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Numerical solutions for generalized trapezoidal fully fuzzy Sylvester matrix equation with sufficient conditions to have a positive solution

Ahmed Abdel Aziz Elsayed[a], Nazihah Ahmad[b,*], Ghassan Malkawi[c], Bassem Saassouh[d], Oluwaseun Adeyeye[b]

[a] Department of Mathematics, Institute of Applied Technology, Mohamed Bin Zayed City 33884, United Arab Emirates.
[b] School of Quantitative Sciences, Universiti Utara Malaysia, Sintok 06010, Kedah, Malaysia.
[c] Faculty of Engineering, Math and Natural Science Division, Higher Colleges of Technology (HCT), Al Ain Campus, Abu Dhabi 17155, United Arab Emirates.
[d] Academic Support Department, Abu Dhabi Polytechnic College, Abu Dhabi 111499, United Arab Emirates.

Abstract

This paper proposes three methods for solving a generalized trapezoidal fully fuzzy Sylvester matrix equation (GTrFFSME) and its special cases. The GTrFFSME is converted to an equivalent system of generalized crisp Sylvester matrix equations based on a new constructed fuzzy multiplication operation between three trapezoidal fuzzy numbers. An analytical solution to the GTrFFSME is obtained by developing a fuzzy matrix vectorization method, and the numerical solution is obtained by developing fuzzy gradient and fuzzy least-squares iterative methods. The necessary and sufficient conditions for the GTrFFSME to have a unique positive fuzzy solution are proved in addition to the convergence for the fuzzy gradient and fuzzy least-square methods. The constructed methods can solve other fuzzy equations such as Sylvester, Lyapunov and Stein matrix equations up to size 100×100. We illustrate the proposed methods by solving numerical examples with different size systems.

Keywords: Generalized fully fuzzy Sylvester matrix equations, gradient iterative, numerical fuzzy solution, least-squares iterative, trapezoidal fuzzy multiplication.

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1. Introduction

Generalized Sylvester matrix equation (GSME) and its special cases, including Sylvester, Lyapunov, and Stein matrix equations, play an important role in the design and analysis of linear control systems [10], reduction of large-scale dynamical systems [43], restoration of noisy images [7, 8], medical imaging data acquisition, model reduction [46] and stochastic control, image processing and filtering [7]. Analytical solutions to the GSME can be obtained by applying the concept of Vec-operator and Kronecker products.

*Corresponding author
Email addresses: ahmed_ahmed_abdel@ahsgs.uum.edu.my (Ahmed Abdel Aziz Elsayed), nazihah@uum.edu.my (Nazihah Ahmad), gmalkawi@hct.ac.ae (Ghassan Malkawi), bassem.saassouh@adpoly.ac.ae (Bassem Saassouh), adeyeye@uum.edu.my (Oluwaseun Adeyeye)

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This approach, however, is limited to GSMEs of small size because it requires getting the inverse of $mn \times mn$ matrices for a GSME of size $m \times n$, which leads to computation complexity for large GSME. Authors in [27] proposed another method for solving the GSME by converting the coefficient matrix into its Schur or Hessenberg form. However, this approach is also limited to small-sized GSMEs. For GSME with large dimensions ($n \geqslant 100$), numerical methods are more practical [45]. Many researchers have proposed numerical approaches for solving GSME and its special cases. For instance, the method of block successive over-relaxation proposed by Starke and Niethammer [47] and the Krylov subspace method by Lin [36], in addition to the method of the truncated low-rank algorithm by Kressner and Sirković [33], and the skew-Hermitian splitting method by [13]. Moreover, Ding and Chen [14, 15] have developed gradient iterative (GI) and least-squares iterative (LSI) algorithms, and Niu et al. [42] suggested a relaxed gradient iterative for finding the solution of the GSME.

However, in many applications, some of the system parameters with incomplete or uncertain values are represented by fuzzy numbers rather than crisp numbers. Fuzzy numbers can present uncertainty problems such as conflicting requirements during the system process, the distraction of any elements and noise. When all parameters of the GSME are in the fuzzy form, it is called a Generalized Fully Fuzzy Sylvester Matrix Equation (GFFSME). Therefore, GFFSME can be used in the stability, controllability and observability analysis of linear time-invariant systems and model order reduction of non-linear control systems and as an equation solver for image restoration, model reduction and medical imaging acquisition systems.

Definition 1.1. The fully fuzzy matrix equation that can be written as

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}, \tag{1.1}$$

where, $\tilde{A} = \left(\tilde{a}_{ij}\right)_{q\times p}$, $\tilde{B} = \left(\tilde{b}_{ij}\right)_{n\times r}$, $\tilde{C} = \left(\tilde{c}_{ij}\right)_{q\times p}$, $\tilde{D} = \left(\tilde{d}_{ij}\right)_{n\times r}$, $\tilde{X} = \left(\tilde{x}_{ij}\right)_{p\times n}$ and $\tilde{E} = \left(\tilde{e}_{ij}\right)_{q\times r}$, is called GFFSME.

The GFFSME in Eq. (1.1) contains the following well-known fully fuzzy matrix equation.

Definition 1.2. If $\tilde{B}$ and $\tilde{C}$ are identity fuzzy matrices, then Eq. (1.1) can be written as

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{D} = \tilde{E}, \tag{1.2}$$

where, $\tilde{A} = \left(\tilde{a}_{ij}\right)_{p\times p}$, $\tilde{D} = \left(\tilde{d}_{ij}\right)_{n\times n}$, $\tilde{X} = \left(\tilde{x}_{ij}\right)_{p\times n}$ and $\tilde{E} = \left(\tilde{e}_{ij}\right)_{p\times n}$ is called a Fully Fuzzy Sylvester Matrix Equation (FFSME).

Definition 1.3. If $\tilde{B}$ and $\tilde{C}$ are identity fuzzy matrices and $\tilde{D} = \tilde{A}^T$, then Eq. (1.1) can be written as

$$\tilde{A}\tilde{X} + \tilde{X}\tilde{A}^T = \tilde{E}, \tag{1.3}$$

where, $\tilde{A} = \left(\tilde{a}_{ij}\right)_{p\times p}$, $\tilde{A}^T = \left(\tilde{a}^T_{ij}\right)_{p\times p}$, $\tilde{X} = \left(\tilde{x}_{ij}\right)_{p\times p}$ and $\tilde{E} = \left(\tilde{e}_{ij}\right)_{p\times p}$ is called a Fully Fuzzy Continuous-Time Lyapunov Matrix Equation (FFCTLME).

Definition 1.4. If $\tilde{A}$ and $\tilde{B}$ are identity fuzzy matrices, then Eq. (1.1) can be written as

$$\tilde{X} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E}, \tag{1.4}$$

where, $\tilde{C} = \left(\tilde{c}_{ij}\right)_{p\times p}$, $\tilde{D} = \left(\tilde{d}_{ij}\right)_{n\times n}$, $\tilde{X} = \left(\tilde{x}_{ij}\right)_{p\times n}$ and $\tilde{E} = \left(\tilde{e}_{ij}\right)_{p\times n}$ is called a Fully Fuzzy Stein Matrix Equation (FFSTME).

Authors in the fuzzy environment have proposed analytical methods for solving the FFSME. The FFSME converted to a system of crisp linear equations using Dubois and Prade's arithmetic operators for multiplication [18]. The solution for such systems is obtained using many classical methods such as matrix inversion, Vec-operator and Kronecker product.

The Triangular Fully Fuzzy Sylvester Matrix Equation (TFFSME) has been studied analytically by [11, 28, 38]. Recently, El Sayed et al. [20, 23, 32] considered the solution of a Trapezoidal Fully Fuzzy Sylvester Matrix Equation (TrFFSME) by transforming the TrFFSME to a system of linear matrix equations where the positive and negative fuzzy solutions are obtained using the Vec-operator and Kronecker product method. However, these methods were restricted only to positive fuzzy numbers and required a long multiplication process and consequently long computational timing. In addition, El Sayed et al. [25] proposed a two-stage algorithm method for solving TrFFSME with arbitrary fuzzy solutions where the first stage algorithm can reduce the search area for the solution and the second stage algorithm can find the fuzzy solution. The two-stage algorithm method successed in obtaining a full arbitrary fuzzy solution to the TrFFSME. However, it required long computational timing to solve the TrFFSME with large sizes ($n > 10$). Furthermore, El Sayed et al. [21] extended the TrFFSMEE to a GTrFFSME and to a Coupled Trapezoidal Fully Fuzzy Sylvester Matrix Equations (CTrFFSME) [22, 24]. The arbitrary fuzzy solutions to the GTrFFSME and the CTrFFSME are obtained by applying new reduced fuzzy multiplication operations which convert the arbitrary GTrFFSME and CTrFFSME into a non-linear system of min-max equations. The system of min-max equations was then converted to an equivalent system of absolute equations where the fuzzy solutions are obtained by solving that system.

There are two main drawbacks of the existing methods in the literature; the first is that the existing analytical methods proposed for solving TFFSME and TrFFSME are based on Kronecker product and Vec-operator and therefore limit the size of the system to (2×2) or (3×3) due to its long multiplication process required to obtain the fuzzy solution. Few researchers considered fuzzy systems with sizes up to (10×10) such as in [37]. The second drawback is that the existence and uniqueness of the fuzzy solution for the TFFSME and TrFFSME are not examined before applying the existing methods. Therefore, there is no guarantee that applying many existing methods will always give the desired fuzzy solution.

In addition, the GTrFFSME is not investigated in the fuzzy literature. Thus, it is important to develop new analytical and numerical methods for solving the GTrFFSME and its special cases with large sizes. To deal with this shortcoming, in this paper, new arithmetic fuzzy multiplication operators between three TrFNs are constructed and then applied to the positive GTrFFSME in Eq. (1.1), which converts the GTrFFSME to an equivalent system of GSME. The Equivalency of the GTrFFSME and the system of GSME is proved, and therefore the consistency of the GTrFFSME can be examined using the system of GSME. In addition, three different methods are proposed for solving the GTrFFSME and its special cases. The exact positive fuzzy solution to the GTrFFSME is obtained analytically by the fuzzy matrix vectorization method. To solve positive GTrFFSME with large sizes, the hierarchical identification principle is applied to construct two new numerical methods, namely Fuzzy Gradient Iterative Methods (FGIM) and Fuzzy Least-Squares Iterative Methods (FLSIM). The numerical methods can obtain the fuzzy solution for large GTrFFSME up to (100×100) with a very small error bound compared to the existing numerical approaches, which were applied to at max (10×10) fuzzy systems [2–6, 12, 19, 26, 29, 48].

Moreover, the proposed methods can also be applied to other fuzzy equations such as Sylvester, Lyapunov and Stein matrix equations in Eqs. (1.2), (1.3), and (1.4) with both Triangular Fuzzy Numbers (TFNs) and Trapezoidal Fuzzy Numbers (TrFNs). To illustrate the effectiveness of the proposed methods for solving the GTrFFSME in Eq. (1.1), we solve two examples with different sizes, small (2×2) and large (100×100). In addition, the performance of the proposed methods is compared by calculating the number of iterations (k), convergence factor (α), error $\delta^l(k)$, error bound (ε), convergence rate, CPU time, real-time and memory usage. In addition to the graphical representation of the error $\delta^l(k)$ when k increases.

This paper is organized as follows. Section 2 introduces preliminary arithmetic operations of trapezoidal fuzzy numbers. In Section 3, new arithmetic multiplication operations between three TrFNs are developed. In Section 4, proposed methods for solving GTrFFSME are developed along with a presentation of its algorithm and necessary theorems for consistency and convergence. In Section 5, two numerical examples are presented to illustrate the proposed methods. Section 6 is dedicated to the conclusion.

2. Preliminaries

The following are basic definitions and results related to TrFNs in fuzzy theory [17, 30, 31, 35] and fuzzy matrix [1, 40, 41].

Definition 2.1. Let X be a universal set. Then, the fuzzy subset $\tilde{A}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \to [0,\ 1]$ which assigns to each element $x \in X$ a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$, where the function value of $\mu_{\tilde{A}}(x)$ represents the grade of membership of x in $\tilde{A}$. A fuzzy set $\tilde{A}$ is written as $\tilde{A} = \{(x,\ \mu_{\tilde{A}}(x)),\ x \in X,\ \mu_{\tilde{A}}(x) \in [0,\ 1]\}$.

Definition 2.2. A fuzzy set $\tilde{A}$, defined on the universal set of real number R, is said to be a fuzzy number if its membership function has the following characteristics:

1. $\tilde{A}$ is convex, i.e.,

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda) x_2 \geqslant \min(\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(x))\ \forall\ x_1, x_2 \in R\ , \forall \lambda \in [0,1]\,;$$

2. $\tilde{A}$ is normal, i.e., $\exists\ x_0 \in R$ such that $\mu_{\tilde{A}}(\ x_0) = 1$;
3. $\mu_{\tilde{A}}$ is piecewise continuous.

Definition 2.3. A fuzzy number $\hat{A} = (a_1,\ a_2,\ a_3,\ a_4)$ is a TrFN if its membership function is:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a_1, \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leqslant x \leqslant a_2, \\ 1, & a_2 \leqslant x \leqslant a_3, \\ \frac{a_4-x}{a_4-a_3}, & a_3 \leqslant x \leqslant a_4, \\ 0, & x > a_4. \end{cases}$$

Figure 1 Represents a TrFN in the form (a_1, a_2, a_3, a_4).

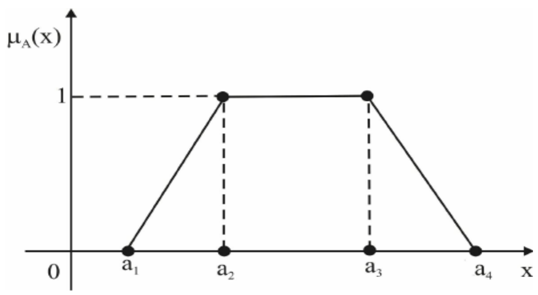

Figure 1: Representation of TrFN (a_1, a_2, a_3, a_4).

Definition 2.4. The sign of the TrFN $\tilde{A} = (a_1, a_2, a_3, a_4)$ can be classified as:

1. $\tilde{A}$ is positive (negative) iff $a_1 \geqslant 0, (a_4 \leqslant 0)$;
2. $\tilde{A}$ is zero iff $(a_1, a_2, a_3$ and $a_4 = 0)$;
3. $\tilde{A}$ is near zero iff $a_1 \leqslant 0 \leqslant a_4$.

Definition 2.5 (Operations of TrFNs)**.** The arithmetic operations of TrFNs are presented as follows, let $\hat{A} = (a_1, a_2, a_3, a_4)$, $\hat{B} = (b_1, b_2, b_3, b_4)$ be two TrFNs then:

1. Addition:

$$\hat{A} + \hat{B} = (a_1, a_2, a_3, a_4) + (b_1, b_2, b_3, b_4) = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4)\,. \tag{2.1}$$

2. Subtraction:

$$\hat{A} - \hat{B} = (a_1, a_2, a_3, a_4) - (b_1, b_2, b_3, b_4) = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1).$$

3. Symmetric image:

$$-\hat{A} = (-a_4, -a_3, -a_2, -a_1).$$

4. Scalar multiplication: Let $\lambda \in \mathbb{R}$, then,

$$\lambda \otimes (a_1, a_2, a_3, a_4) = \begin{cases} (\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4), & \lambda \geqslant 0, \\ (\lambda a_4, \lambda a_3, \lambda a_2, \lambda a_1), & \lambda < 0. \end{cases}$$

5. Multiplication: The multiplication between fuzzy numbers is neither commutative nor associative. Thus, TrFNs multiplication operations can be classified as follows:

Case I: If $\hat{A} = (a_1, a_2, a_3, a_4)$, $\hat{B} = (b_1, b_2, b_3, b_4)$ be two arbitrary TrFNs then:

$$\tilde{A}\tilde{B} = (a, h, m, d), \tag{2.2}$$

where,

$$a = \min(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4), \qquad h = \min(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3),$$
$$m = \max(a_2 b_2, a_2 b_3, a_3 b_2, a_3 b_3), \qquad d = \max(a_1 b_1, a_1 b_4, a_4 b_1, a_4 b_4).$$

Case II: If $\hat{A}, \hat{B} > 0$, then:

$$\tilde{A}\tilde{B} = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4). \tag{2.3}$$

Case III: If $\hat{A}, \hat{B} < 0$, then:

$$\tilde{A}\tilde{B} = (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1).$$

Case IV: If $\hat{A} > 0$ and $\hat{B} < 0$, then:

$$\tilde{A}\tilde{B} = (a_4 b_1, a_3 b_2, a_2 b_3, a_1 b_4).$$

Case V: If $\hat{A} < 0$ and $\hat{B} > 0$, then:

$$\tilde{A}\tilde{B} = (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1). \tag{2.4}$$

6. Equality: The fuzzy numbers $\hat{A} = (a_1, a_2, a_3, a_4)$ and $\hat{B} = (b_1, b_2, b_3, b_4)$ are equal iff

$$a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3, \text{ and } a_4 = b_4. \tag{2.5}$$

Definition 2.6. A matrix $\tilde{A} = \left(\tilde{a}_{ij}\right)_{m \times n}$ is called a trapezoidal fuzzy matrix, if each element of $\tilde{A}$ is a TrFN.

Definition 2.7. A fuzzy matrix $\tilde{A}$ will be:

1. positive (negative) and denoted by $\tilde{A} > 0$, $\left(\tilde{A} < 0\right)$ if each element of $\tilde{A}$ is positive (negative) TrFN;
2. non-negative (non-positive) and denoted by $\tilde{A} \geqslant 0$, $\left(\tilde{A} \leqslant 0\right)$ if each element of $\tilde{A}$ is non-negative (non-positive) TrFNs;
3. arbitrary, if at least one element of $\tilde{A}$ is near zero TrFNs.

Definition 2.8. The Vec-operator generates a column vector from a matrix A by stacking the column vectors of $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ as $\mathrm{Vec}(A) = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nn} \end{pmatrix}$. In addition, if $A = \mathrm{Vec}^{-1}\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{nn} \end{pmatrix}$, then $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$.

Definition 2.9 ([39])**.** A block diagonal matrix is invertible if and only if each of its main-diagonal blocks is invertible, and in this case, its inverse is another block diagonal matrix given by

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}^{-1} = \begin{pmatrix} A_1^{-1} & 0 & \cdots & 0 \\ 0 & A_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^{-1} \end{pmatrix}.$$

Definition 2.10 ([39])**.** The determinant of the block diagonal matrix $A = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{pmatrix}$ is,

$$\det(A) = \det(A_1) \times \cdots \times \det(A_n).$$

Theorem 2.11 ([16])**.** *If the crisp linear matrix equation $AXB = E$ has a unique solution X, then the gradient iterative solution $\hat{X}(k)$ given by $\hat{X}(k) = \hat{X}(k-1) + \alpha \bullet (A)^T \left(E - A\hat{X}(k-1)B\right)(B)^T$ converges to X or $\lim_{k\to\infty}\left(\hat{X}(k)\right) = X$ for any initial value $\hat{X}(0)$.*

Theorem 2.12 ([44])**.** *If the crisp linear matrix equation $AXB = E$ has a unique solution X, then the gradient iterative solution $\hat{X}(k)$ given by $\hat{X}(k) = \hat{X}(k-1) + \alpha \bullet \left((A)^T \bullet A\right)^{-1} (A)^T \left(E - A\hat{X}(k-1)B\right)(B)^T((B(B)^T)^{-1}$ converges to X or $\lim_{k\to\infty}\left(\hat{X}(k)\right) = X$ for any initial value $\hat{X}(0)$.*

In the following section, the arithmetic multiplication operations for TrFNs are discussed.

3. Trapezoidal fuzzy numbers multiplication

The multiplication between fuzzy numbers is neither commutative nor associative. Therefore, in the following propositions, the arithmetic multiplication operations between three TrFNs are discussed based on the multiplication operations in (2.2)-(2.4).

Proposition 3.1 (Multiplication of three arbitrary trapezoidal fuzzy numbers)**.** *If $\hat{A} = (a_1, a_2, a_3, a_4)$, $\hat{B} = (b_1, b_2, b_3, b_4)$, and $\hat{X} = (x_1, x_2, x_3, x_4)$ be three arbitrary TrFNs, then*

$$\tilde{A}\tilde{X}\tilde{B} = (b, l, k, h), \tag{3.1}$$

where

$$\begin{aligned} b = \min\big[&\min(a_1x_1, a_1x_4, a_4x_1, a_4x_4)\cdot b_1, \min(a_1x_1, a_1x_4, a_4x_1, a_4x_4)\cdot b_4, \\ &\max(a_1x_1, a_1x_4, a_4x_1, a_4x_4)\cdot b_1, \max(a_1x_1, a_1x_4, a_4x_1, a_4x_4)\cdot b_4\big], \\ l = \min\big[&\min(a_2x_2, a_2x_3, a_3x_2, a_3x_3)\cdot b_2, \min(a_2x_2, a_2x_3, a_3x_2, a_3x_3)\cdot b_3, \end{aligned}$$

$$
\begin{aligned}
&\max\left(a_2x_2, a_2x_3, a_3x_2, a_3x_3\right)\cdot b_2, \max(a_2x_2, a_2x_3, a_3x_2, a_3x_3)\cdot b_3\big],\\
k = \max\big[&\min(a_2x_2, a_2x_3, a_3x_2, a_3x_3)\cdot b_2, \min\left(a_2x_2, a_2x_3, a_3x_2, a_3x_3\right)\cdot b_3,\\
&\max\left(a_2x_2, a_2x_3, a_3x_2, a_3x_3\right)\cdot b_2, \max(a_2x_2, a_2x_3, a_3x_2, a_3x_3)\cdot b_3\big],\\
h = \max\big[&\min(a_1x_1, a_1x_4, a_4x_1, a_4x_4)\cdot b_1, \min\left(a_1x_1, a_1x_4, a_4x_1, a_4x_4\right)\cdot b_4,\\
&\max\left(a_1x_1, a_1x_4, a_4x_1, a_4x_4\right)\cdot b_1, \max\left(a_1x_1, a_1x_4, a_4x_1, a_4x_4\right)\cdot b_4\big].
\end{aligned}
$$

Proof. Straightforward using Eq. (2.2) in Definition 2.5. □

Proposition 3.2 (Multiplication of three positive trapezoidal fuzzy numbers)**.** *If* $\hat{A}=(a_1, a_2, a_3, a_4)$, $\hat{X}=(x_1, x_2, x_3, x_4)$, *and* $\hat{B}=(b_1, b_2, b_3, b_4)$ *be three positive TrFNs respectively, then,*

$$\tilde{A}\tilde{X}\tilde{B}=(a_1x_1b_1, a_2x_2b_2, a_3x_3b_3, a_4x_4b_4).$$

Proof. From Definition 2.5 and by Eq. (2.2), we have:

$$\tilde{A}\tilde{X} = (a, h, m, d).$$

Since $\hat{A}, \hat{X} > 0$, and by Eq. (2.3) in Definition 2.5, $\tilde{A}\tilde{B}$ can be reduced as follows:

$$
\begin{aligned}
a &= \min(a_1x_1, a_1x_4, a_4x_1, a_4x_4) = a_1x_1, & h &= \min(a_2x_2, a_2x_3, a_3x_2, a_3x_3)= a_2x_2,\\
m &= \max\left(a_2x_2, a_2x_3, a_3x_2, a_3x_3\right) = a_3x_3, & d &= \max\left(a_1x_1, a_1x_4, a_4x_1, a_4x_4\right) = a_4x_4.
\end{aligned}
$$

Thus, Eq. (3.1) can be reduced as follows:

$$
\begin{aligned}
b &= \min[a_1x_1b_1, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4], & l &= \min[a_2x_2b_2, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3],\\
k &= \max[a_2x_2b_2, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3], & h &= \max[a_1x_1b_1, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4].
\end{aligned}
\tag{3.2}
$$

Since $\hat{B} = (b_1, b_2, b_3, b_4)$ is a positive fuzzy number, by Definition 2.6, the following can be concluded:

$$0 < b_1 \leqslant b_2 \leqslant b_3 \leqslant b_4.$$

Thus, Eq. (3.2) can be reduced to:

$$
\begin{aligned}
a &= \min[a_1x_1b_1, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4] = a_1x_1b_1, & h &= \min[a_2x_2b_2, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3] = a_2x_2b_2,\\
m &= \max[a_2x_2b_2, a_2x_2b_3, a_3x_3b_2, a_3x_3b_3] = a_3x_3b_3, & d &= \max[a_1x_1b_1, a_1x_1b_4, a_4x_4b_1, a_4x_4b_4] = a_4x_4b_4,
\end{aligned}
$$

and therefore,

$$\tilde{A}\tilde{X}\tilde{B} = (a_1x_1b_1, a_2x_2b_2, a_3x_3b_3, a_4x_4b_4).$$

□

The following section proposes three new methods for solving the GTrFFSME in Eq. (1.1) based on the arithmetic multiplication operation proposed in Proposition 3.2.

4. The solution of generalized trapezoidal fully fuzzy Sylvester matrix equation

In this section, the positive GTrFFSME is converted to an equivalent system of GSME, and the solution to the system of GSME is obtained by three different methods. In the following subsection, we first prove the equivalency between the GTrFFSME and the system of GSME and derive the sufficient and necessary conditions to have a unique positive fuzzy solution.

4.1. Systems of generalized Sylvester matrix equations

To obtain the positive fuzzy solution to the positive GTrFFSME in Eq. (1.1), the developed arithmetic fuzzy multiplication in Section 3 is applied to the positive GTrFFSME to convert it to an equivalent system of GSME. The exact fuzzy solution is obtained analytically using the fuzzy matrix vectorization method (FMVM) and approximated numerically using FGIM and FLSIM. There are five steps involved in the construction of the methods. Figure 2 displays the flow chart of the constructed methods for solving positive GTrFFSME.

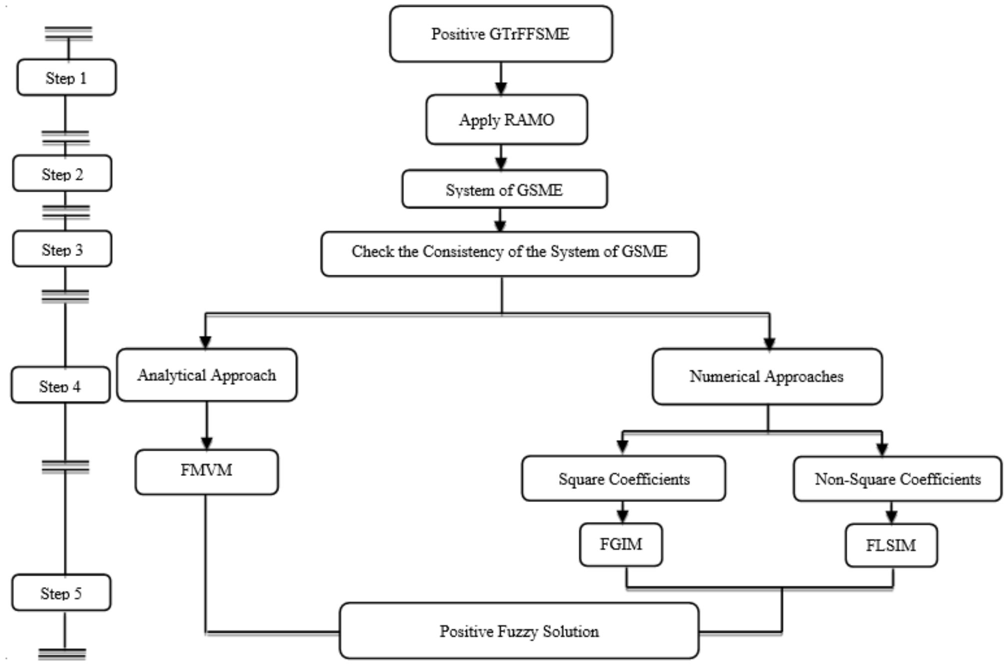

Figure 2: Flow chart of the constructed methods for solving positive GTrFFSME.

In the following Theorem 4.1, the positive GTrFFSME is converted to an equivalent system of GSME.

Theorem 4.1 (Fundamental theorem of generalized trapezoidal fully fuzzy Sylvester matrix equation)**.** *In the GTrFFSME in Eq.* (1.1)*, if* $\tilde{A} = (\tilde{a}_{ij})_{q\times p} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)}) > 0$, $\tilde{C} = (\tilde{c}_{ij})_{q\times p} = (c_{ij}^{(1)}, c_{ij}^{(2)}, c_{ij}^{(3)}, c_{ij}^{(4)}) > 0$, $\forall 1 \leqslant i,j \leqslant q,p$ *and* $\tilde{B} = (\tilde{b}_{ij})_{n\times r} = (b_{ij}^{(1)}, b_{ij}^{(2)}, b_{ij}^{(3)}, b_{ij}^{(4)}) > 0$, $\tilde{D} = (\tilde{d}_{ij})_{n\times r} = (d_{ij}^{(1)}, d_{ij}^{(2)}, d_{ij}^{(3)}, d_{ij}^{(4)}) > 0$, $\forall 1 \leqslant i,j \leqslant n,r$ *and* $\tilde{X} = (\tilde{x}_{ij})_{p\times n} = (x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}) > 0, \forall 1 \leqslant i,j \leqslant p,n,$ *and* $\tilde{E} = (\tilde{e}_{ij})_{q\times r} = (e_{ij}^{(1)}, e_{ij}^{(2)}, e_{ij}^{(3)}, e_{ij}^{(4)}) > 0$, $\forall 1 \leqslant i,j \leqslant q,r$, *then, the positive GTrFFSME in Eq.* (1.1) *is equivalent to the following system of GSME:*

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)} + c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)} + c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)} + c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)} + c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases} \tag{4.1}$$

Proof. The ij^{th} equation of the GTrFFSME in Eq. (1.1) can be written as follows:

$$\sum_{t=1}^{j}\sum_{k=1}^{p} \tilde{a}_{ik}\tilde{x}_{kt}\tilde{b}_{tj} + \sum_{t=1}^{j}\sum_{k=1}^{p} \tilde{c}_{ik}\tilde{x}_{kt}\tilde{d}_{tj} = \tilde{e}_{ij}, \quad \forall 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant r.$$

Since $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{X}$ in Eq. (1.1) are positive trapezoidal fully fuzzy matrices respectively, then Proposition 3.2 is used to find $\tilde{a}_{ij}\tilde{x}_{ij}\tilde{b}_{ij}$ and $\tilde{c}_{ij}\tilde{x}_{ij}\tilde{d}_{ij}$ as follows:

$$\tilde{A}\tilde{X}\tilde{B} = \sum_{t=1}^{j}\left(a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)}, a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)}, a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)}\right)$$

and

$$\tilde{C}\tilde{X}\tilde{D} = \sum_{t=1}^{j}\left(c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)}, c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)}, c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)}\right),$$

$\forall 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant r$. By Definition 2.5 and Eq. (2.1), we get,

$$\begin{aligned}\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \sum_{t=1}^{j}\Big(& a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)}, a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)}, a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)}, a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)} \\ &+ c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)}, c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)}, c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)}, c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)}\Big),\ \forall 1 \leqslant i \leqslant q,\ 1 \leqslant j \leqslant r.\end{aligned}$$

By Definition 2.5 and Eq. (2.5), the GTrFFSME in Eq. (1.1) is equivalent to the following:

$$\begin{cases} a_{ij}^{(1)}x_{ij}^{(1)}b_{ij}^{(1)} + c_{ij}^{(1)}x_{ij}^{(1)}d_{ij}^{(1)} = e_{ij}^{(1)}, \\ a_{ij}^{(2)}x_{ij}^{(2)}b_{ij}^{(2)} + c_{ij}^{(2)}x_{ij}^{(2)}d_{ij}^{(2)} = e_{ij}^{(2)}, \\ a_{ij}^{(3)}x_{ij}^{(3)}b_{ij}^{(3)} + c_{ij}^{(3)}x_{ij}^{(3)}d_{ij}^{(3)} = e_{ij}^{(3)}, \\ a_{ij}^{(4)}x_{ij}^{(4)}b_{ij}^{(4)} + c_{ij}^{(4)}x_{ij}^{(4)}d_{ij}^{(4)} = e_{ij}^{(4)}. \end{cases}$$

□

In the following definition, the trapezoidal positive fuzzy solution matrix in general form is presented.

Definition 4.2. The trapezoidal fuzzy matrix $\tilde{X} = \left(x_{ij}^{(1)}, x_{ij}^{(2)}, x_{ij}^{(3)}, x_{ij}^{(4)}\right)$ is called an exact positive fuzzy solution of GTrFFSME in Eq. (4.1) if $x_{ij}^{(4)} \geqslant x_{ij}^{(3)} \geqslant x_{ij}^{(2)} \geqslant x_{ij}^{(1)} > 0, \forall 1 \leqslant i, j \leqslant n, m$.

To solve the positive GTrFFSME in Eq. (1.1), we consider the corresponding system of GSME in Eq. (4.1). Now, in the next Theorem 4.3, sufficient conditions for the system of GSME in Eq. (4.1) to have a unique positive solution are discussed.

Theorem 4.3 (Uniqueness of positive solution to system of GSME)**.** *The system of GSME in Eq.* (4.1) *has a unique positive solution if:*

1. $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$ *and* $\det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 *and* r_4 *are invertible matrices, where,*

$$r_1 = (b_{ij}^{(1)})^T \bigotimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \bigotimes c_{ij}^{(1)}, \qquad r_2 = (b_{ij}^{(2)})^T \bigotimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \bigotimes c_{ij}^{(2)},$$
$$r_3 = (b_{ij}^{(3)})^T \bigotimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \bigotimes c_{ij}^{(3)}, \qquad r_1 = (b_{ij}^{(4)})^T \bigotimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \bigotimes c_{ij}^{(4)};$$

2. $r_1^{-1}, r_2^{-1}, r_3^{-1}$ *and* $r_4^{-1} > 0$.

Proof.

1. By Theorem 4.1, the positive GTrFFSME in Eq. (1.1) is converted to an equivalent system of GSME in Eq. (4.1). Applying the concept of Vec-operator and Kronecker product on the system of GSME in Eq. (4.1) yield a system of linear matrix equations as follows:

$$\begin{cases} ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})\mathrm{vec}(x_{ij}^{(1)}) = \mathrm{vec}(e_{ij}^{(1)}), \\ ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)})\mathrm{vec}(x_{ij}^{(2)}) = \mathrm{vec}(e_{ij}^{(2)}), \\ ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)})\mathrm{vec}(x_{ij}^{(3)}) = \mathrm{vec}(e_{ij}^{(3)}), \\ ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})\mathrm{vec}(x_{ij}^{(4)}) = \mathrm{vec}(e_{ij}^{(4)}). \end{cases} \tag{4.2}$$

If we let

$$r_1 = (b_{ij}^{(1)})^T \bigotimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \bigotimes c_{ij}^{(1)}, \qquad r_2 = (b_{ij}^{(2)})^T \bigotimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \bigotimes c_{ij}^{(2)},$$
$$r_3 = (b_{ij}^{(3)})^T \bigotimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \bigotimes c_{ij}^{(3)}, \qquad r_1 = (b_{ij}^{(4)})^T \bigotimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \bigotimes c_{ij}^{(4)},$$

then, this system in Eq. (4.2) can be written as

$$RS = T, \tag{4.3}$$

or in a matrix form as,

$$\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix},$$

where,

$$R = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}, \quad S = \begin{pmatrix} \mathrm{vec}\left(x_{ij}^{(1)}\right) \\ \mathrm{vec}\left(x_{ij}^{(2)}\right) \\ \mathrm{vec}\left(x_{ij}^{(3)}\right) \\ \mathrm{vec}\left(x_{ij}^{(4)}\right) \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} \mathrm{vec}\left(e_{ij}^{(1)}\right) \\ \mathrm{vec}\left(e_{ij}^{(2)}\right) \\ \mathrm{vec}\left(e_{ij}^{(3)}\right) \\ \mathrm{vec}\left(e_{ij}^{(4)}\right) \end{pmatrix} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}.$$

Matrix R is a block diagonal matrix, by Definition 2.10, the determinant of R ($\det(R)$) is obtained as follows:

$$\det(R) = \det\left[\begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}\right] = \det(r_1) \times \det(r_2) \times \det(r_3) \times \det(r_4).$$

The linear matrix equation $RS = T$ has a unique solution if and only if $\det(R) \neq 0$. Therefore, the system of GSME in Eq. (4.1) has a unique positive solution if: $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$ and $\det(r_4) \neq 0$, i.e., r_1, r_2, r_3, and r_4 are invertible matrices.

2. If $r_1^{-1}, r_2^{-1}, r_3^{-1}$, and $r_4^{-1} > 0$ then the system of GSME in Eq. (4.1) has a positive solution, and the proof is straightforward.

From 1 and 2, the system of GSME in Eq. (4.1) has a unique positive solution if: $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$, and $\det(r_4) \neq 0$, i.e., r_1, r_2, r_3 and r_4 are invertible matrices; and $r_1^{-1}, r_2^{-1}, r_3^{-1}$, and $r_4^{-1} > 0$. □

Since the system of GSME obtained in Eq. (4.1) consists of four crisp GSMEs, in the following remark, the system of GSME is represented in a more general form.

Remark 4.4. Based on Eq. (4.1), the GTrFFSME in Eq. (1.1) can be written as follows: for $1 \leqslant l \leqslant 4$ we have:

$$a_{ij}^{(l)} x_{ij}^{(l)} b_{ij}^{(l)} + c_{ij}^{(l)} x_{ij}^{(l)} d_{ij}^{(l)} = e_{ij}^{(l)}. \tag{4.4}$$

Based on Theorem 4.1, the GTrFFSME in Eq. (1.1) is transferred to an equivalent linear system of GSME in crisp form, which can be solved analytically and numerically. The main advantage of the analytical method is that the exact fuzzy solution to the GTrFFSME in Eq. (1.1) can be obtained. In addition, the analytical methods can be applied to the GTrFFSME in Eq. (1.1) with square and non-square fuzzy coefficient matrices.

However, the conversion of $m \times n$ GTrFFSME vec-operator and Kronecker product increases the dimension of the system by $mn \times mn$, which makes the computational more complex and impracticable. Normally, previous researchers limited their examples to small sizes ($n < 10$). For GTrFFSME with large dimensions ($n \geqslant 10$), iterative algorithms to find an approximated solution are more practical [9]. Therefore, three different methods are proposed in the following sections for solving the GTrFFSME in Eq. (1.1). The first method aims to find the exact fuzzy solution by extending the matrix vec-operator and Kronecker product concept. In addition, two iterative methods are developed to approximate the positive fuzzy solution of the positive GTrFFSME with large dimensions.

4.1.1. *Fuzzy matrix vectorization method for GTrFFSME*

In this method, the GTrFFSME in Eq. (1.1) is solved analytically using vec-operator and Kronecker product. The detail of the constructed method is presented in the following steps.

Step 1: Decompose $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, $\tilde{E}$ and $\tilde{X}$ into $a_{ij}^{(l)}$, $b_{ij}^{(l)}$, $c_{ij}^{(l)}$, $d_{ij}^{(l)}$, $e_{ij}^{(l)}$ and $x_{ij}^{(l)}$, where $l = 1, 2, 3, 4$, respectively, and convert the GTrFFSME in Eq. (1.1) to the system of linear matrix equations in Eq. (4.1) using Theorem 4.1.

Step 2: Applying the vec-operator and Kronecker product on Eq. (4.1) gives:

$$\begin{cases} ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})\mathrm{vec}(x_{ij}^{(1)}) = \mathrm{vec}(e_{ij}^{(1)}), \\ ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)})\mathrm{vec}(x_{ij}^{(2)}) = \mathrm{vec}(e_{ij}^{(2)}), \\ ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)})\mathrm{vec}(x_{ij}^{(3)}) = \mathrm{vec}(e_{ij}^{(3)}), \\ ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})\mathrm{vec}(x_{ij}^{(4)}) = \mathrm{vec}(e_{ij}^{(4)}). \end{cases}$$

Step 3: Multiply the system of linear matrix equation in step 2 by matrix multiplicative inverse as follows:

$$\begin{cases} \mathrm{vec}\left(x_{ij}^{(1)}\right) = ((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})^{-1} \mathrm{vec}\left(e_{ij}^{(1)}\right), \\ \mathrm{vec}\left(x_{ij}^{(2)}\right) = ((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)})^{-1} \mathrm{vec}\left(e_{ij}^{(2)}\right), \\ \mathrm{vec}\left(x_{ij}^{(3)}\right) = ((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)})^{-1} \mathrm{vec}\left(e_{ij}^{(3)}\right), \\ \mathrm{vec}\left(x_{ij}^{(4)}\right) = ((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})^{-1} \mathrm{vec}\left(e_{ij}^{(4)}\right). \end{cases}$$

Step 4: Multiplying the system of linear matrix equation in step 3 by vec^{-1} gives the following positive fuzzy solutions:

$$\begin{cases} x_{ij}^{(1)} = \mathrm{vec}^{-1}(((b_{ij}^{(1)})^T \otimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \otimes c_{ij}^{(1)})^{-1} \mathrm{vec}(e_{ij}^{(1)})), \\ x_{ij}^{(2)} = \mathrm{vec}^{-1}\left(((b_{ij}^{(2)})^T \otimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \otimes c_{ij}^{(2)}\right)^{-1} \mathrm{vec}(e_{ij}^{(2)})), \\ x_{ij}^{(3)} = \mathrm{vec}^{-1}\left(((b_{ij}^{(3)})^T \otimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \otimes c_{ij}^{(3)}\right)^{-1} \mathrm{vec}(e_{ij}^{(3)})), \\ x_{ij}^{(4)} = \mathrm{vec}^{-1}(((b_{ij}^{(4)})^T \otimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \otimes c_{ij}^{(4)})^{-1} \mathrm{vec}(e_{ij}^{(4)})). \end{cases} \tag{4.5}$$

Step 5: Combine the positive fuzzy solutions obtained in Eq. (4.5) as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{p1}^{(1)}, x_{p1}^{(2)}, x_{p1}^{(3)}, x_{p1}^{(4)}\right) & \cdots & \left(x_{pn}^{(1)}, x_{pn}^{(2)}, x_{pn}^{(3)}, x_{pn}^{(4)}\right) \end{pmatrix}.$$

In the following remark, the system of equations in Step 4 is written in a general form.

Remark 4.5. The positive fuzzy solution in Eq. (4.5) to the positive GTrFFSME in Eq. (1.1) can be written as follows: for $1 \leqslant l \leqslant 4$ we have

$$x_{ij}^{(l)} = \mathrm{vec}^{-1}(((b_{ij}^{(l)})^T \bigotimes a_{ij}^{(l)} + (d_{ij}^{(l)})^T \bigotimes c_{ij}^{(l)})^{-1} \mathrm{vec}(e_{ij}^{(l)})). \tag{4.6}$$

In the next Theorem 4.6, the relation between the positive fuzzy solution obtained in Eq. (4.6) to the positive GTrFFSME in Eq. (1.1) and the solution to the system of crisp linear matrix equation in Eq. (4.3) is discussed.

Theorem 4.6. *The solution for the system of linear matrix equations of the* $RS = T$ *and the positive fuzzy solution to the GTrFFSME are equivalent if:*

1. $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$ *and* $\det(r_4) \neq 0$*, i.e.,* r_1*,* r_2*,* r_3 *and* r_4 *are invertible matrices.*
2. $r_1^{-1}, r_2^{-1}, r_3^{-1}$*, and* $r_4^{-1} > 0$*;*
3. ${r_1}^{-1}t_1 \leqslant {r_2}^{-1}t_2 \leqslant {r_3}^{-1}t_3 \leqslant {r_4}^{-1}t_4$.

Proof. By Theorem 4.3, the positive GTrFFSME is converted to the system of linear matrix equations in Eq. (4.3). Multiplying both sides of Eq. (4.3) by R^{-1} gives:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 0 \\ 0 & 0 & 0 & r_4 \end{pmatrix}^{-1} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}.$$

Since R^{-1} is a block diagonal matrix, R^{-1} can be evaluated by Definition 2.9 as follows:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} {r_1}^{-1} & 0 & 0 & 0 \\ 0 & {r_2}^{-1} & 0 & 0 \\ 0 & 0 & {r_3}^{-1} & 0 \\ 0 & 0 & 0 & {r_4}^{-1} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{pmatrix}.$$

The right-hand side can be simplified to the following:

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{pmatrix} = \begin{pmatrix} {r_1}^{-1}t_1 \\ {r_2}^{-1}t_2 \\ {r_3}^{-1}t_3 \\ {r_4}^{-1}t_4 \end{pmatrix}. \tag{4.7}$$

By Theorem 4.3, the positive GTrFFSME has a positive fuzzy solution only if r_1, r_2, r_3 *and* r_4 are invertible. Thus, the solution to the system of linear matrix equations $RS = T$ in Eq. (4.3) is as follows: for $1 \leqslant l \leqslant 4$, we have

$$s_l = {r_l}^{-1} t_l,$$

where,

$$r_l = (b_{ij}^{(l)})^T \bigotimes a_{ij}^{(l)} + (d_{ij}^{(l)})^T \bigotimes c_{ij}^{(l)}, \quad s_l = \text{vec}\left(x_{ij}^{(l)}\right), \quad \text{and} \quad t_l = \text{vec}\left(e_{ij}^{(l)}\right).$$

To get a unique positive solution, the following conditions must be met.

1. $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$, and $\det(r_4) \neq 0$, i.e., r_1, r_2, r_3 and r_4 are invertible matrices;
2. $r_1^{-1}, r_2^{-1}, r_3^{-1}$, and $r_4^{-1} > 0$.

For the obtained solution to be fuzzy, the following condition must be met.

1. ${r_1}^{-1} t_1 \leqslant {r_2}^{-1} t_2 \leqslant {r_3}^{-1} t_3 \leqslant {r_4}^{-1} t_4$.

Therefore, the positive fuzzy solution obtained in Eq. (4.5) to the positive GTrFFSME in Eq. (1.1) and the solution to the system of crisp linear matrix equation in Eq. (4.3) is equivalent. □

Corollary 4.7 (Uniqueness of fuzzy solution to positive GTrFFSME)**.** *The positive GTrFFSME in Eq. (1.1) has a unique positive fuzzy solution if the corresponding system of GSME in Eq. (4.1) has a unique solution (i.e.,* $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$, *and* $\det(r_4) \neq 0$, *i.e.,* r_1, r_2, r_3 *and* r_4 *are invertible matrices).*

Proof. The solution to the positive GTrFFSME in Eq. (1.1) is equivalent to the solution system of GSME in Eq. (4.1) by Theorem 4.6. Therefore, the positive GTrFFSME in Eq. (1.1) has a unique positive fuzzy solution if the corresponding system of GSME in Eq. (4.1) has a unique solution. Therefore, by Theorem 4.3, the positive GTrFFSME in Eq. (1.1) has a unique positive fuzzy solution if: $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$, and $\det(r_4) \neq 0$, i.e., r_1, r_2, r_3 and r_4 are invertible matrices. □

In the next Theorem 4.8, necessary and sufficient conditions for positive GTrFFSME to have a positive fuzzy solution are discussed.

Theorem 4.8 (Existence of positive fuzzy solution to positive GTrFFSME)**.** *The positive GTrFFSME has a positive fuzzy solution when matrices* r_1, r_2, r_3 *and* r_4 *are invertible, and if:*

1. $r_1^{-1}, r_2^{-1}, r_3^{-1}$, *and* $r_4^{-1} > 0$;
2. ${r_1}^{-1} t_1 > 0$, ${r_2}^{-1} t_2 > 0$, ${r_3}^{-1} t_3 > 0$, *and* ${r_4}^{-1} t_4 > 0$;
3. ${r_1}^{-1} t_1 \leqslant {r_2}^{-1} t_2 \leqslant {r_3}^{-1} t_3 \leqslant {r_4}^{-1} t_4$.

Proof. Part 1 and 2 can be proved as follows. By Corollary 4.7, the positive GTrFFSME has a unique positive solution only if r_1, r_2, r_3 *and* r_4 are positive and invertible. And by Theorem 4.6, the solution for the system of linear matrix equations $RS = T$ and the GTrFFSME is equivalent. Thus, from Eq. (4.7), the positive GTrFFSME has a positive solution only if

$${r_1}^{-1} t_1 > 0, \quad {r_2}^{-1} t_2 > 0, \quad {r_3}^{-1} t_3 > 0, \quad {r_4}^{-1} t_4 > 0.$$

By the definition of positive fuzzy solution matrix in Definition 4.2, the positive GTrFFSME has a unique positive fuzzy solution if the following condition is satisfied,

$${r_1}^{-1} t_1 \leqslant {r_2}^{-1} t_2 \leqslant {r_3}^{-1} t_3 \leqslant {r_4}^{-1} t_4.$$

Now we proceed to the feasibility of a positive fuzzy solution to the positive GTrFFSME.

Based on the conditions for a feasible fuzzy solution by [34], the obtained positive fuzzy solution in Eq. (4.5) to the positive GTrFFSME in Eq. (1.1) is feasible (strong fuzzy solution) if the following conditions are satisfied: for $1 \leqslant l \leqslant 4$,

$$x_{ij}^{(l)} > 0, \ \ \forall \{1 \leqslant i, j \leqslant p, n\}, \quad \text{and} \quad x_{ij}^{(4)} \geqslant x_{ij}^{(3)} \geqslant x_{ij}^{(2)} \geqslant x_{ij}^{(1)}, \ \forall \{1 \leqslant i, j \leqslant p, n\}. \tag{4.8}$$

□

Remark 4.9. If the solution fails to satisfy the feasibility conditions, it is infeasible (weak fuzzy solution).

The algorithm of the FMVM for solving the CTrFFMSE in Eq. (1.1) is given in the following five steps.

Algorithm 4.10 (Fuzzy matrix vectorization and Kronecker product algorithm for solving GTrFFSME)**.**

Step 1: Convert the GTrFFSME in Eq. (1.1) to a system of linear matrix equations using Theorem 4.1.
Step 2: Apply vec-operator and Kronecker product on the system obtained in Step 1.
Step 3: Multiply both sides of the system obtained in Step 2 by the multiplicative inverse of

$$((b_{ij}^{(l)})^T \bigotimes a_{ij}^{(l)} + (d_{ij}^{(l)})^T \bigotimes c_{ij}^{(l)}),\ \forall 1 \leqslant l \leqslant 4.$$

Step 4: Multiply both sides of the system obtained in Step 3 by vec^{-1}.
Step 5: Solving the system of matrix equations in Step 4 and write the positive fuzzy solution as follows:

$$\tilde{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \cdots & \left(x_{1n}^{(1)}, x_{1n}^{(2)}, x_{1n}^{(3)}, x_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(x_{p1}^{(1)}, x_{p1}^{(2)}, x_{p1}^{(3)}, x_{p1}^{(4)}\right) & \cdots & \left(x_{pn}^{(1)}, x_{pn}^{(2)}, x_{pn}^{(3)}, x_{pn}^{(4)}\right) \end{pmatrix}.$$

Approximating the positive fuzzy solution to the positive GTrFFSME in Eq. (1.1) is more practical than getting the exact fuzzy solution by FMVM, especially if the GTrFFSME's size is more than 10. In addition, since the positive GTrFFSME in Eq. (1.1) is converted to a system of four crisp GSME equations in Eq. (4.1), many methods can numerically approximate the fuzzy solution to the positive GTrFFSME. The following section extends the GI for solving the positive GTrFFSME in Eq. (1.1).

4.1.2. Fuzzy gradient-iterative method for GTrFFSME

In this section, the positive fuzzy solution to the positive GTrFFSME in Eq. (1.1) is approximated iteratively by extending the GI method in Theorem 2.11 to the system of GSME in Eq. (4.1) or equivalently in Eq. (4.4). The hierarchical identification principle is used to split the system of GSME in Eq. (4.1) into two subsystems, where the GI method is applied in obtaining the solution.

Using the hierarchical identification principle and Remark 4.1, the system of GSME in Eq. (4.4) can be decomposed into two subsystems. For $1 \leqslant l \leqslant 4$,

$$\xi_1^{(l)} = e_{ij}^{(l)} - a_{ij}^{(l)} x_{ij}^{(l)} b_{ij}^{(l)} \quad \text{and} \quad \xi_2^{(l)} = e_{ij}^{(l)} - c_{ij}^{(l)} x_{ij}^{(l)} d_{ij}^{(l)}, \tag{4.9}$$

where the iterative positive solution to the system of GSME in Eq. (4.4) is the average of the iterative solution for the subsystems.

From Eqs. (4.4) and (4.9), the following can be obtained: for $1 \leqslant l \leqslant 4$,

$$\xi_2^{(l)} = a_{ij}^{(l)} x_{ij}^{(l)} b_{ij}^{(l)} \quad \text{and} \quad \xi_1^{(l)} = c_{ij}^{(l)} x_{ij}^{(l)} d_{ij}^{(l)}. \tag{4.10}$$

The iterative positive solution to the system of equations in (4.10) can be obtained by Theorem 2.11 as follows:

$$\hat{x}_1^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \bullet \left(a_{ij}^{(l)}\right)^T \left(\xi_2^{(l)} - a_{ij}^{(l)} \hat{x}^{(l)}(k-1)\, b_{ij}^{(l)}\right) \left(b_{ij}^{(l)}\right)^T, \tag{4.11}$$

$$\hat{x}_2^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \bullet \left(c_{ij}^{(l)}\right)^T \left(\xi_1^{(l)} - c_{ij}^{(l)} \hat{x}^{(l)}(k-1)\, d_{ij}^{(l)}\right) \left(d_{ij}^{(l)}\right)^T. \tag{4.12}$$

Substitute Eq. (4.9) into (4.11) and (4.12) as follows:

$$\hat{x}_1^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \bullet \left(a_{ij}^{(l)}\right)^T \left(e_{ij}^{(l)} - c_{ij}^{(l)} \hat{x}^{(l)}(k-1)\, d_{ij}^{(l)} - a_{ij}^{(l)} \hat{x}^{(l)}(k-1)\, b_{ij}^{(l)}\right) \left(b_{ij}^{(l)}\right)^T, \tag{4.13}$$

$$\hat{x}_2^{(l)}(k) = \hat{x}^{(l)}(k-1) + \alpha_l \bullet \left(c_{ij}^{(l)}\right)^T \left(e_{ij}^{(l)} - a_{ij}^{(l)}\hat{x}^{(l)}(k-1)\, b_{ij}^{(l)} - c_{ij}^{(l)}\hat{x}^{(l)}(k-1)\, d_{ij}^{(l)}\right)\left(d_{ij}^{(l)}\right)^T. \tag{4.14}$$

If we let

$$s^l(k-1) = e^{(l)} - a^{(l)}\hat{x}^{(l)}(k-1)\, b^{(l)} - c^{(l)}\hat{x}^{(l)}(k-1)\, d^{(l)},$$

then, the average of the two iterative positive solutions in Eqs. (4.13) and (4.14) is

$$\hat{x}^{(l)}(k) = \frac{\hat{x}_1^{(l)}(k) + \hat{x}_2^{(l)}(k)}{2}.$$

Therefore, for $1 \leqslant l \leqslant 4$ the iterative positive solution to the system of GSME in Eq. (4.4) is

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T \left(s^l(k-1)\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T \left(s^l(k-1)\right)\left(d^{(l)}\right)^T\right), \tag{4.15}$$

where the convergence factor (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{\max}\left[\left(a^{(l)}\right)^T a^{(l)}\right]\lambda_{\max}\left[b^{(l)}\left(b^{(l)}\right)^T\right] + \lambda_{\max}\left[\left(c^{(l)}\right)^T c^{(l)}\right]\lambda_{\max}\left[d^{(l)}\left(d^{(l)}\right)^T\right]}. \tag{4.16}$$

It can also be obtained as follows,

$$0 < \alpha_l < \frac{2}{\|a^{(l)}\|^2\|b^{(l)}\|^2 + \|c^{(l)}\|^2\|d^{(l)}\|^2}, \tag{4.17}$$

where, $\|a^{(l)}\|^2 = \mathrm{tr}\left[a^{(l)} \bullet \left(a^{(l)}\right)^T\right]$. If we let $ff_0 = \|a^{(l)}\|^2\|b^{(l)}\|^2 + \|c^{(l)}\|^2\|d^{(l)}\|^2$, then

$$0 < \alpha_l < \frac{2}{ff_0}. \tag{4.18}$$

At step k^{th} of the iteration, the following error is considered:

$$\delta^l(k) = \|s^l(k-1)\|_2.$$

The obtained iterative positive solution in Eq. (4.15), can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{X} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \cdots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \tag{4.19}$$

In the next theorem, we prove that the iterative solution obtained by the FGIM converges to the positive solution of the positive GTrFFSME for any initial value.

Theorem 4.11. *If the positive GTrFFSME in Eq. (1.1), or equivalently the system of GSME in Eq. (4.4), has a unique positive solution $X^{(l)}$, then the iterative solution $\hat{x}^{(l)}(k)$ in Eq. (4.19) converges to $X^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e., if $k \to \infty$, then $X^{(l)} = \hat{X}^{(l)}(k)$).*

Proof. Let, $\psi(k)$ be the error at each k, for $k = 1, \ldots, n$ and $1 \leqslant l \leqslant 4$ as

$$\psi(k) = X^{(l)} - \hat{X}^{(l)}(k). \tag{4.20}$$

From (4.4), (4.15), and (4.20), the following is obtained:

$$\begin{aligned}\psi(k) = \psi(k-1) + \frac{\alpha_l}{2}\Bigg(&\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(b^{(l)}\right)^T \\ &+\left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(d^{(l)}\right)^T\Bigg).\end{aligned} \tag{4.21}$$

Taking $\|.\|^2$ to both sides of Eq. (4.21) gives

$$\begin{aligned}\|\psi(k)\|^2 = \Bigg\|&\psi(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(b^{(l)}\right)^T\right. \\ &\left.+\left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(d^{(l)}\right)^T\right)\Bigg\|^2.\end{aligned} \tag{4.22}$$

Apply the following formula

$$\|A+B\|^2 = \mathrm{tr}\left((A+B)^T(A+B)\right) = \|A\|^2 + 2\mathrm{tr}\left(A^TB\right) + \|B\|^2.$$

to Eq. (4.22) we get,

$$\begin{aligned}\|\psi(k)\|^2 = \|\psi(k-1)\|^2 &+ \alpha_l \mathrm{tr}\Big[\psi^T(k-1)\Big(\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(b^{(l)}\right)^T \\ &+\left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(d^{(l)}\right)^T\Big)\Big] \\ &+\frac{\alpha_l^2}{4}\Big\|\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(b^{(l)}\right)^T \\ &+\left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(d^{(l)}\right)^T\Big\|^2.\end{aligned}$$

Applying norm properties gives:

$$\begin{aligned}\|\psi(k)\|^2 \leqslant \|\psi(k-1)\|^2 + \alpha_l \mathrm{tr}\Big[&\left(\psi^T(k-1)\left(a^{(l)}\right)^T\left(b^{(l)}\right)^T + \psi^T(k-1)\left(c^{(l)}\right)^T\left(d^{(l)}\right)^T\right) \\ &\times\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\Big] \\ &+\frac{\alpha_l^2}{4}\Big\|\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(b^{(l)}\right)^T \\ &+\left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(d^{(l)}\right)^T\Big\|^2.\end{aligned}$$

And since $\|A\|^2 = \mathrm{tr}\left[(A)^TA\right]$, then

$$\begin{aligned}\|\psi(k)\|^2 \leqslant \|\psi(k-1)\|^2 &- \alpha_l\left\|a^{(l)}\psi(k-1)\,b^{(l)} + c^{(l)}\psi(k-1)\,d^{(l)}\right\|^2 \\ &+\frac{\alpha_l^2}{4}\Big\|\left(a^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(b^{(l)}\right)^T \\ &+\left(c^{(l)}\right)^T\left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(d^{(l)}\right)^T\Big\|^2.\end{aligned}$$

Applying norm properties gives:

$$\begin{aligned}\|\psi(k)\|^2 \leqslant \|\psi(k-1)\|^2 &- \alpha_l \left\|a^{(l)}\psi(k-1)\,b^{(l)} + c^{(l)}\psi(k-1)\,d^{(l)}\right\|^2 \\ &+ \frac{\alpha_l^2}{4}\left(\left\|a^{(l)}\right\|^2\left\|b^{(l)}\right\|^2 + \left\|c^{(l)}\right\|^2\left\|d^{(l)}\right\|^2\right)\left\|a^{(l)}\psi(k-1)\,b^{(l)} + c^{(l)}\psi(k-1)\,d^{(l)}\right\|^2, \\ \|\psi(k)\|^2 \leqslant \|\psi(k-1)\|^2 & \\ &+ \left(-\alpha_l + \frac{\alpha_l^2}{4}\left(\left\|a^{(l)}\right\|^2\left\|b^{(l)}\right\|^2 + \left\|c^{(l)}\right\|^2\left\|d^{(l)}\right\|^2\right)\right)\left\|a^{(l)}\psi(k-1)\,b^{(l)} + c^{(l)}\psi(k-1)\,d^{(l)}\right\|^2.\end{aligned}$$

By (4.18), the following can be obtained:

$$\|\psi(k)\|^2 \leqslant \|\psi(k-1)\|^2 + \left(-\alpha_l + \frac{\alpha_l^2}{4} \times \frac{2}{\alpha_0}\right)\left\|a^{(l)}\psi(k-1)\,b^{(l)} + c^{(l)}\psi(k-1)\,d^{(l)}\right\|^2.$$

At $k = 1$, $\|\psi(1)\|^2 \leqslant \|\psi(0)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(0)\,b^{(l)} + c^{(l)}\psi(0)\,d^{(l)}\right\|^2$. At $k = 2$, $\|\psi(2)\|^2 \leqslant \|\psi(1)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(1)\,b^{(l)} + c^{(l)}\psi(1)\,d^{(l)}\right\|^2$. At $k = 3$,

$$\|\psi(3)\|^2 \leqslant \|\psi(2)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(2)\,b^{(l)} + c^{(l)}\psi(2)\,d^{(l)}\right\|^2.$$

At $k = n-1$, $\|\psi(n-1)\|^2 \leqslant \|\psi(n-2)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(n-2)\,b^{(l)} + c^{(l)}\psi(n-2)\,d^{(l)}\right\|^2$. At $k = n$, $\|\psi(n)\|^2 \leqslant \|\psi(n-1)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\left\|a^{(l)}\psi(n-1)\,b^{(l)} + c^{(l)}\psi(n-1)\,d^{(l)}\right\|^2$. Therefore, the following is obtained,

$$\|\psi(k)\|^2 \leqslant \|\psi(0)\|^2 - \alpha_l\left(1 - \frac{\alpha_l}{2\alpha_0}\right)\sum_{k=1}^{n}\left(\left\|a^{(l)}\psi(k)\,b^{(l)} + c^{(l)}\psi(k)\,d^{(l)}\right\|^2\right).$$

If the convergence factor ff is chosen to (4.18) and $k \to \infty$, then

$$\sum_{k=1}^{\infty}\left(\left\|a^{(l)}\psi(k)\,b^{(l)} + c^{(l)}\psi(k)\,d^{(l)}\right\|^2\right) < \infty.$$

Therefore,

$$\lim_{k\to\infty}\left(a^{(l)}\psi(k)\,b^{(l)} + c^{(l)}\psi(k)\,d^{(l)}\right) = 0.$$

Since $a^{(l)} > 0$, $b^{(l)} > 0$, $c^{(l)} > 0$, and $d^{(l)} > 0$, then,

$$\lim_{k\to\infty}\psi(k) = 0.$$

By Eq. (4.20), the following is obtained,

$$\lim_{k\to\infty}\left(X^{(l)} - \hat{X}^{(l)}(k)\right) = 0.$$

Consequently, if $k \to \infty$, then $X^{(l)} = \hat{X}^{(l)}(k)$ and therefore, the positive GTrFFSME in Eq. (1.1), or equivalently the system of GSME in Eq. (4.4) has a unique positive solution $X^{(l)}$, then the iterative solution $\hat{x}^{(l)}(k)$ in Eq. (4.19) converges to $X^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ and for $1 \leqslant l \leqslant 4$. □

The convergence rate of the FGIM algorithm in Eq. (4.20) is slow. To improve the convergence speed, in the following section, we derive a FLSIM. It is worth mentioning that the FGIM can only be applied to GTrFFSME with square coefficients; however, the FLSIM algorithm can be applied to GTrFFSME with non-square coefficients.

4.1.3. Fuzzy least-square iterative method for GTrFFSME

The development of the FLSIM is similar to the FGIM. However, to improve the convergence rate of the FGIM in Section 4.1.2, the least-square term of the coefficients in Eq. (4.4) should be added to the FGIM algorithm obtained in Eq. (4.15). Therefore, by Theorem 2.12 and Eq. (4.15), the following can be obtained: for $1 \leqslant l \leqslant 4$ we have:

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\Bigg(\left(\left(\left(a^{(l)}\right)^T \bullet a^{(l)}\right)^{-1} \bullet \left(a^{(l)}\right)^T \left(s^l(k-1)\right)\left(b^{(l)}\right)^T \left(\left(b^{(l)}\left(b^{(l)}\right)^T\right)^{-1}\right) \\ + \left(\left(c^{(l)}\right)^T \bullet c^{(l)}\right)^{-1}\left(c^{(l)}\right)^T \left(s^l(k-1)\right)\left(d^{(l)}\right)^T \left(\left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\right)\Bigg).$$

Where the convergence factor (step size) is given by,

$$0 < \alpha_l < 4.$$

As step k^{th} of the iteration, the following error is considered:

$$\delta^l(k) = \left\|s^l(k-1)\right\|_2.$$

The obtained iterative positive solution in Eq. (4.15), can be expressed as,

$$\hat{x} = \left(\hat{x}^{(1)}, \hat{x}^{(2)}, \hat{x}^{(3)}, \hat{x}^{(4)}\right).$$

It can also be written in matrix form as,

$$\hat{X} = \begin{pmatrix} \left(\hat{x}_{11}^{(1)}, \hat{x}_{11}^{(2)}, \hat{x}_{11}^{(3)}, \hat{x}_{11}^{(4)}\right) & \cdots & \left(\hat{x}_{1n}^{(1)}, \hat{x}_{1n}^{(2)}, \hat{x}_{1n}^{(3)}, \hat{x}_{1n}^{(4)}\right) \\ \vdots & \ddots & \vdots \\ \left(\hat{x}_{p1}^{(1)}, \hat{x}_{p1}^{(2)}, \hat{x}_{p1}^{(3)}, \hat{x}_{p1}^{(4)}\right) & \cdots & \left(\hat{x}_{pn}^{(1)}, \hat{x}_{pn}^{(2)}, \hat{x}_{pn}^{(3)}, \hat{x}_{pn}^{(4)}\right) \end{pmatrix}. \tag{4.23}$$

In the next theorem, we prove that the iterative solution obtained by the FLSIM method converges to the positive solution of the positive GTrFFSME for any initial value.

Theorem 4.12. *If the positive GTrFFSME in Eq. (1.1), or equivalently the system of GSME in Eq. (4.4), has a unique positive solution $X^{(l)}$, then the iterative solution $\hat{x}^{(l)}(k)$ in Eq. (4.23) converges to $X^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e., if $k \to \infty$, then $X^{(l)} = \hat{X}^{(l)}(k)$).*

Proof. Let, $\psi(k)$ be the error at each k, for $k = 1, \ldots, n$ and $1 \leqslant l \leqslant 4$ as

$$\psi(k) = X^{(l)} - \hat{X}^{(l)}(k). \tag{4.24}$$

From (4.4), (4.15), and (4.24), the following is obtained:

$$\psi(k) = \psi(k-1) + \frac{\alpha_l}{2}\Bigg(\left(\left(\left(a^{(l)}\right)^T \bullet a^{(l)}\right)^{-1}\left(a^{(l)}\right)^T \left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right) \\ \times \left(b^{(l)}\right)^T \left(b^{(l)}\left(b^{(l)}\right)^T\right)^{-1} + \left(\left(c^{(l)}\right)^T \bullet c^{(l)}\right)^{-1}\left(c^{(l)}\right)^T \\ \times \left(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\right)\left(d^{(l)}\right)^T \left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\Bigg). \tag{4.25}$$

Taking $\|.\|^2$ to both sides of Eq. (4.25) gives

$$\begin{aligned}\|\psi(k)\|^2 = \Bigg\| \psi(k-1) + \frac{\alpha_l}{2}\Bigg(\Big(\Big(a^{(l)}\Big)^T \bullet a^{(l)}\Big)^{-1}\Big(a^{(l)}\Big)^T\Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big) \\ \times \Big(b^{(l)}\Big)^T\Big(b^{(l)}\Big(b^{(l)}\Big)^T\Big)^{-1} + \Big(\Big(c^{(l)}\Big)^T \bullet c^{(l)}\Big)^{-1}\Big(c^{(l)}\Big)^T \\ \times \Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big)\Big(d^{(l)}\Big)^T\Big(d^{(l)}\Big(d^{(l)}\Big)^T\Big)^{-1}\Bigg)\Bigg\|^2.\end{aligned} \tag{4.26}$$

Applying the following formula

$$\begin{aligned}&\left\| A\left(X + \left((A)^T \bullet A\right)^{-1} Y\left(B(B)^T\right)^{-1}\right)B\right\|^2 \\ &= \mathrm{tr}\left(\left(\left(X + \left((A)^T \bullet A\right)^{-1} Y\left(B(B)^T\right)^{-1}\right)B\right)^T\left(\left(X + \left((A)^T \bullet A\right)^{-1} Y\left(B(B)^T\right)^{-1}\right)B\right)\right) \\ &= \|AXB\|^2 + 2\mathrm{tr}\left(X^T Y\right) + \left\|\left(A\left((A)^T \bullet A\right)^{-1} Y\left(B(B)^T\right)^{-1}\right)B\right\|^2.\end{aligned}$$

to Eq. (4.26) we get,

$$\begin{aligned}\left\|a^{(l)}\psi(k)\,b^{(l)}\right\|^2 = &\left\|a^{(l)}\psi(k-1)\,b^{(l)}\right\|^2 \\ &+ \alpha_l \mathrm{tr}\Bigg[\psi^T(k-1)\Big(\Big(a^{(l)}\Big)^T\Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big)\Big(b^{(l)}\Big)^T \\ &+ \Big(c^{(l)}\Big)^T\Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big)\Big(d^{(l)}\Big)^T\Big)\Bigg] \\ &+ \frac{\alpha_l^2}{4}\Bigg\|\Bigg(\Big(\Big(a^{(l)}\Big)^T \bullet a^{(l)}\Big)^{-1}\Big(a^{(l)}\Big)^T a^{(l)}\Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big) \\ &\times b^{(l)}\Big(b^{(l)}\Big)^T\Big(b^{(l)}\Big(b^{(l)}\Big)^T\Big)^{-1} + \Big(\Big(c^{(l)}\Big)^T \bullet c^{(l)}\Big)^{-1}\Big(c^{(l)}\Big)^T c^{(l)} \\ &\times \Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big)d^{(l)}\Big(d^{(l)}\Big)^T\Big(d^{(l)}\Big(d^{(l)}\Big)^T\Big)^{-1}\Bigg)\Bigg\|^2.\end{aligned}$$

Applying norm properties, we get

$$\begin{aligned}\left\|a^{(l)}\psi(k)\,b^{(l)}\right\|^2 \leqslant &\left\|a^{(l)}\psi(k-1)\,b^{(l)}\right\|^2 \\ &+ 2\alpha_l \mathrm{tr}\Bigg[\psi^T(k-1)\Big(\Big(a^{(l)}\Big)^T\Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big)\Big(b^{(l)}\Big)^T \\ &+ \Big(c^{(l)}\Big)^T\Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big)\Big(d^{(l)}\Big)^T\Big)\Bigg] \\ &+ \frac{\alpha_l^2}{4}\Bigg\|\Bigg(\Big(\Big(a^{(l)}\Big)^T \bullet a^{(l)}\Big)^{-1}\Big(a^{(l)}\Big)^T a^{(l)}\Big(-a^{(l)}\psi(k-1)\,b^{(l)} - c^{(l)}\psi(k-1)\,d^{(l)}\Big) \\ &\times b^{(l)}\Big(b^{(l)}\Big)^T\Big(b^{(l)}\Big(b^{(l)}\Big)^T\Big)^{-1} + \Big(\Big(c^{(l)}\Big)^T \bullet c^{(l)}\Big)^{-1}\Big(c^{(l)}\Big)^T c^{(l)}\end{aligned}$$

$$\times\left(-a^{(l)}\psi(k-1)b^{(l)}-c^{(l)}\psi(k-1)d^{(l)}\right)d^{(l)}\left(d^{(l)}\right)^{T}\left(d^{(l)}\left(d^{(l)}\right)^{T}\right)^{-1}\Bigg)\Bigg\|^{2},$$

which can be written as,

$$\begin{aligned}\left\|a^{(l)}\psi(k)b^{(l)}\right\|^{2}\leqslant{}&\left\|a^{(l)}\psi(k-1)b^{(l)}\right\|^{2}\\&+2\alpha_{l}\mathrm{tr}\Big[(\psi^{T}(k-1)\left(a^{(l)}\right)^{T}\left(b^{(l)}\right)^{T}+\psi^{T}(k-1)\left(c^{(l)}\right)^{T}\left(d^{(l)}\right)^{T})\\&\times\left(-a^{(l)}\psi(k-1)b^{(l)}-c^{(l)}\psi(k-1)d^{(l)}\right)\Big]\\&+\frac{\alpha_{l}^{2}}{4}\Bigg\|\Bigg(\left(\left(a^{(l)}\right)^{T}\bullet a^{(l)}\right)^{-1}\left(a^{(l)}\right)^{T}a^{(l)}\left(-a^{(l)}\psi(k-1)b^{(l)}-c^{(l)}\psi(k-1)d^{(l)}\right)\\&\times b^{(l)}\left(b^{(l)}\right)^{T}\left(b^{(l)}\left(b^{(l)}\right)^{T}\right)^{-1}+\left(\left(c^{(l)}\right)^{T}\bullet c^{(l)}\right)^{-1}\left(c^{(l)}\right)^{T}c^{(l)}\\&\times\left(-a^{(l)}\psi(k-1)b^{(l)}-c^{(l)}\psi(k-1)d^{(l)}\right)d^{(l)}\left(d^{(l)}\right)^{T}\left(d^{(l)}\left(d^{(l)}\right)^{T}\right)^{-1}\Bigg)\Bigg\|^{2}.\end{aligned}$$

Applying norm properties, we get

$$\begin{aligned}\left\|a^{(l)}\psi(k)b^{(l)}\right\|^{2}\leqslant{}&\left\|a^{(l)}\psi(k-1)b^{(l)}\right\|^{2}\\&+2\alpha_{l}\ \mathrm{tr}\Big[(\psi^{T}(k-1)\left(a^{(l)}\right)^{T}\left(b^{(l)}\right)^{T}+\psi^{T}(k-1)\left(c^{(l)}\right)^{T}\left(d^{(l)}\right)^{T})\\&\times\left(-a^{(l)}\psi(k-1)b^{(l)}-c^{(l)}\psi(k-1)d^{(l)}\right)\Big]\\&+\frac{\alpha_{l}^{2}}{4}\left\|-2\left(a^{(l)}\psi(k-1)b^{(l)}+c^{(l)}\psi(k-1)d^{(l)}\right)\right\|^{2}.\end{aligned}$$

And since $\|A\|^{2}=\mathrm{tr}\left[(A)^{T}A\right]$, then

$$\begin{aligned}\left\|a^{(l)}\psi(k)b^{(l)}\right\|^{2}\leqslant{}&\left\|a^{(l)}\psi(k-1)b^{(l)}\right\|^{2}-2\alpha_{l}\left\|a^{(l)}\psi(k-1)b^{(l)}+c^{(l)}\psi(k-1)d^{(l)}\right\|^{2}\\&+\frac{\alpha_{l}^{2}}{2}\left\|a^{(l)}\psi(k-1)b^{(l)}+c^{(l)}\psi(k-1)d^{(l)}\right\|^{2},\\\|\psi(k)\|^{2}\leqslant{}&\|\psi(k-1)\|^{2}-2\alpha_{l}\left(1-\frac{\alpha_{l}}{4}\right)\left\|a^{(l)}\psi(k-1)b^{(l)}+c^{(l)}\psi(k-1)d^{(l)}\right\|^{2}.\end{aligned}$$

At $k=1$, $\|\psi(1)\|^{2}\leqslant\|\psi(0)\|^{2}-2\alpha_{l}\left(1-\frac{\alpha_{l}}{4}\right)\left\|a^{(l)}\psi(0)b^{(l)}+c^{(l)}\psi(0)d^{(l)}\right\|^{2}$. At $k=2$,

$$\|\psi(2)\|^{2}\leqslant\|\psi(1)\|^{2}-2\alpha_{l}\left(1-\frac{\alpha_{l}}{4}\right)\left\|a^{(l)}\psi(1)b^{(l)}+c^{(l)}\psi(1)d^{(l)}\right\|^{2}.$$

At $k=3$,

$$\|\psi(3)\|^{2}\leqslant\|\psi(2)\|^{2}-2\alpha_{l}\left(1-\frac{\alpha_{l}}{4}\right)\left\|a^{(l)}\psi(2)b^{(l)}+c^{(l)}\psi(2)d^{(l)}\right\|^{2}.$$

At $k=n-1$, $\|\psi(n-1)\|^{2}\leqslant\|\psi(n-2)\|^{2}-2\alpha_{l}\left(1-\frac{\alpha_{l}}{4}\right)\left\|a^{(l)}\psi(n-2)b^{(l)}+c^{(l)}\psi(n-2)d^{(l)}\right\|^{2}$. At $k=n$, $\|\psi(n)\|^{2}\leqslant\|\psi(n-1)\|^{2}-2\alpha_{l}\left(1-\frac{\alpha_{l}}{4}\right)\left\|a^{(l)}\psi(n-1)b^{(l)}+c^{(l)}\psi(n-1)d^{(l)}\right\|^{2}$. Consequently,

$$\|\psi(k)\|^{2}\leqslant\|\psi(0)\|^{2}-2\alpha_{l}\left(1-\frac{\alpha_{l}}{4}\right)\sum_{k=1}^{n}\left(\left\|a^{(l)}\psi(k)b^{(l)}+c^{(l)}\psi(k)d^{(l)}\right\|^{2}\right),$$

$$\|\psi(k)\|^2 \leqslant \|\psi(0)\|^2 - 2\alpha_l\left(1-\frac{\alpha_l}{4}\right)\sum_{k=1}^{n}\left(\left\|a^{(l)}\psi(k)\,b^{(l)} + c^{(l)}\psi(k)\,d^{(l)}\right\|^2\right),$$

if the convergence factor α is chosen to satisfy

$$0 < \alpha_l < 4,$$

and $n \to \infty$, then

$$\sum_{k=1}^{\infty}\left(\left\|a^{(l)}\psi(k)\,b^{(l)} + c^{(l)}\psi(k)\,d^{(l)}\right\|^2\right) < \infty,$$

then

$$\lim_{k\to\infty}\left(a^{(l)}\psi(k)\,b^{(l)} + c^{(l)}\psi(k)\,d^{(l)}\right) = 0,$$

therefore, since $a^{(l)} > 0,\ b^{(l)} > 0,\ c^{(l)} > 0,$ and $d^{(l)} > 0$, then,

$$\lim_{k\to\infty}\psi(k) = 0, \quad \lim_{k\to\infty}\left(X^{(l)} - \hat{X}^{(l)}(k)\right) = 0.$$

Consequently, if $n \to \infty$, then $X^{(l)} = \hat{X}^{(l)}(k)$. Thus, the positive GTrFFSME in Eq. (1.1), or equivalently the system of GSME in Eq. (4.4), has a unique positive solution $X^{(l)}$, then the iterative solution $\hat{x}^{(l)}(k)$ in Eq. (4.23) converges to $X^{(l)}$ for any initial values $\hat{x}^{(l)}(0)$ (i.e., if $k \to \infty$, then $X^{(l)} = \hat{X}^{(l)}(k)$) for $1 \leqslant l \leqslant 4$. □

Below is Algorithm 4.13 for the FGIM. This algorithm can be used by different software for solving the positive GTrFFSME in Eq. (1.1).

Algorithm 4.13 (Fuzzy gradient algorithm for GTrFFSME)**.** Input $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, *and* $\tilde{E}$. Split each matrix into 4 matrices (e.g., $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $a^{(4)}$). For $l = 1,2,3,4$: choose α_l, ε, $x^{(l)}(k) = 0$ # 0 is the zero matrix with the same dimension as $x^{(l)}(k)$.

While $k = 0,1,2,\ldots,n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(s^l(k-1)\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(s^l(k-1)\right)\left(d^{(l)}\right)^T\right),$$
$$s^{(l)} = e^{(l)} - a^{(l)}x^{(l)}(k)\,b^{(l)} - c^{(l)}x^{(l)}(k)\,d^{(l)},$$
$$\alpha_l = \left\|s^{(l)}\right\|_2.$$

If $\alpha_l < \varepsilon$ **then**

$$\text{print}\ (x^{(l)}(k));\text{print}\ (\text{“number of iterations ="},\ k),$$

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T\left(s^l(k-1)\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T\left(s^l(k-1)\right)\left(d^{(l)}\right)^T\right).$$

Update k.

end

$$\text{print}\ (x^{(l)}(k));\text{print}\ (\text{“number of iterations ="},\ k),$$

end

Below is the algorithm for the FLSIM. This algorithm can be used by different software for solving the positive GTrFFSME in Eq. (1.1).

Algorithm 4.14 (Fuzzy least-square algorithm for GTrFFSME)**.** Input $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$ and $\tilde{E}$ # Split each matrix into 4 matrices (e.g., $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $a^{(4)}$) , for $l = 1,2,3,4$: choose α_l, ε, $x^{(l)}(k) = 0$ # 0 is the zero matrix with the same dimension as $x^{(l)}(k)$.

While $k = 0,\ 1,\ 2, \ldots,\ n$ **do**

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(\left(\left(a^{(l)}\right)^T \bullet a^{(l)}\right)^{-1} \bullet \left(a^{(l)}\right)^T \left(s^l(k-1)\right)\left(b^{(l)}\right)^T \left(\left(b^{(l)}\left(b^{(l)}\right)^T\right)^{-1}\right)\right.\right.$$

$$\left. + \left(\left(c^{(l)}\right)^T \bullet c^{(l)}\right)^{-1} \left(c^{(l)}\right)^T \left(s^l(k-1)\right)\left(d^{(l)}\right)^T \left(\left(d^{(l)}\left(d^{(l)}\right)^T\right)^{-1}\right)\right),$$

$$s^{(l)} = e^{(l)} - a^{(l)}x^{(l)}(k)\,b^{(l)} - c^{(l)}x^{(l)}(k)\,d^{(l)},$$

$$\alpha_l = \left\|s^{(l)}\right\|_2.$$

If $\alpha_l < \varepsilon$ **then**

print $(x^{(l)}(k))$; print (“number of iterations =”, k) ,

else

$$\hat{x}^{(l)}(k) = \hat{x}^{(l)}(k-1) + \frac{\alpha_l}{2}\left(\left(a^{(l)}\right)^T \left(s^l(k-1)\right)\left(b^{(l)}\right)^T + \left(c^{(l)}\right)^T \left(s^l(k-1)\right)\left(d^{(l)}\right)^T\right),$$

Update k.

end

print $(x^{(l)}(k))$; print (“number of iterations =”, k) ,

end

Computational complexity analysis

The computational complexity of the algorithms developed in this article is discussed in this section. The computations in these algorithms include Kronecker product of matrices and other matrix algebra operations where the matrices under consideration are considered as square $m\times m$ matrices. The computational complexity of the Kronecker product of two matrices of size $m \times m$ is $O(m^4)$ with resulting matrix of size $m^2\times m^2$, while the computational complexity to take the inverse of a square matrix of size $m \times m$ using the classical Gaussian elimination method is roughly $O(m^3)$. For matrix multiplication of one $n\times m$ matrix and one $m\times p$, the computational complexity is $O(nmp)$, the computational complexity of computing the transpose of an $m \times m$ matrix is $O(m^2)$, and matrix algebra of addition and subtraction between two $m\times m$ involves a computational complexity of $O(m^2)$. The computational complexity of taking the p-norm of an $m\times m$ matrix depends on the value of P. If $p= 1$ or $p=\infty$, the computational complexity of calculating the p-norm is $O(m^2)$, since we need to iterate over all the elements of the matrix at least once, while for other values of p, the computational complexity of calculating the p-norm is $O(m^3)$ because we need to first calculate the sum of squares of all the elements in the matrix, which takes $O(m^2)$ time, and then take the square root, which takes $O(m)$ time. Therefore, the overall computational complexity is $O(m^2+m) = O(m^2)$.

In Algorithm 4.10, which is the fuzzy matrix vectorization and Kronecker product, the steps require a total complexity of $l\left(m^2\left(O\left(m^2\right)\right)\right) + O\left(\left(m^2\right)^3\right) + O\left(\left(m^2\right)^3\right)$. Upon simplification, this implies that Algorithm 4.10 has a computational complexity order of $O(m^6)$. For Algorithm 4.13, the maximum possible computational complexity is $n\left(11\left(O\left(m^2\right)\right) + 6\left(O\left(m^2\right)\right) + O\left(m^3\right) + 11\left(O\left(m^2\right)\right)\right)$ which simplifies to $O(m^3)$. Whereas Algorithm 4.14 requires a maximum computational complexity of $n(20\left(O\left(m^2\right)\right) + 6\left(O\left(m^2\right)\right) + O\left(m^3\right) + 11\left(O\left(m^2\right)\right))$ which also simplifies to $O(m^3)$ like Algorithm 4.13. This implies that, among the developed algorithms, Algorithms 4.13 and 4.14 have the same complexity of $O(m^3)$, while Algorithm 4.10 has a higher complexity of $O(m^6)$. Therefore, Algorithms 4.13 and Algorithm 4.14

are better than Algorithm 4.10 in terms of computational complexity. However, between Algorithms 4.13 and 4.14, Algorithm 4.13 has a lower constant factor compared to Algorithm 4.14. Therefore, Algorithm 4.13 is better than Algorithm 4.14 in terms of computational complexity.

In the following section, the three proposed methods for solving the GTrFFSME in Eq. (1.1) are applied to different fuzzy systems in (1.2), (1.3), and (1.4), respectively.

4.1.4. Applications of the proposed methods to other fuzzy systems and fuzzy numbers

The proposed methods for the GTrFFSME in (1.1) can be modified to solve the fuzzy systems in (1.2), (1.3), and (1.4), respectively. In the following method, the FMVM for GTrFFSEM is applied to the fuzzy systems in (1.2), (1.3), and (1.4).

Fuzzy matrix vectorization method (FMVM)

The FMVM method is reduced to solve different fuzzy systems. FMVM is applied to FFSME in Eq. (1.2) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.5) and Definition 1.2, we get:

$$x_{ij}^{(l)} = \mathrm{vec}^{-1}\left(I_{ij}^{(l)} \bigotimes a_{ij}^{(l)} + (d_{ij}^{(l)})^{T} \bigotimes I_{ij}^{(l)}\right)^{-1} \mathrm{vec}(e_{ij}^{(l)})).$$

FMVM is applied to FFCTLME in Eq. (1.3) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.5) and Definition 1.3, we have:

$$x_{ij}^{(l)} = \mathrm{vec}^{-1}\left(I_{ij}^{(l)} \bigotimes a_{ij}^{(l)} + a_{ij}^{(l)} \bigotimes I_{ij}^{(l)}\right)^{-1} \mathrm{vec}(e_{ij}^{(l)})).$$

FMVM is applied to FFSTME in Eq. (1.4) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.5) and Definition 1.4, we have:

$$x_{ij}^{(l)} = \mathrm{vec}^{-1}\left(I_{ij}^{(l)} \bigotimes I_{ij}^{(l)} + (d_{ij}^{(l)})^{T} \bigotimes c_{ij}^{(l)}\right)^{-1} \mathrm{vec}(e_{ij}^{(l)})).$$

In the following method, the FGIM for GTrFFSEM is applied to the fuzzy systems in (1.2), (1.3), and (1.4).

Fuzzy gradient-iterative method (FGIM)

The FGIM method is reduced to solve different fuzzy systems. FGIM is applied to FFSME in Eq. (1.2) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.15) and Definition 1.2, we have:

$$\begin{aligned}\hat{X}^{(l)}(K) = \hat{X}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\left(A^{(l)}\right)^{T}\left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)\,D^{(l)}\right) \\ + \left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)\,D^{(l)}\right)\left(D^{(l)}\right)^{T}\Big),\end{aligned}$$

where the convergence factor (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{\max}\left[\left(A^{(l)}\right)^{T} A^{(l)}\right] + \lambda_{\max}\left[D^{(l)}\left(D^{(l)}\right)^{T}\right]}.$$

FGIM is applied to FFCTLME in Eq. (1.3) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.15) and Definition 1.3, we have

$$\begin{aligned}\hat{X}^{(l)}(K) = \hat{X}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(\left(A^{(l)}\right)^{T}\left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)\left(A^{(l)}\right)^{T}\right) \\ + \left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)\left(A^{(l)}\right)^{T}\right) A^{(l)}\Big),\end{aligned}$$

where the convergence factor (step size) is given by,

$$0 < \alpha_l < \frac{1}{\lambda_{\max}\left[\left(A^{(l)}\right)^T A^{(l)}\right]}.$$

FGIM is applied to FFSTME in Eq. (1.4) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.15) and Definition 1.4, we have

$$\begin{aligned}\hat{X}^{(l)}(K) = \hat{X}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(&\left(E^{(l)} - \hat{X}^{(l)}(k-1) - C^{(l)}\hat{X}^{(l)}(k-1)D^{(l)}\right)\\ &+ \left(C^{(l)}\right)^T\left(E^{(l)} - \hat{X}^{(l)}(k-1) - C^{(l)}\hat{X}^{(l)}(k-1)D^{(l)}\right)\left(D^{(l)}\right)^T\Big),\end{aligned}$$

where the convergence factor (step size) is given by,

$$0 < \alpha_l < \frac{2}{\lambda_{\max}\left[\left(C^{(l)}\right)^T C^{(l)}\right]\lambda_{\max}\left[D^{(l)}\left(D^{(l)}\right)^T\right]}.$$

In the following method, the FLSIM for GTrFFSME is applied to the fuzzy systems in (1.2), (1.3), and (1.4).

Fuzzy least-square iterative method (FLSIM)

The FLSIM method is reduced to solve different fuzzy systems.

FLSIM is applied to FFSME in Eq. (1.2) as follows: for $1 \leqslant l \leqslant 4$ and by applying Eqs. (4.11)-(4.14) and Definition 1.2, we have

$$\begin{aligned}\hat{X}^{(l)}(K) = \hat{X}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(&\left(\left(A^{(l)}\right)^T \bullet A^{(l)}\right)^{-1} \bullet \left(A^{(l)}\right)^T\left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)D^{(l)}\right)\\ &+ \left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)D^{(l)}\right)\left(D^{(l)}\right)^T\left(\left(D^{(l)}\left(D^{(l)}\right)^T\right)^{-1}\right)\Big),\end{aligned}$$

where the convergence factor (step size) is given by,

$$0 < \alpha_l < 2.$$

FLSIM is applied to FFCTLME in Eq. (1.3) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.22) and Definition 1.3, we have

$$\begin{aligned}\hat{X}^{(l)}(K) = \hat{X}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(&\left(\left(A^{(l)}\right)^T \bullet A^{(l)}\right)^{-1} \bullet \left(A^{(l)}\right)^T\left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)\left(A^{(l)}\right)^T\right)\\ &+ \left(E^{(l)} - A^{(l)}\hat{X}^{(l)}(k-1) - \hat{X}^{(l)}(k-1)\left(A^{(l)}\right)^T\right)A^{(l)}\left(\left(A^{(l)}\right)^T \bullet A^{(l)}\right)^{-1}\Big),\end{aligned}$$

where the convergence factor (step size) is given by,

$$0 < \alpha_l < 2.$$

FLSIM is applied to FFSTME in Eq. (1.4) as follows: for $1 \leqslant l \leqslant 4$ and by modifying the solution in Eq. (4.22) and Definition 1.4, we have

$$\begin{aligned}\hat{X}^{(l)}(K) = \hat{X}^{(l)}(k-1) + \frac{\alpha_l}{2}\Big(&\left(E^{(l)} - \hat{X}^{(l)}(k-1) - C^{(l)}\hat{X}^{(l)}(k-1)D^{(l)}\right) + \left(\left(C^{(l)}\right)^T \bullet C^{(l)}\right)^{-1}\left(C^{(l)}\right)^T\\ &\times\left(E^{(l)} - \hat{X}^{(l)}(k-1) - C^{(l)}\hat{X}^{(l)}(k-1)D^{(l)}\right)\left(D^{(l)}\right)^T\left(\left(D^{(l)}\left(D^{(l)}\right)^T\right)^{-1}\right)\Big),\end{aligned}$$

where the convergence factor (step size) is given by,

$$0 < \alpha_l < 2.$$

The next remark shows that the proposed methods can also be applied to different fuzzy systems with triangular fuzzy numbers.

Remark 4.15. The proposed methods for the GTrFFSME in Eq. (1.1) and their special cases in (1.2)-(1.4) can be applied to the same systems with TFNs whenever the mean values in the TrFNs used in the previous systems are equal.

5. Numerical examples

To illustrate the accuracy and effectiveness of the proposed methods for solving the GTrFFSME in Eq. (1.1), we consider GTrFFSME with (2×2) and (100×100). Analytical solutions are found by Algorithm 4.10 for FMVM, and then we compare the performance of Algorithm 4.13 and 4.14 for FGIM and FLSIM for approximating that solution by calculating the number of iterations (k), convergence factor (α), error bound (ε), convergence rate, CPU time, real-time and memory usage. In addition to the graphical representation of the error $\delta^{l}(k)$ when k increases.

In the following example, the proposed methods are applied to small GTrFFSME (2×2).

Example 5.1. Solve the following 2×2 GTrFFSME:

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E},$$

given that

$$\tilde{A} = \begin{pmatrix} (4,6,7,8) & (1,3,4,5) \\ (1,2,3,4) & (3,5,6,7) \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} (4,6,7,9) & (2,3,4,6) \\ (1,3,4,5) & (3,5,6,7) \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} (5,6,7,8) & (1,3,4,5) \\ (2,4,5,6) & (4,6,7,9) \end{pmatrix}, \qquad \tilde{D} = \begin{pmatrix} (4,5,6,8) & (1,2,3,4) \\ (1,3,4,5) & (2,5,6,7) \end{pmatrix},$$

$$\tilde{E} = \begin{pmatrix} (95,474,952,1890) & (66,390,828,1680) \\ (76,504,980,1960) & (76,430,867,1730) \end{pmatrix}.$$

To solve the given positive GTrFFSME, the necessary and sufficient conditions in Corollary 4.7 and Theorem 4.8 for having a unique positive fuzzy solution must be examined first.

The uniqueness of the positive fuzzy solutions

By Corollary 4.7, the given positive GTrFFSME has a unique fuzzy solution if and only if: $\det(r_1) \neq 0, \det(r_2) \neq 0, \det(r_3) \neq 0$, and $\det(r_4) \neq 0$ *i.e* r_1, r_2, r_3 and r_4 are invertible matrices. The determinants of r_1, r_2, r_3, and r_4 can be calculated as follows:

$$r_1 = (b_{ij}^{(1)})^T \bigotimes a_{ij}^{(1)} + (d_{ij}^{(1)})^T \bigotimes c_{ij}^{(1)} = \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix} \text{ and } \det(r_1) = 224694 \neq 0,$$

$$r_2 = (b_{ij}^{(2)})^T \bigotimes a_{ij}^{(2)} + (d_{ij}^{(2)})^T \bigotimes c_{ij}^{(2)} = \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix} \text{ and } \det(r_2) = 3686400 \neq 0,$$

$$r_3 = (b_{ij}^{(3)})^T \bigotimes a_{ij}^{(3)} + (d_{ij}^{(3)})^T \bigotimes c_{ij}^{(3)} = \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix} \text{ and } \det(r_3) = 8708400 \neq 0,$$

$$r_4 = (b_{ij}^{(4)})^T \bigotimes a_{ij}^{(4)} + (d_{ij}^{(4)})^T \bigotimes c_{ij}^{(4)} = \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix} \text{ and } \det(r_4) = 29062800 \neq 0.$$

Thus, the given positive GTrFFSME has a unique solution.

Existence of the positive fuzzy solution of positive GTrFFSME

By Theorem 4.8, the given positive GTrFFSME has a positive fuzzy solution if and only if:

1. ${r_1}^{-1}t_1 > 0$, ${r_2}^{-1}t_2 > 0$, ${r_3}^{-1}t_3 > 0$, and ${r_4}^{-1}t_4 > 0$,

$${r_1}^{-1}t_1 = \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix}^{-1} \begin{pmatrix} 95 \\ 76 \\ 66 \\ 76 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} > 0,$$

$${r_2}^{-1}t_2 = \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix}^{-1} \begin{pmatrix} 474 \\ 504 \\ 390 \\ 430 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 4 \end{pmatrix} > 0,$$

$${r_3}^{-1}t_3 = \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix}^{-1} \begin{pmatrix} 952 \\ 828 \\ 980 \\ 867 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \\ 5 \end{pmatrix} > 0,$$

$${r_4}^{-1}t_4 = \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix}^{-1} \begin{pmatrix} 1890 \\ 1960 \\ 1680 \\ 1730 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \\ 6 \end{pmatrix} > 0;$$

2. ${r_1}^{-1}t_1 \leqslant {r_2}^{-1}t_2 \leqslant {r_3}^{-1}t_3 \leqslant {r_4}^{-1}t_4$,

$$\begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix} < \begin{pmatrix} 3 \\ 4 \\ 2 \\ 4 \end{pmatrix} < \begin{pmatrix} 4 \\ 5 \\ 3 \\ 5 \end{pmatrix} < \begin{pmatrix} 5 \\ 6 \\ 5 \\ 6 \end{pmatrix}.$$

Since the first and second conditions hold for each corresponding element, the positive fuzzy solution to the given positive GTrFFSME exists. Therefore, the developed FMVM in Section 4.1.1 can now be applied to obtain the positive fuzzy solution to the given positive GTrFFSME. The details of the illustration of the FMVM are as follows.

Step 1: Decompose $\tilde{A}$, $\tilde{X}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, and $\tilde{E}$ into $a_{ij}^{(1)} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}$, $b_{ij}^{(1)} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$, $c_{ij}^{(1)} = \begin{pmatrix} 5 & 1 \\ 2 & 4 \end{pmatrix}$, $d_{ij}^{(1)} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$, $e_{ij}^{(1)} = \begin{pmatrix} 95 & 66 \\ 76 & 76 \end{pmatrix}$, $a_{ij}^{(2)} = \begin{pmatrix} 6 & 3 \\ 2 & 5 \end{pmatrix}$, $b_{ij}^{(2)} = \begin{pmatrix} 6 & 4 \\ 3 & 6 \end{pmatrix}$, $c_{ij}^{(2)} = \begin{pmatrix} 6 & 3 \\ 4 & 6 \end{pmatrix}$, $d_{ij}^{(2)} = \begin{pmatrix} 5 & 2 \\ 3 & 5 \end{pmatrix}$, $e_{ij}^{(2)} = \begin{pmatrix} 474 & 390 \\ 504 & 430 \end{pmatrix}$, $a_{ij}^{(3)} = \begin{pmatrix} 7 & 4 \\ 3 & 6 \end{pmatrix}$, $b_{ij}^{(3)} = \begin{pmatrix} 7 & 4 \\ 4 & 6 \end{pmatrix}$, $c_{ij}^{(3)} = \begin{pmatrix} 7 & 4 \\ 5 & 7 \end{pmatrix}$, $d_{ij}^{(3)} = \begin{pmatrix} 6 & 3 \\ 4 & 6 \end{pmatrix}$, $e_{ij}^{(3)} = \begin{pmatrix} 952 & 828 \\ 980 & 867 \end{pmatrix}$, $a_{ij}^{(4)} = \begin{pmatrix} 8 & 5 \\ 4 & 7 \end{pmatrix}$, $b_{ij}^{(4)} = \begin{pmatrix} 9 & 6 \\ 5 & 7 \end{pmatrix}$, $c_{ij}^{(4)} = \begin{pmatrix} 8 & 5 \\ 6 & 9 \end{pmatrix}$, $d_{ij}^{(4)} = \begin{pmatrix} 8 & 4 \\ 5 & 7 \end{pmatrix}$, and $e_{ij}^{(4)} = \begin{pmatrix} 1890 & 1680 \\ 1960 & 1730 \end{pmatrix}$.

Step 2: Applying the vec-operator and Kronecker product on Eq. (4.1) gives:

$$\begin{cases} \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix} \begin{pmatrix} x_{11}^{(1)} \\ x_{21}^{(1)} \\ x_{12}^{(1)} \\ x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 95 \\ 76 \\ 66 \\ 76 \end{pmatrix}, \\ \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix} \begin{pmatrix} x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{12}^{(2)} \\ x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 474 \\ 504 \\ 390 \\ 430 \end{pmatrix}, \\ \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix} \begin{pmatrix} x_{11}^{(3)} \\ x_{21}^{(3)} \\ x_{12}^{(3)} \\ x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 952 \\ 828 \\ 980 \\ 867 \end{pmatrix}, \\ \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix} \begin{pmatrix} x_{11}^{(4)} \\ x_{21}^{(4)} \\ x_{12}^{(4)} \\ x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 1890 \\ 1960 \\ 1680 \\ 1730 \end{pmatrix}. \end{cases} \tag{5.1}$$

Step 3: Multiply the system of linear matrix equation in Eq. (5.1) by multiplicative matrix inverse as follows:

$$\begin{cases} \begin{pmatrix} x_{11}^{(1)} \\ x_{21}^{(1)} \\ x_{12}^{(1)} \\ x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 36 & 8 & 9 & 2 \\ 12 & 28 & 3 & 7 \\ 13 & 3 & 22 & 5 \\ 4 & 10 & 7 & 17 \end{pmatrix}^{-1} \begin{pmatrix} 95 \\ 76 \\ 66 \\ 76 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{12}^{(2)} \\ x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 66 & 33 & 36 & 18 \\ 32 & 60 & 18 & 33 \\ 36 & 18 & 66 & 33 \\ 16 & 32 & 32 & 60 \end{pmatrix}^{-1} \begin{pmatrix} 474 \\ 504 \\ 390 \\ 430 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(3)} \\ x_{21}^{(3)} \\ x_{12}^{(3)} \\ x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 91 & 52 & 56 & 32 \\ 51 & 84 & 32 & 52 \\ 49 & 28 & 84 & 48 \\ 27 & 45 & 48 & 78 \end{pmatrix}^{-1} \begin{pmatrix} 952 \\ 828 \\ 980 \\ 867 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(4)} \\ x_{21}^{(4)} \\ x_{12}^{(4)} \\ x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 136 & 85 & 80 & 50 \\ 84 & 135 & 50 & 80 \\ 80 & 50 & 112 & 70 \\ 48 & 78 & 70 & 112 \end{pmatrix}^{-1} \begin{pmatrix} 1890 \\ 1960 \\ 1680 \\ 1730 \end{pmatrix}. \end{cases} \tag{5.2}$$

Step 4: Using matrix multiplication on the system in Eq. (5.2), the positive fuzzy solution to the given positive GTrFFSME is as follows:

$$\begin{cases} \begin{pmatrix} x_{11}^{(1)} \\ x_{21}^{(1)} \\ x_{12}^{(1)} \\ x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} x_{11}^{(2)} \\ x_{21}^{(2)} \\ x_{12}^{(2)} \\ x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 2 \\ 4 \end{pmatrix}, \\ \begin{pmatrix} x_{11}^{(3)} \\ x_{21}^{(3)} \\ x_{12}^{(3)} \\ x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 3 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} x_{11}^{(4)} \\ x_{21}^{(4)} \\ x_{12}^{(4)} \\ x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \\ 6 \end{pmatrix}. \end{cases}$$

By Definition 2.8, the obtained fuzzy solution can be written as:

$$\begin{pmatrix} x_{11}^{(1)} & x_{12}^{(1)} \\ x_{21}^{(1)} & x_{22}^{(1)} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} x_{11}^{(2)} & x_{12}^{(2)} \\ x_{21}^{(2)} & x_{22}^{(2)} \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 4 \end{pmatrix},$$
$$\begin{pmatrix} x_{11}^{(3)} & x_{12}^{(3)} \\ x_{21}^{(3)} & x_{22}^{(3)} \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 5 & 5 \end{pmatrix}, \quad \begin{pmatrix} x_{11}^{(4)} & x_{12}^{(4)} \\ x_{21}^{(4)} & x_{22}^{(4)} \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 6 & 6 \end{pmatrix}.$$

Step 5: By combining the obtained positive fuzzy solution in Step 4, the positive fuzzy solution to Example 5.1 is

$$\hat{X} = \begin{pmatrix} \left(x_{11}^{(1)}, x_{11}^{(2)}, x_{11}^{(3)}, x_{11}^{(4)}\right) & \left(x_{12}^{(1)}, x_{12}^{(2)}, x_{12}^{(3)}, x_{12}^{(4)}\right) \\ \left(x_{21}^{(1)}, x_{21}^{(2)}, x_{21}^{(3)}, x_{21}^{(4)}\right) & \left(x_{22}^{(1)}, x_{22}^{(2)}, x_{22}^{(3)}, x_{22}^{(4)}\right) \end{pmatrix} = \begin{pmatrix} (2,\ 3,\ 4,\ 5) & (1,\ 2,\ 3,\ 5) \\ (1,\ 4,\ 5,\ 6) & (3,\ 4,\ 5,\ 6) \end{pmatrix}. \tag{5.3}$$

The solution for the given GTrFFSME is obtained by the proposed methods as follows.

This solution is approximated using Algorithms 4.13 and 4.14 as follows.

Fuzzy gradient-iterative method (FGIM) and Fuzzy least-square iterative method (FLSIM)

Algorithms 4.13 and 4.14 for FGIM and FLSIM are applied to compute the approximated solution $\hat{X}^{(l)}(k)$ for the given GTrFFSME using the following initial value for $1 \leqslant l \leqslant 4$, $\hat{X}^{(l)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. The approximated fuzzy solution $\tilde{X}$ is shown in Table 1 with the convergence factor (α), error bound (ε), and a total number of iterations (k). While Table 2 shows the computational time and memory usage for FGIM and FLSIM.

Table 1: Comparison between FMVM, FGIM, and FLSIM, for Example 5.1.

	Method	Analytical solution-approximated solution	α	ε	k
$\hat{X}^{(1)}$	FMVM	$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 1.99949855017298 & 1.00098875551005 \\ 1.00079141255138 & 2.99843234122586 \end{pmatrix}$	0.0005	10^{-4}	147
	FLSIM	$\begin{pmatrix} 1.99985414539036 & 0.999963165192173 \\ 1.00002909178272 & 2.99966344885767 \end{pmatrix}$	0.2	10^{-4}	7
$\hat{X}^{(2)}$	FMVM	$\begin{pmatrix} 3 & 4 \\ 3 & 4 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 2.99974677876014 & 4.00031607313094 \\ 3.00032352010653 & 3.99970338494345 \end{pmatrix}$	0.0001	10^{-4}	244
	FLSIM	$\begin{pmatrix} 2.99999955606239 & 3.99999959009359 \\ 2.99999667934673 & 3.99999527502009 \end{pmatrix}$	0.2	10^{-4}	8
$\hat{X}^{(3)}$	FMVM	$\begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 3.98757299663079 & 5.01285881972725 \\ 4.01339431585115 & 4.98608683548303 \end{pmatrix}$	0.00002	10^{-4}	144
	FLSIM	$\begin{pmatrix} 3.99999941493081 & 4.999999571129046 \\ 3.99999570858620 & 4.999994185596878 \end{pmatrix}$	0.2	10^{-4}	8
$\hat{X}^{(4)}$	FMVM	$\begin{pmatrix} 5 & 6 \\ 5 & 6 \end{pmatrix}$	NA	0	NA
	FGIM	$\begin{pmatrix} 5.00156518405719 & 5.99802708526725 \\ 5.00018074252759 & 5.99996176085665 \end{pmatrix}$	0.00001	10^{-4}	133
	FLSIM	$\begin{pmatrix} 4.99999460351041 & 5.99999962240993 \\ 5.00018074252759 & 5.99990855571824 \end{pmatrix}$	0.2	10^{-4}	8

Table 2: Computational time and memory usage.

	Method	k	CPU time	Real-time	Memory usage
$\hat{X}^{(1)}$	FGIM	147	21.59 ms	20.45 ms	3.70 MiB
	FLSIM	7	17.50 ms	19.38 ms	4.01 MiB
$\hat{X}^{(2)}$	FGIM	244	12.23 ms	11.93 ms	2.17 MiB
	FLSIM	8	11.75 ms	11.62 ms	2.43 MiB
$\hat{X}^{(3)}$	FGIM	144	12.26 ms	12.40 ms	2.17 MiB
	FLSIM	8	13.75 ms	11.88 ms	2.43 MiB
$\hat{X}^{(4)}$	FGIM	133	12.10 ms	12.24 ms	2.17 MiB
	FLSIM	8	17.62 ms	19.62 ms	2.43 MiB

Figure 3 shows the change in the error $\delta^l(k)$, when k increases up to $k = 20$.

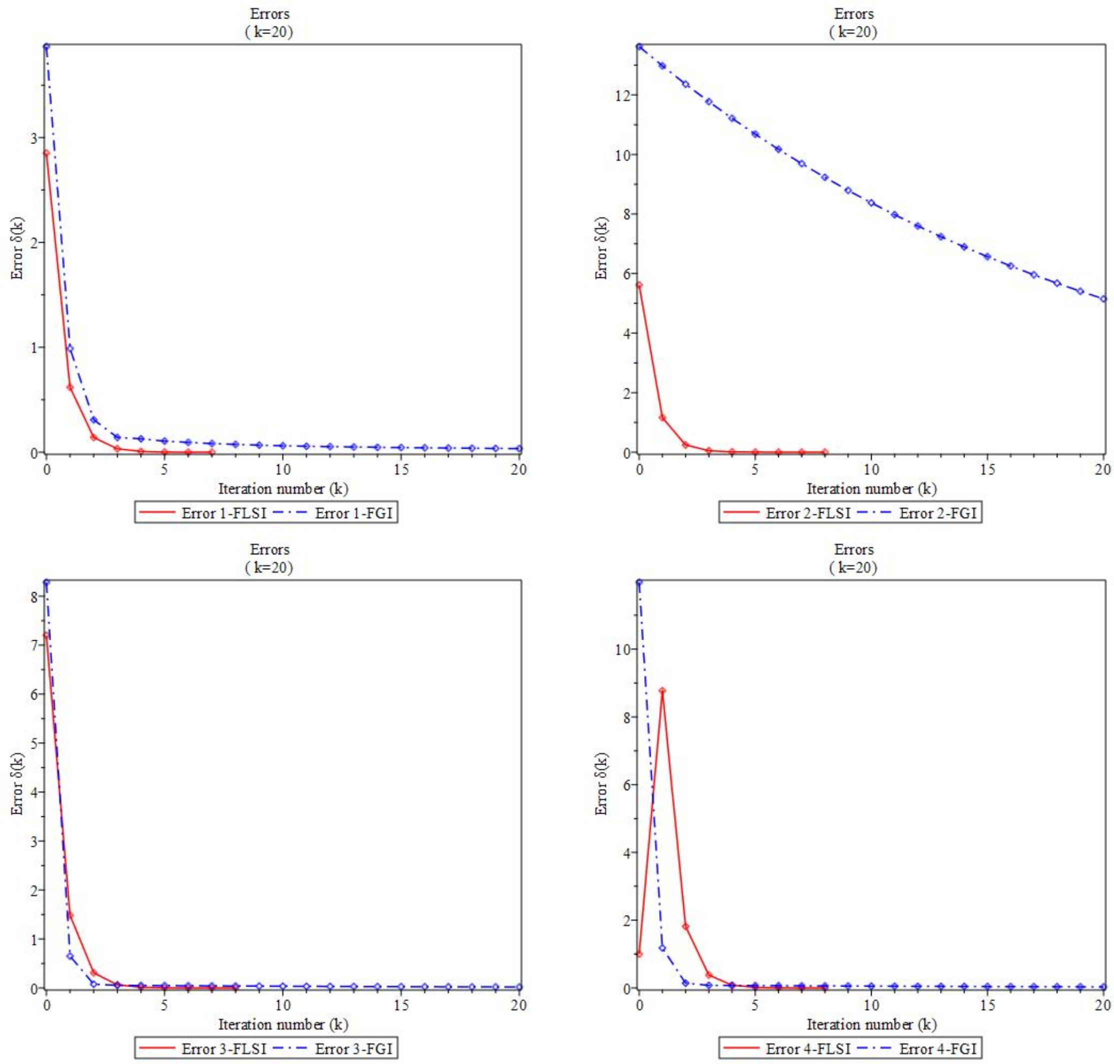

Figure 3: Comparison between $\delta^l(k)$ of FGIM and FLSIM for the first 20 iterations.

From Tables 1 and 2 and Figure 3, it is obvious that the error $\delta^l(k)$ is reducing as k increases. This indicates that the proposed algorithm is effective and convergent for the given GTrFFSME. In addition, the FLSIM takes more computational time and more memory compared to FGIM. However, in terms of accuracy, error, and number of iterations, FLSIM is superb compared to FGIM. The analyses of the obtained positive solution in Eq. (5.3) for the positive GTrFFSME in Example 5.1 are discussed. The analysis of the obtained positive fuzzy solution in Eq. (5.3) for the given positive GTrFFSME in Example 5.1 includes verification of the solution, representation of the solution and checking the feasibility conditions for the solution, details of explanation are given as follows.

Verification of positive fuzzy solution to positive GTrFFSME

To verify the obtained positive fuzzy solution in Eq. (5.3) for the positive GTrFFSME in Example 5.1, we first multiply $\tilde{A}\tilde{X}\tilde{B}$ as follows:

$$\begin{aligned}\tilde{A}\tilde{X}\tilde{B} &= \begin{pmatrix} (4,\ 6,\ 7,\ 8) & (1,\ 3,\ 4,\ 5) \\ (1,\ 2,\ 3,\ 4) & (3,\ 5,\ 6,\ 7) \end{pmatrix} \begin{pmatrix} (2,\ 3,\ 4,\ 5) & (1,\ 2,\ 3,\ 5) \\ (1,\ 4,\ 5,\ 6) & (3,\ 4,\ 5,\ 6) \end{pmatrix} \begin{pmatrix} (4,\ 6,\ 7,\ 9) & (2,\ 3,\ 4,\ 6) \\ (1,\ 3,\ 4,\ 5) & (3,\ 5,\ 6,\ 7) \end{pmatrix} \\ &= \begin{pmatrix} (43,\ 252,\ 500,\ 980) & (39,\ 210,\ 438,\ 910) \\ (30,\ 228,\ 450,\ 868) & (40,\ 198,\ 402,\ 806) \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\tilde{C}\tilde{X}\tilde{D} &= \begin{pmatrix} (5,\ 6,\ 7,\ 8) & (1,\ 3,\ 4,\ 5) \\ (2,\ 4,\ 5,\ 6) & (4,\ 6,\ 7,\ 9) \end{pmatrix} \begin{pmatrix} (2,\ 3,\ 4,\ 5) & (1,\ 2,\ 3,\ 5) \\ (1,\ 4,\ 5,\ 6) & (3,\ 4,\ 5,\ 6) \end{pmatrix} \begin{pmatrix} (4,\ 5,\ 6,\ 8) & (1,\ 2,\ 3,\ 4) \\ (1,\ 3,\ 4,\ 5) & (2,\ 5,\ 6,\ 7) \end{pmatrix} \\ &= \begin{pmatrix} (52,\ 222,\ 452,\ 910) & (27,\ 180,\ 390,\ 770) \\ (46,\ 276,\ 530,\ 1092) & (36,\ 232,\ 465,\ 924) \end{pmatrix}.\end{aligned}$$

Therefore,

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \begin{pmatrix} (95,\ 474,\ 952,\ 1890) & (66,\ 390,\ 828,\ 1680) \\ (76,\ 504,\ 980,\ 1960) & (76,\ 430,\ 867,\ 1730) \end{pmatrix} = \tilde{E}.$$

This means the obtained positive fuzzy solution in Eq. (5.3) satisfies the positive GTrFFSME in Example 5.1.

Representation of positive fuzzy solution to positive GTrFFSME

The positive fuzzy solution, for Example 5.1, is represented in Figure 4.

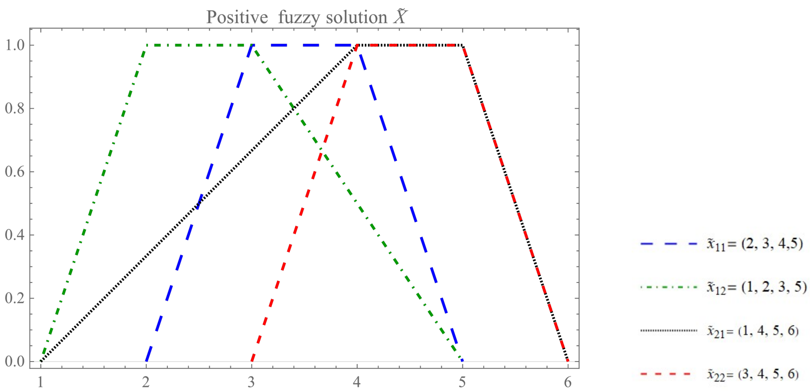

Figure 4: Positive Fuzzy Solution for Example 5.1.

Figure 4 shows that each fuzzy number in the obtained fuzzy solution in Eq. (5.3) is positive TrFN. This means that the FMVM can provide an exact unique positive fuzzy solution to the given GTrFFSME.

Feasibility of positive fuzzy solution to positive GTrFFSME

Based on (4.8), the feasibility conditions are checked as follows:

1. $x_{ij}^{(l)} > 0, \forall \{1 \leqslant i, j \leqslant p, n\}, x_{ij}^{(1)} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} > 0,$

$$x_{ij}^{(2)} = \begin{pmatrix} 3 & 2 \\ 4 & 4 \end{pmatrix} > 0, \quad x_{ij}^{(3)} = \begin{pmatrix} 4 & 3 \\ 5 & 5 \end{pmatrix} > 0, \quad x_{ij}^{(4)} = \begin{pmatrix} 5 & 5 \\ 6 & 6 \end{pmatrix} > 0;$$

2. $x_{ij}^{(4)} \geqslant x_{ij}^{(3)} \geqslant x_{ij}^{(2)} \geqslant x_{ij}^{(1)}, \forall \{1 \leqslant i, j \leqslant p, n\}$,

$$\begin{pmatrix} 5 & 5 \\ 6 & 6 \end{pmatrix} \geqslant \begin{pmatrix} 4 & 3 \\ 5 & 5 \end{pmatrix} \geqslant \begin{pmatrix} 3 & 2 \\ 4 & 4 \end{pmatrix} \geqslant \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

The feasibility conditions are satisfied, and therefore, the obtained positive fuzzy solution is feasible.

The verification, representation, and feasibility of the obtained positive solution satisfy the given positive GTrFFSME, and it is a strong positive fuzzy solution. In the following example, the method is applied to 5×5 positive GTrFFSME.

In the following example, we tested the proposed method on 100×100 GTrFFSME, obtaining the following results.

Example 5.2. Solve the following 100×100 GTrFFSME:

$$\tilde{A}\tilde{X}\tilde{B} + \tilde{C}\tilde{X}\tilde{D} = \tilde{E},$$

where

$$\begin{aligned}
A^{(1)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 1 \cdots 2)\,, \\
B^{(1)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 1 \cdots 2)\,, \\
C^{(1)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 1 \cdots 2)\,, \\
D^{(1)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 1 \cdots 2)\,, \\
E^{(1)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\left(100,\ 100,\ \text{generator} = 2 \times 10^5 \cdots 3 \times 10^5\right), \\
A^{(2)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 3 \cdots 4)\,, \\
B^{(2)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 3 \cdots 4)\,, \\
C^{(2)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 3 \cdots 4)\,, \\
D^{(2)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 3 \cdots 4)\,, \\
E^{(2)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\left(100,\ 100,\ \text{generator} = 3 \times 10^6 \cdots 4 \times 10^6\right), \\
A^{(3)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 5 \cdots 6)\,, \\
B^{(3)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 5 \cdots 6)\,, \\
C^{(3)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 5 \cdots 6)\,, \\
D^{(3)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 5 \cdots 6)\,, \\
E^{(3)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\left(100,\ 100,\ \text{generator} = 1 \times 10^8 \cdots 2 \times 10^8\right), \\
A^{(4)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 7 \cdots 8)\,, \\
B^{(4)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 7 \cdots 8)\,, \\
C^{(4)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 7 \cdots 8)\,, \\
D^{(4)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\,(100,\ 100,\ \text{generator} = 7 \cdots 8)\,, \\
E^{(4)} &= \text{LinearAlgebra} : -\text{RandomMatrix}\left(100,\ 100,\ \text{generator} = 3 \times 10^8 \cdots 4 \times 10^8\right).
\end{aligned}$$

The fuzzy solution to the given GTrFFSME is obtained by the proposed methods as follows.

Fuzzy matrix vectorization method (FMVM)

To apply the FMVM, we need to find the inverse of the 10000×10000 matrix, which requires long computational timing and huge memory. Thus, FMVM is not a practical approach for such a large dimensional system. However, FGIM and FLSIM can be used to obtain an approximated solution as follows.

Fuzzy gradient-iterative method (FGIM) and fuzzy least-square iterative method (FLSIM)

Algorithms 4.13 and 4.14 for FGIM and FLSIM are applied to compute the approximated solution $\hat{X}^{(l)}(k)$, using the following initial value for $1 \leqslant l \leqslant 4$,

$$\hat{X}^{(l)}(0) = \text{LinearAlgebra} : -\text{RandomMatrix}(100, 100, \text{generator} = 0).$$

FLSIM can get the solution in just 4 iterations with $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.25)$. However, FGIM needs thousands of iterations to give the approximated solution using $(\alpha_1 = 10^{-12}, \alpha_2 = 10^{-13}, \alpha_3 = 10^{-14}, \alpha_4 = 10^{-15})$. In the following Table 3, the step size, computational time and memory usage for the first 20 iterations for FLSIM and FGIM are compared.

Table 3: Comparison between FGIM and FLSIM, for Example 5.2.

	Method	Step size α	Number of iteration	CPU time	Real-time	Memory usage
$\hat{X}^{(1)}$	FGIM	10^{-12}	20	14.22 s	11.40 s	2.57 GiB
	FLSIM	0.25	4	116.49 s	107.21 s	15.80 GiB
$\hat{X}^{(2)}$	FGIM	10^{-13}	20	15.99 s	12.97 s	2.84 GiB
	FLSIM	0.25	3	119.12 s	108.52 s	16.01 GiB
$\hat{X}^{(3)}$	FGIM	10^{-14}	20	16.82 s	13.61 s	3.17 GiB
	FLSIM	0.25	3	120.03 s	111.34 s	16.30 GiB
$\hat{X}^{(4)}$	FGIM	10^{-15}	20	18.01 s	16.35 s	4.12 GiB
	FLSIM	0.25	3	121.18 s	112.45 s	16.52 GiB

The following Figure 5 shows the change in the error $\delta^{l}(k)$, when k increases up to $k = 20$.

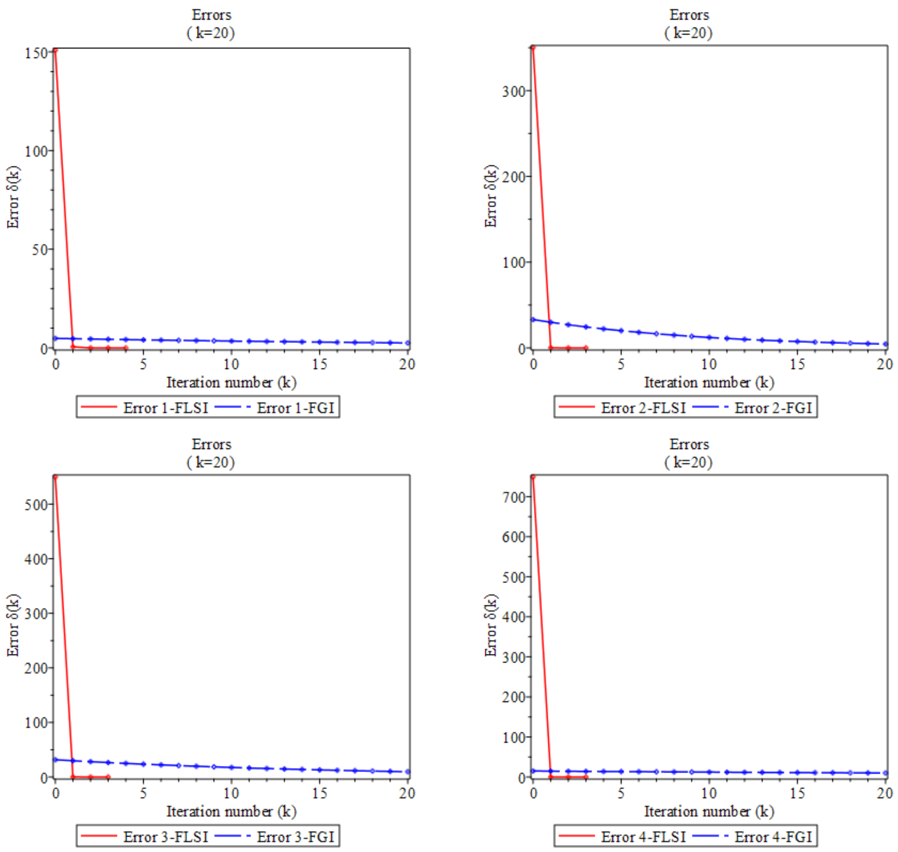

Figure 5: Comparison between $\delta^{l}(k)$ of FGIM and FLSIM for the first 20 iterations.

From Table 3 and Figure 5, it is obvious that FLSIM converges to the solution in 4 steps, and the error is reduced to almost zero. However, FGIM requires very long computational time and memory to converge to the solution.

Remark 5.3. The construction and solution to the positive GTrFFSME in Examples 5.1 and 5.2 are done by Maple 2019.0.

Comparison between the methods for solving positive GTrFFSME

In the following Table 4, a complete comparison between the advantages and disadvantages of FMVM, FGIM, and FLSIM are discussed.

Table 4: Comparison between the advantages and disadvantages of FMVM, FGIM and FLSIM.

Method	Advantages	Disadvantages
FMVM	An exact fuzzy solution can be found. Does not require initial values.	It required getting the inverse of $mn \times mn$ matrices for a system of size $m \times n$ and therefore limited to small systems.
FGIM	Gives an accurate fuzzy approximation. It can be applied to large GTrFFSME. It takes any initial value.	It is limited to GTrFFSME with square coefficients. The convergence rate is very small ($\alpha < 10^{-5}$) which means it takes many iterations to give the desired fuzzy solution.
FLSIM	It is applied to large GTrFFSME. Takes any initial value. Gives an accurate fuzzy approximation. It can be applied to systems with square and non-square coefficients.	The convergence rate is big ($\alpha > 10^{-1}$) compared to the FGIM. It requires getting the inverse of the least square term, which means it takes longer computational time and memory usage compared to the FGIM

6. Conclusion

In this paper, three different methods are proposed for solving positive GTrFFSME and its special cases. The FMVM aims to find the analytical positive fuzzy solution to the positive GTrFFSME with square and non-square coefficients. However, it is limited for small-sized systems, while FGIM and FLSIM aim to approximate the positive fuzzy solution numerically for large GTrFFSME. The numerical examples analysis indicates that the iterative fuzzy solutions obtained by both FGIM and FLSIM algorithms converge to the exact fuzzy solution for any initial value and any size of the matrix system (up to 100×100). However, FLSIM requires getting the inverse of the least square term, which takes longer computational time and memory usage than the FGIM. The major differences between our methods from other methods are following.

1. For the first time, unified analytical and numerical methods are developed for solving a family of large fully fuzzy systems with TrFNs and TFNs, based on new reduced arithmetic fuzzy multiplication operations.
2. The necessary and sufficient theorems for the GTrFFSME to have a unique positive fuzzy solution are checked before applying the proposed methods.
3. The feasibility conditions to have strong positive fuzzy solutions are derived.
4. The obtained positive fuzzy solution analyses are presented, including verification and graphical representation of the obtained fuzzy solution and the feasibility conditions.

For future works, the proposed methods will be applied to GTrFFSME with arbitrary coefficients.

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