J. Math. Computer Sci., 31 (2023), 95–101

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Nörlund $\mathcal{I}$-convergent double sequence spaces via Orlicz function

Nazneen Khan[a,*], Manal Al-labadi[b], Areej Al-Balawi[a], Eman Almuhur[c]

[a] Dept. of Mathematics, College of Science, Taibah University, Madina, Saudi Arabia.

[b] Dept. of Basic Science and Humanities, Faculty of Arts and Science, University of Petra, Amman, Jordan.

[c] Dept. of Basic Science and Humanities, Faculty of Arts and Science, Applied Science Private University, Amman, Jordan.

Abstract

In the present article, we introduce some new classes of $\mathcal{I}$-convergent double sequence spaces defined by Orlicz function using the Nörlund mean. We also find the condition for any sequence in these spaces to be $\mathcal{I}$-convergent and draw some inclusion relations between them.

Keywords: Nörlund sequences, Orlicz function, ideal, filter, ideal-convergence.

2020 MSC: 40A05, 40A35, 40G15.

1. Introduction

Orlicz Sequence space l^M was first introduced by Lindenstrauss and Tzafriri [18], in the year 1971. They proved that this sequence space contains a subspace isomorphic to l_p $(1 \leqslant p < \infty)$. Later, Orlicz function was employed by several authors to construct different classes of sequences [12, 21]. Let c_0, c, and l_∞denote the spaces of all null, convergent, and bounded sequence, respectively, with the usual sup-norm defined by

$$\|x\|_\infty = \sup_k |x_k| \text{ for each } k \in \mathbb{N}.$$

Orlicz function is defined as $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$, [18, 21, 24, 25]. The integral representation of the Orlicz function M is given as $M(x) = \int_0^x \eta(t)dt$ where η is the kernel of M, $\eta(t) > 0$, η is non-decreasing and right differentiable for $t \geqslant 0$. Also $\eta(t) \to \infty$ as $t \to \infty$, [6].

A double sequence, denoted by $x = (x_{ij})$ is a double infinite array of elements $x_{kl} \in \mathbb{R}$ for all $k, l \in \mathbb{N}$. For a detailed study of double sequences, one can go through numerous articles available, naming a few [3, 8, 14, 16]. A double sequence (x_{jk}) is called statistically convergent to L if

$$\lim_{(m,n)\to\infty} \frac{1}{mn} |(j,k) : |x_{jk} - L| \geqslant \varepsilon, j \leqslant m, k \leqslant n| = 0,$$

*Corresponding author

Email address: nkkhan@taibahu.edu.sa (Nazneen Khan)

doi: 10.22436/jmcs.031.01.08

Received: 2023-01-29 *Revised:* 2022-02-17 *Accepted:* 2022-03-03

where the vertical bars indicate the number of elements in the set.

Furthermore, let $2^{\omega} = \{x = (x_{n,m}) : x_{n,m} \in \mathbb{R}\}$ denote the linear space of all real double sequences. If λ and μ denote two double sequence spaces and $A = (a_{n,k})$ is an infinite matrix of real numbers $(a_{n,k})$ for $n, k \in \mathbb{N}$, then A determines a matrix transformation from λ into μ that is $A : \lambda \to \mu$ such that for every double sequence $x = (x_{jk}) \in \lambda$, the A-transform of x, which is $Ax = A_{mn}(x)$, is in μ. The double sequence $A_{mn}(x)$ is given as

$$A(x) = \sum_{k=0}^{\infty} a^k_{mn} x_{mn}, \quad \text{for each } (m,n) \in \mathbb{N} \times \mathbb{N}.$$

In 1949, Hardy [7], introduced the notion of sequence of non-negative real numbers (t_k) and constructed the sequence T_n, the sequence of the partial sums as $T_n = \sum_{k=0}^{n} t_n$ for all $n \in \mathbb{N}$ and $t_0 > 0$. Later, the Nörlund mean for the sequence t_k was defined as $N^t = (a^t_{nk})$ for all $k, n \in \mathbb{N}$. This concept of Nörlund mean was used to construct sequence spaces by Wang in the year 1971, [26]. Further studies on Nörlund mean can be found in [1, 9, 10, 13, 15, 23].

From [19], the double transform of T_n is given as $T_{m,n} = \sum_{j=0}^{m} \sum_{k=0}^{n} t_{jk}$ for all $(m,n), (j,k) \in \mathbb{N} \times \mathbb{N}$ with $t_{00} > 0$. Therefore, the Nörlund mean for the sequence $t = t_{jk}$ is given by $N^t = (a^t_{mnjk})$ as

$$a^t_{nk} = \begin{cases} \frac{t_{m-j,n-k}}{T_{m,n}}, & \text{if } 0 \leqslant j \leqslant m \text{ and } 0 \leqslant k \leqslant n, \\ 0, & \text{if } j > m \text{ and } k > n, \end{cases}$$

for all (j,k) and $(m,n) \in \mathbb{N} \times \mathbb{N}$. It can be proved that the Nörlund matrix with respect to N_t-transform are in the space l_∞ as follows:

$$l_\infty(N^t) := \left\{ x = (x_{m,n}) \in 2^{\omega} : \sup_{(m,n) \in \mathbb{N} \times \mathbb{N}} \left| \frac{1}{T_{m,n}} \sum_{j=0}^{m} \sum_{k=0}^{n} t_{m-j,n-k} x_{jk} \right| < \infty \right\}.$$

The notion of $\mathcal{I}$-convergence has been generalized from statistical convergence given by [5]. Initially, it was studied by Kostyrko et al. [17]. Later, it was extensively used by others, [2, 4, 8, 11, 22]. Henceforth, we give some useful definitions about $\mathcal{I}$-convergence.

Definition 1.1. Let X be a non empty set. Then a family of sets $\mathcal{I} \subseteq 2^X$ (power set of X) is said to be an ideal in X if $\emptyset \in \mathcal{I}$, $\mathcal{I}$ is additive, i.e., $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$ and $\mathcal{I}$ is hereditary, i.e., $A \in \mathcal{I}$, $B \subseteq A \Rightarrow B \in \mathcal{I}$, [3, 20].

Definition 1.2. An ideal $\mathcal{I} \subseteq 2^X$ is called non-trivial if $\mathcal{I} \neq 2^X$. A non-trivial ideal $\mathcal{I} \subseteq 2^X$ is called admissible if $\{\{x\} : x \in X\} \subseteq \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is maximal if there cannot exist any non-trivial ideal $\mathcal{J} \neq \mathcal{I}$ containing $\mathcal{I}$ as a subset [8].

Definition 1.3. For each ideal $\mathcal{I}$, there is a filter $\mathcal{F}(\mathcal{I})$ corresponding to $\mathcal{I}$, that is a non-empty family of sets $\mathcal{F}(\mathcal{I}) \subseteq 2^X$ is said to be filter on X if and only if $\Phi \notin \mathcal{F}(\mathcal{I})$, for $A, B \in \mathcal{F}(\mathcal{I})$, we have $A \cap B \in \mathcal{F}(\mathcal{I})$, and for each $A \in (\mathcal{F}(\mathcal{I}))$ and $A \subseteq B$ implies $B \in \mathcal{F}(\mathcal{I})$, [8].

Definition 1.4. A double sequence $(x_{ij}) \in_2 \omega$ is said to be $\mathcal{I}$-convergent to a number L if for every $\epsilon > 0$ we have $\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geqslant \epsilon\} \in \mathcal{I}$, [19]. In this case we write $\mathcal{I} - \lim x_{ij} = L$.

Definition 1.5. A double sequence $(x_{ij}) \in_2 \omega$ is said to be $\mathcal{I}$-Cauchy if for every $\epsilon > 0$, there exists $(m,n)(\epsilon)$ such that we have $\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geqslant \epsilon\} \in \mathcal{I}$, [19].

Definition 1.6. A sequence space E is said to be solid or normal if for every sequence of scalars α_{ij} with $|\alpha_{ij}| < 1$, we have $\alpha_{ij}(x_{ij}) \in E$, [19].

Definition 1.7. A sequence space E is said to be convergence free if for $(x_{ij}), y_{ij} \in E$ we have $y_{ij} = 0$ implies $(x_{ij}) = 0$, [19].

Lemma 1.8. *Every solid space is monotone.*

In [15], Khan et al., introduced the Nörlund ideal convergent sequence spaces $c_0^{\mathcal{I}}(N^t)$, $c^{\mathcal{I}}(N^t)$, and $l_\infty^{\mathcal{I}}(N^t)$, which were later extended to double sequences in [19]. Hence we get our motivation to use Orlicz function on these spaces and study its topological and algebraic properties.

2. Main results

In this section we introduce the new sequence spaces ${}_2c_0^{\mathcal{I}}(N^t, M)$, ${}_2c^{\mathcal{I}}(N^t, M)$, and ${}_2l_\infty^{\mathcal{I}}(N^t, M)$. Moreover, we study topological and algebraic properties of double sequences in these spaces and introduce some inclusion relations between them. Throughout the article, we assume that $\mathcal{I}_2$ is an admissible ideal in $\mathbb{N} \times \mathbb{N}$ and $N^t_{m,n}(x)$ is the N^t-transform of double sequence $x = (x_{jk}) \in {}_2\omega$, where $N^t_{m,n}(x)$ is given by

$$N^t_{m,n}(x) := \frac{1}{T_{m,n}}\left(\sum_{j=0}^{m}\sum_{k=0}^{n} t_{m-j,n-k}x_{jk}\right).$$

Define,

$${}_2c^{\mathcal{I}}(N^t, M) := \Bigg\{(x_{jk}) \in {}_2\omega : \{(j,k) \in \mathbb{N} \times \mathbb{N} : \left|\frac{1}{T_{m,n}}\sum_{j=0}^{m}\sum_{k=0}^{n} M(t_{m-j,n-k}x_{jk}) - L\right| \geqslant \epsilon \text{ for some } L \in \mathbb{R}\} \in \mathcal{I}_2\Bigg\},$$

$${}_2c_0^{\mathcal{I}}(N^t, M) := \left\{(x_{jk}) \in {}_2\omega : \{(j,k) \in \mathbb{N} \times \mathbb{N} : \left|\frac{1}{T_{m,n}}\sum_{j=0}^{m}\sum_{k=0}^{n} M(t_{m-j,n-k}x_{jk})\right| \geqslant \epsilon\} \in \mathcal{I}_2\right\},$$

$${}_2l_\infty^{I}(N^t, M) := \left\{(x_{jk}) \in {}_2\omega : \{(j,k) \in \mathbb{N} \times \mathbb{N} : \left|\frac{1}{T_{m,n}}\sum_{j=0}^{m}\sum_{k=0}^{n} M(t_{m-j,n-k}x_{jk})\right| \geqslant K\} \in \mathcal{I}_2\right\}.$$

Henceforth, for convenience we will write

$$M(N^t_{m,n}(x)) = \left|\frac{1}{T_{m,n}}\sum_{j=0}^{m}\sum_{k=0}^{n} M(t_{m-j,n-k}x_{jk})\right|.$$

Also we construct the spaces

$${}_2m^{\mathcal{I}}(N^t, M) := {}_2c^{\mathcal{I}}(N^t, M) \cap {}_2l_\infty^{\mathcal{I}}(N^t, M), \quad {}_2m_0^{\mathcal{I}}(N^t, M) := {}_2c_0^{\mathcal{I}}(N^t, M) \cap {}_2l_\infty^{\mathcal{I}}(N^t, M).$$

Definition 2.1. Let $\mathcal{I}_2$ is an admissible ideal of a subset of $\mathbb{N} \times \mathbb{N}$. If for each $\epsilon > 0$, there exists a number $(m, n) \in \mathbb{N}$ depending on ϵ such that $\{(j,k) \in \mathbb{N} \times \mathbb{N} : |N^t_{j,k}(x) - (x)N^t_{m,n}(x)| \geqslant \epsilon\} \in \mathcal{I}$, then a sequence $\{x = (x_{jk} \in {}_2\omega\}$ is called Nörlund $\mathcal{I}$-Cauchy double sequence.

Example 2.2 Consider a non-trivial ideal $\mathcal{I}_{2_d} = \{A \times A \subseteq \mathbb{N} \times \mathbb{N} : d(A) = 0\}$, where $d(A)$ denotes the natural density of the set A. In this case, ${}_2c^{\mathcal{I}_d}(N_t, M) = S(N_t, M)$, where $S(N_t, M)$ is the space of all Nörlund statistically convergent double sequences under Orlicz function. It is defined as

$$S(N^t, M) := \{x = (x_{j,k}) \in {}_2\omega : d(\{(m,n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{m,n}(x)) - L \geqslant \epsilon\}) = 0, \text{ for some } L \in \mathbb{R}\}.$$

Theorem 2.2. *The sequence spaces* ${}_2c_0^{\mathcal{I}}(N^t, M)$, ${}_2c^{\mathcal{I}}(N^t, M)$, ${}_2l_\infty^{\mathcal{I}}(N^t, M)$, ${}_2m^{\mathcal{I}}(N^t, M)$, *and* ${}_2m_0^{\mathcal{I}}(N^t, M)$ *are all linear vector spaces.*

Proof. Let $x = (x_{jk})$ and $y = (y_{jk})$ be two arbitrary sequences in ${}_2c^{\mathcal{I}}(N^t, M)$ and α, β are two scalars. By the definition of ${}_2c^{\mathcal{I}}(N^t, M)$, for some $\epsilon > 0$ there exist L_1, $L_2 \in \mathbb{R}$ such that

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left|M(N^t_{mn}(x)) - L_1\right| \geqslant \frac{\epsilon}{2}\right\} \in I, \quad \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left|M(N^t_{mn}(y)) - L_2\right| \geqslant \frac{\epsilon}{2}\right\} \in I.$$

Now let

$$A_1 = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \left|M(N^t_{mn}(x)) - L_1\right| < \frac{\epsilon}{2\alpha}\right\} \in \mathcal{F}(\mathcal{I}),$$

$$A_2 = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \left|M(N^t_{mn}(y)) - L_2\right| < \frac{\epsilon}{2\beta}\right\} \in \mathcal{F}(\mathcal{I}),$$

such that $A_1^c, A_2^c \in \mathcal{I}$. Then

$$A_3 = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \left|M(\alpha N^t_{mn}(x)) + M(\beta N^t_{mn}(y)) - (\alpha L_1 + \beta L_2)\right| < \epsilon\right\} \supseteq \{A_1 \cap A_2\}.$$

Since, the set on the right is in filter $\mathcal{F}(\mathcal{I})$, so the complement of the set on left is in the ideal $\mathcal{I}$. Therefore, $(\alpha x + \beta y) \in {}_2c^{\mathcal{I}}(N^t, M)$. For other sequence spaces, result follows in a similar way. This completes the proof. □

Theorem 2.3. *A sequence $x = (x_{jk}) \in {}_2\omega$ is Nörlund $\mathcal{I}$-convergent if and only if for every $\epsilon > 0$, there exists $(r,s) \in \mathbb{N}\times\mathbb{N}$ depending on ϵ such that*

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |M(N^t_{rs}(x)) - M(N^t_{mn}(x))| < \epsilon\} \in \mathcal{F}(\mathcal{I}). \tag{2.1}$$

Proof. Let the double sequence $x = (x_{jk}) \in {}_2\omega$ be a Nörlund $\mathcal{I}$- sequence to some number $L \in \mathbb{R}$. Then, for a given $\epsilon > 0$,

$$A_\epsilon = \{(m,n) \in \mathbb{N}\times\mathbb{N} : |M(N^t_{mn}(x) - L)| < \frac{\epsilon}{2}\} \in \mathcal{F}(\mathcal{I}).$$

For any two fixed numbers $(r,s) \in \mathbb{N}\times\mathbb{N}$ depending on ϵ, we have

$$M\Big(|N^t_{mn}(x) - N^t_{rs}(x)|\Big) \leqslant M|(N^t_{mn}(x) - L)| + M|L - N^t_{rs}(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all $m,n \in A_\epsilon$. Hence (2.1) holds.

Conversely, assume that (2.1) holds for all $\epsilon > 0$. Then

$$E_\epsilon = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : N^t_{mn}(x) \in [N^t_{mn}(x) - \epsilon, N^t_{mn}(x) + \epsilon]\right\} \in \mathcal{F}(\mathcal{I}), \quad \forall \ \epsilon > 0.$$

Let $J_\epsilon = [N^t_{mn}(x) - \epsilon, N^t_{mn}(x) + \epsilon]$. Hence we have $E_\epsilon \in \mathcal{F}(\mathcal{I})$ and also $E_{\frac{\epsilon}{2}} \in \mathcal{F}(\mathcal{I})$ for fixed $\epsilon > 0$. Therefore $E_\epsilon \cap E_{\frac{\epsilon}{2}} \in \mathcal{F}(\mathcal{I})$. This implies that $J = J_\epsilon \cap J_{\frac{\epsilon}{2}} \neq \phi$ and thus, $\text{diam}(J) \leqslant \frac{1}{2}\text{diam}(j_\epsilon)$, where diam of J denotes the length of the interval J. Proceeding inductively, we get a sequence of closed intervals

$$J_\epsilon = \mathcal{I} \supseteq I_1 \supseteq I_2 \supseteq I_3 \cdots \supseteq I_k \supseteq \cdots,$$

where $\text{diam} I_n \leqslant \text{diam} I_{n-1}$ for $n = (1,2,3,\ldots)$ and

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : |M(N^t_{mn}(x))| < \epsilon\} \in \mathcal{F}(\mathcal{I}).$$

Then there exists a number $\xi \in \cap_{k\in\mathbb{N}}(\mathcal{I})$ such that $\xi = (\mathcal{I}) - \lim(N^t_{mn}(x))$, which implies $L = M(\xi) = (\mathcal{I}) - \lim M(N^t_{mn}(x))$. Therefore the double sequence $x = (x_{jk}) \in {}_2\omega$ is Nörlund-$\mathcal{I}_2$ convergent. □

Theorem 2.4. *Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ be a non trivial ideal. Then the inclusion ${}_2c(N^t, M) \subset {}_2c^{\mathcal{I}}(N^t, M)$ is strict.*

Proof. The inclusion ${}_2c \subset {}_2c^{\mathcal{I}}$ is obvious and for any spaces X and Y, if $X \subseteq Y$, then $X(N^t) \subseteq Y(N^t)$. Therefore we have ${}_2c(N^t) \subseteq {}_2c^{\mathcal{I}}(N^t)$. □

Consider the following example to verify strict inclusion.

Example 2.5. Define the sequence $x = (x_{jk}) \in {}_2\omega$ such that

$$N^t_{mn} = \begin{cases} \sqrt{(mn)}, & \text{if } m = j^2, n = k^2 \text{ for } (j,k) \in \mathbb{N}\times\mathbb{N}, \\ mn, & \text{otherwise}. \end{cases}$$

Then $x \in {}_2c^{\mathcal{I}_d}(N^t, M)$ but $x \notin {}_2c(N^t, M)$.

Theorem 2.6. *The inclusions* ${}_2c_0^{\mathcal{I}}(N^t, M) \subset {}_2c^{\mathcal{I}}(N^t, M) \subset {}_2l_\infty^{\mathcal{I}}(N^t, M)$ *hold.*

Proof.
(i) ${}_2c_0^{\mathcal{I}}(N^t, M) \subset {}_2c^{\mathcal{I}}(N^t, M)$. The inclusion ${}_2c_0^{\mathcal{I}}(N^t, M) \subset {}_2c^{\mathcal{I}}(N^t, M)$ is obvious. To show the strictness of the inclusion, consider the double sequence $x = (x_{j,k}) \in {}_2\omega$ such that $N^t_{m,n}(x) = 1$, which gives $M(N^t_{m,n}(x)) = M(1) > 0$. Therefore the double sequence $x = (x_{jk}) \in {}_2c^{\mathcal{I}}(N^t, M) \setminus {}_2c_0^{\mathcal{I}}(N^t, M)$.

(ii) ${}_2c^{\mathcal{I}}(N^t, M) \subset {}_2l_\infty^{\mathcal{I}}(N^t, M)$. Let $x = (x_{jk}) \in {}_2c^{\mathcal{I}}(N^t, M)$, then there exists a number $L \in \mathbb{R}$ such that $\mathcal{I} - \lim M|N^t_{m,n} - L| = 0$. That is

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{m,n}) - L| \geqslant \epsilon\} \in \mathcal{I}.$$

Therefore we have

$$|M(N^t_{m,n})| = |M(N^t_{m,n}) - L + L| \leqslant |M(N^t_{m,n}) - L| + |L|.$$

This implies that the double sequence $x = (x_{jk})$ must be in ${}_2l_\infty^{\mathcal{I}}(N^t, M)$. □

For strict inclusion consider the example given below.

Example 2.7. Define the sequence $x = (x_{jk}) \in {}_2\omega$ such that

$$N^t_{mn} = \begin{cases} \sqrt{(mn)}, & \text{if } mn \text{ is square,} \\ 1, & \text{if } (mn) \text{ is odd non-square,} \\ 0, & \text{if } (mn) \text{ is even non-square.} \end{cases}$$

Then double sequence $N^t_{mn}(x) \in {}_2l_\infty^{\mathcal{I}}$ but $N^t_{mn}(x) \notin {}_2c^{\mathcal{I}}$, which implies that $x \in {}_2l_\infty^{\mathcal{I}}(N^t, M) \setminus {}_2c^{\mathcal{I}}(N^t, M)$. Hence the inclusions are strict.

Theorem 2.8. *The spaces* ${}_2m^{\mathcal{I}}(N^t, M)$ *and* ${}_2m_0^{\mathcal{I}}(N^t, M)$ *are Banach spaces with the given norm*

$$\|x\|_{X(N^t)} = \sup |M(N^t_{mn}(x)|, \textit{ where } X \in \{{}_2m^{\mathcal{I}}, {}_2m_0^{\mathcal{I}}\}.$$

Proof. Since the double sequence spaces ${}_2m^{\mathcal{I}}(N^t)$ and ${}_2m_0^{\mathcal{I}}(N^t)$ are Banach spaces with the norm

$$\|x\|_{X(N^t)} = \sup |(N^t_{mn}(x)|, \text{ where } X \in \{{}_2m^{\mathcal{I}}, {}_2m_0^{\mathcal{I}}\}$$

and Orlicz function is a bounded function, therefore the sequence spaces ${}_2m^{\mathcal{I}}(N^t, M)$ and ${}_2m_0^{\mathcal{I}}(N^t, M)$ are also Banach under the given norm. □

Theorem 2.9. *The sequence spaces* $c_0^{\mathcal{I}(N^t,M)}$ *and* $m_0^{\mathcal{I}(N^t,M)}$ *are solid and monotone.*

Proof. We prove the result for $c_0^{\mathcal{I}(N^t,M)}$. Let $x = (x_{jk} \in c_0^{\mathcal{I}(N^t,M)}$ for $\epsilon > 0$, the set

$$\{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(x))| \geqslant \epsilon\} \in \mathcal{I}. \tag{2.2}$$

For a double sequence of scalars $\alpha = (\alpha_{jk})$ with $|\alpha| \leqslant 1$ for all $(j, k) \in \mathbb{N} \times \mathbb{N}$, and the Orlicz function being non-decreasing, we have

$$M|N^t_{mn}(\alpha.x)| = M|\alpha.N^t_{mn}(x)| \leqslant (|\alpha|)M|N^t_{mn}(x)| \leqslant M|N^t_{mn}(x)|, \text{ for all } (m, n) \in \mathbb{N} \times \mathbb{N}.$$

Using this inequality in (2.2) we get

$$\big\{\{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(\alpha.x))| \geqslant \epsilon\} \subseteq \{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(x))| \geqslant \epsilon\big\}\big\} \in \mathcal{I}.$$

This implies

$$\{\{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(\alpha.x))| \geqslant \epsilon\big\} \in \mathcal{I}.$$

Therefore the sequence $\alpha.(x_{jk}) \in {}_2c_0^{\mathcal{I}}(N^t, M)$. Hence the space ${}_2c_0^{\mathcal{I}}(N^t, M)$ is solid, which by Lemma 1.8 becomes monotone as well. □

Theorem 2.10. *The spaces* $c^{\mathcal{I}}(N^t, M)$ *and* $c_0^{\mathcal{I}}(N^t, M)$ *are not convergence free.*

Proof. The result is established by considering the following example. Let $\mathcal{I} = \mathcal{I}_d$. Then consider the double sequences $(x_{jk}) = \frac{1}{jk}$ and $(y_{jk}) = jk$, for all $(j, k) \in \mathbb{N} \times \mathbb{N}$. Then we see that $(x_{jk}) \in c^{\mathcal{I}}(N^t, M)$ and $c_0^{\mathcal{I}}(N^t, M)$ but $(y_{jk}) \notin c^{\mathcal{I}}(N^t, M)$ and $c_0^{\mathcal{I}}(N^t, M)$. Therefor the spaces are not convergence free. □

Theorem 2.11. *The function* $g : {}_2m^{\mathcal{I}}(N^t, M) \to \mathbb{R}$ *defined by* $g(x) = |\mathcal{I} - \lim M(N^t_{mn}(x))|$*, where*

$$ {}_2m^I(N^t, M) := {}_2c^I(N^t, M) \cap {}_2l^I_\infty(N^t, M), $$

is a Lipschitz function and hence it is uniformly continuous.

Proof. First, we show that g is well defined. Let $x, y \in {}_2m^{\mathcal{I}}(N^t, M)$ such that $x = y$, then

$$ N^t_{mn}(x) = N^t_{mn}(y), \quad M(N^t_{mn}(x)) = M(N^t_{mn}(y)), \quad |\mathcal{I} - M(N^t_{mn}(x))| = |\mathcal{I} - M(N^t_{mn}(y))|, \quad g(x) = g(y). $$

This shows that g is well defined. Next, we check if g satisfies Lipschitz criterion. Consider the double sequences $x = (x_{jk})$ and $y = (y_{jk})$ both in ${}_2m^{\mathcal{I}}(N^t, M)$ such that $x \neq y$. Then

$$ A_1 = \{(m, n) \in \mathbb{N} \times \mathcal{N} : |M(N^t_{mn}(x)) - g(x)| \geqslant |x - y|_*\} \in \mathcal{I}, $$
$$ A_2 = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(y)) - g(y)| \geqslant |x - y|_*\} \in \mathcal{I}, $$

where

$$ |x - y|_* = \sup_{(m,n)} |M(N^t_{mn}(x)) - M(N^t_{mn}(y))|. $$

Therefore

$$ B_1 = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(y)) - g(y)| < |x - y|_*\} \in \mathcal{F}(\mathcal{I}), $$

and

$$ B_1 = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(x)) - g(x)| < |x - y|_*\} \in \mathcal{F}(\mathcal{I}). $$

Hence we get $B = B_1 \cap B_2 \in \mathcal{F}(\mathcal{I})$, so that B is non-empty. Let $(m, n) \in B$ such that

$$ |g(x) - g(y)| \leqslant |g(x) - M(N^t_{mn}(x))| + |M(N^t_{mn}(x)) - M(N^t_{mn}(y))| + |M(N^t_{mn}(y)) - g(y)| \leqslant 3|x - y|_*. $$

Thus we find the Lipschitz constant $L = 3$ and g satisfies Lipschitz condition therefore it is uniformly continuous. □

Theorem 2.12. *If* $x = (x_{jk}), y = (y_{jk}) \in {}_2m^{\mathcal{I}}(N^t, M)$ *with*

$$ M(N^t_{mn}(x.y)) = M(N^t_{mn}(x)).M(N^t_{mn}(y)), \tag{2.3} $$

then $(x.y) \in 2m^{\mathcal{I}}(N^t, M)$ *and* $g(x.y) = g(x).g(y)$*, where* $g : 2m^{\mathcal{I}}(N^t, M) \to \mathbb{R}$ *is defined as* $g(x) = |\mathcal{I} - \lim M(N^t_{mn}(x))|$.

Proof. For $\epsilon > 0$,

$$ A = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(x)) - g(x)| < \epsilon\} \in \mathcal{F}(\mathcal{I}), $$

and

$$ B = \{(m, n) \in \mathbb{N} \times \mathbb{N} : |M(N^t_{mn}(y)) - g(y)| < \epsilon\} \in \mathcal{F}(\mathcal{I}), $$

where

$$ \epsilon = |x - y|_* = \sup_{(m,n)} |M(N^t_{mn}(x)) - M(N^t_{mn}(y))|. $$

Consider

$$ |M(N^t_{mn}(xy)) - g(x)g(y)| = |M(N^t_{mn}(x))M(N^t_{mn}(y)) - M(N^t_{mn}(x))g(y) + M(N^t_{mn}(x))g(y) - g(x)g(y)| $$

$$\leqslant |M(N^t_{mn}(x))||M(N^t_{mn}(y))g(y)| + |g(y)||M(N^t_{mn}(x)) - g(x)|,$$

as ${}_2m^{\mathcal{I}}(N^t, M) \subset {}_2l_\infty(N^t, M)$. Therefore there exists a number $L \in \mathbb{R}$ such that $|M(N^t_{mn})| < L$. Therefore from (2.3), we have

$$|M(N^t_{mn}(xy)) - g(x)g(y)| = |M(N^t_{mn}(x))M(N^t_{mn}(y)) - g(x)g(y)| \leqslant L_\epsilon + |g(y)|_\epsilon = \epsilon_1$$

for all $(m, n) \in A \cap B \in \mathcal{F}(\mathcal{I})$. Hence $(x.y) \in {}_2m^{\mathcal{I}}(N^t, M)$ and $g(x.y) = g(x).g(y)$. □

Acknowledgment

The authors are grateful to the referees and the editor for their careful reading and valuable comments that improved the presentation of the article.

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