenon is best explained by the center manifold theory as explained in [10]. To implement this theory, we transform the variable described in Table 2 as $x_i : i = 1, \dots, 4$ and determine the bifurcation parameter at $R_{UE} = 1$. We further determine if the eigenvalues of the Jacobian matrix of the UPPIFE has a zero as single eigenvalue while other eigenvalues are real and negative by investigating the left and right eigenvalues.

We consider the r.h.s of the equations (2.1)-(2.4) and re-represent as seen in equation (5.5)

$$\begin{aligned} B1 &= \Lambda-\beta\cdot x_2\cdot x_1+\alpha_4\cdot x_4+\alpha_5\cdot x_2+\alpha_6\cdot x_3-\mu\cdot x_1,\\ B2 &= \beta\cdot x_2\cdot x_1-\alpha_1\cdot x_2-\alpha_2\cdot x_2-\alpha_5\cdot x_2-\mu\cdot x_2,\\ B3 &= \alpha_1\cdot x_2-\alpha_3\cdot x_3-\mu\cdot x_3-\alpha_6\cdot x_3,\\ B4 &= \alpha_2\cdot x_2+\alpha_3\cdot x_3-\alpha_4\cdot x_4-\mu\cdot x_4. \end{aligned} \tag{5.5}$$

The Jacobian matrix of equation (5.5) is

$$J_{\Xi^1} = \begin{bmatrix} -\beta x_2 - \mu & -\beta x_1 + \alpha_5 & \alpha_6 & \alpha_4 \\ \beta x_2 & \beta x_1 - \mu - \alpha_1 - \alpha_2 - \alpha_5 & 0 & 0 \\ 0 & \alpha_1 & -\mu - \alpha_3 - \alpha_6 & 0 \end{bmatrix}.$$

Evaluating the Jacobian matrix at

$$x_1 = \frac{\Lambda}{\mu}, \quad x_2 = 0, x_3 = 0, \quad x_4 = 0, \quad \beta = \frac{\mu(\alpha_1 + \alpha_2 + \alpha_5 + \mu)}{\Lambda}.$$

By evaluation, its imperative to show if the part of the eigenvalues is zero.

$$J_{\Xi^2} = \begin{bmatrix} -\mu & -\alpha_1 - \alpha_2 - \mu & \alpha_6 & \alpha_4 \\ 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & -\mu - \alpha_3 - \alpha_6 & 0 \\ 0 & \alpha_2 & \alpha_3 & -\mu - \alpha_4 \end{bmatrix}.$$

In doing this, we obtain eigenvalues evaluated at J_{Ξ^2} are obtained

$$\lambda_1 = -\mu, \quad \lambda_2 = 0, \quad \lambda_3 = -\mu - \alpha_3 - \alpha_6, \quad -\lambda_4 = \mu - \alpha_4$$

and it has been shown that part of the eigenvalues has a zero from r.h.s.. The same idea was implemented for the l.h.s, and it suffices to obtain the same values of λ with a simple zero eigenvalue present. According to the center manifold theorem, we need to compute the coefficients of a and b under the condition of the signs of coefficients are always positive. We should note that if:

- $a < 0$ and $b > 0$, then its transactional (forward) bifurcation; and
- $a > 0$ and $b > 0$, then its subcritical (backward) bifurcation.

From the eigenvalues obtained, we construct a 4×1 column vector matrix which is defined by w_i : $i = 1, \ldots, 4$ and perform a matrix multiplication with the eigenvalues obtained at J_{Ξ^2}. Let the matrix multiplication be represented as J_Φ such that:

$$J_\Phi = \begin{bmatrix} -\mu w_1 + (-\alpha_1 - \alpha_2 - \mu) w_2 + \alpha_6 w_3 + \alpha_4 w_4 \\ 0 \\ \alpha_1 w_2 + (-\mu - \alpha_3 - \alpha_6) w_3 \\ \alpha_2 w_2 + \alpha_3 w_3 + (-\mu - \alpha_4) w_4 \end{bmatrix},$$

where

$$\begin{aligned} g0 &= \mu w_1 + (-\alpha_1 - \alpha_2 - \mu) w_2 + \alpha_6 w_3 + \alpha_4 w_4, \\ g1 &= 0, \\ g2 &= \alpha_1 w_2 + (-\mu - \alpha_3 - \alpha_6) w_3, \\ g3 &= \alpha_2 w_2 + \alpha_3 w_3 + (-\mu - \alpha_4) w_4. \end{aligned}$$

Solving for w_1, w_2, w_3, w_4, we obtain as follows:

$$\begin{aligned} w_1 &= -\frac{(\mu^2 + \mu\alpha_1 + \mu\alpha_2 + \mu\alpha_3 + \mu\alpha_4 + \mu\alpha_6 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_6 + \alpha_3\alpha_4 + \alpha_4\alpha_6) w_3}{\alpha_1(\mu + \alpha_4)}, \\ w_2 &= \frac{(\mu + \alpha_3 + \alpha_6) w_3}{\alpha_1}, \\ w_3 &= w_3, \end{aligned}$$

$$w_4 = \frac{(\mu\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_2\alpha_6)w_3}{\alpha_1(\mu + \alpha_4)}.$$

Considering the l.h.s, we implement the same procedure as the r.h.s at J_{Ξ^2}, equation (4.11) is obtained as well. We perform a matrix multiplication of 1×4 row vector defined by $v_i : i = 1, \ldots, 4$. By this multiplication, obtain the following:

$$g0 = -v_1\mu, \qquad g1 = v_1(-\alpha_1 - \alpha_2 - \mu) + v_3\alpha_1 + v_4\alpha_2,$$
$$g2 = v_1\alpha_6 + v_3(-\mu - \alpha_3 - \alpha_6) + v_4\alpha_3, \qquad g3 = v_1\alpha_4 + v_4(-\mu - \alpha_4).$$

We further employ the center manifold theorem to determine the coefficient values of a and b. This is computed using Maple to obtain expression for the a and b is obtained as:

$$a = -\frac{2v_2(\mu^2 + \mu\alpha_1 + \mu\alpha_2 + \mu\alpha_3 + \mu\alpha_4 + \mu\alpha_6 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_6 + \alpha_3\alpha_4 + \alpha_4\alpha_6)w_3^2(\mu + \alpha_3 + \alpha_6)\beta}{\alpha_1^2(\mu + \alpha_4)} < 0,$$
$$b = \frac{v_2(\mu + \alpha_3 + \alpha_6)w_3\Lambda}{\alpha_1\mu} > 0.$$

It can be concluded that the coefficients of $a < 0$ and $b > 0$, hence, the local dynamics of the University-PhD-Postdoc-Industry equilibrium is transcritical, i.e., forward bifurcation which implies that the equilibrium is stable if $R_{UE} > 1$.

6. Numerical analysis

In this section, we would numerically simulate by providing positive values to the parameters and initial condition of the compartments. To numerically simulate, we take the values of the parameters presented in Table 3 and the graphs can be seen in Figures 2 and 3.

Parameter	Values	Parameter	Values
Λ	0.5	β	0.0075
α_1	0.0005	μ	0.045
α_2	0.015	α_4	0.05
α_3	0.015	α_5	0
α_6	0.015		

Table 3: Parameters of new mathematical model for unemployment and migration.

For the purpose of initial simulation, we investigate the effect of α_5 when increased, α_5 is observed at when $\alpha_5 = 0.01, 0.02, 0.03, 0.04$ and the initial conditions are to be taken as: $U(0) = 7, P_h(0) = 1, P_d(0) = 1, i(0) = 1$.

From the university population graph, it can be inferred that when $\alpha_5 = 0$, there is an increase in unemployment but no migration of the PhD population, which is symbolized by red line. The university population increased (both population and presence of migration are observed) at $\alpha_5 = 0.01.0.02, 0.03$, and 0.04 as PhD holders gain employment whereas the PhD population in the PhD compartments falls as migration occur.

The graph makes it clear that there is no justification for the migration of PhD graduates to the Postdoc and industry sector segments because unemployment in universities has decreased as a result of their acceptance of PhD graduates.

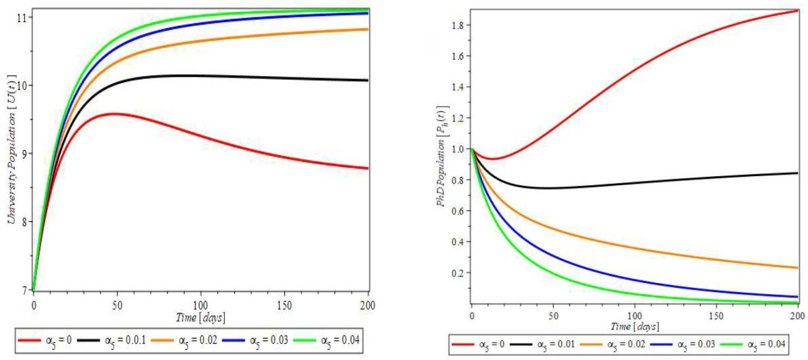

Figure 2: Graph showing the impact of $alpha5$ on the university and PhD population.

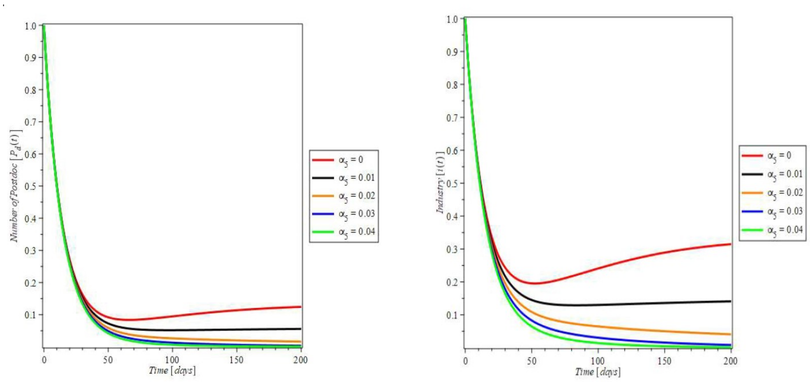

Figure 3: Graph showing the impact of $alpha5$ on Postdoc and industry population.

7. Conclusion and recommendation

From the stability analysis of UPPIFE denoted by Ξ^0, we were able to obtain the recruitment number R_{UE}, Ξ^0 is locally asymptotically stable whenever $R_{UE} < 1$ and employing the use of Lyapunov function, we show that the global stability denoted of Ξ^0 is stable if $R_{UE} \leqslant 1$. For the stability analysis of UPPIEE denoted by Ξ^* which depicts the region where the compartments co-exist and represented in the form of UPPIFE. From investigation, the eigenvalues contain a simple zero (0) which indicates a saddle node and the presence of Bifurcation, which requires more information to determine the stability. To investigate this further, we employ the idea of Center Manifold Theorem (CMT) that requires the coefficients ($a < 0$) and ($b > 0$). The concluding result indicated that Ξ^* is transcritical as a forward bifurcation and stable if $R_{UE} > 1$.

From the numerical simulation, we carefully assume the value of parameters using the the Maple and Mathematica software to investigate the impact of increasing rate of PhD graduates absorbed by the university and its effect on the postdoc and industry compartments. The model developed illustrates migration within 4 compartments (university, PhD, postdoc, and industry), and we looked into the model's stability analysis and the impact of increasing α_5. According to Figures 2 and 3, as more PhD graduates get absorbed at the university, the more migration tends towards the university. As a result, there is lesser movement to the postdoc and industrial compartments.

The university will, nevertheless, be perceived as a location to support researchers' career objectives and aspirations if financial stability, employment security, suitable incentives, effective manpower management, and more attention is paid to the mental health of researchers (PhD grads, postdocs, early careers).

In order to stop the brain-depot from academia and the lack of interest in research that would warrant interest in non-academic positions, it is imperative that policymakers implement attractive pay, equality, and the availability of positions free from racism; otherwise, the industry runs the risk of acting as a brain-depot. The model will be improved in the future by taking political and chaotic terms into account for both sectors. However, the impact of this research is not limited to South Africa but we extend this work with to other countries and this will be done with the use of data and fitting the parameters to study possible effect and solution to sector considered. Future work could be detail on improving the model by considering chaotic term and political impact on both sectors.

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