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Ostrowski like inequalities for $(\alpha,\beta,\gamma,\delta)$-convex functions via fuzzy Riemann integrals

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Abstract

In this paper, we present first time the generalised notion of $(\alpha,\beta,\gamma,\delta)$-convex (concave) functions in mixed kind, which is the generalisation of functions: convex (concave), P-convex (concave), quasi-convex (concave), s-convex (concave) in 1^{st} kind, s-convex (concave) in 2^{nd} kind, (s,r)-convex (concave) in mixed kind, (α,β)-convex (concave) in 1^{st} kind, (α,β)-convex (concave) in 2^{nd} kind. Our aim is to establish Ostrowski like inequalities via fuzzy Riemann integrals for $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind by applying several techniques involving power mean inequality and Hölder's inequality. Moreover, we would obtain various consequences with respect to the convexity of function as corollaries and remarks.

Keywords: Ostrowski inequality, convex functions, fuzzy set, power mean inequality, Hölder's inequality.

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1. Introduction

To generalise the Ostrowski inequality, we require to generalise the concept of convex functions, by using this way we may easily see generalisations and its particular cases. From the history, we recall few definitions for several convex functions [2]. For more study about convex functions see [11, 12, 14–17].

Definition 1.1. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is called convex (concave), if

$$h(\zeta y+(1-\zeta)z) \leqslant (\geqslant) \zeta h(y)+(1-\zeta)h(z),$$

$\forall\, y,z \in K, \zeta \in [0,1]$.

Here we have P-convex (concave) function which is extracted from [4].

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Definition 1.2. Any function $h : K \subseteq \mathbb{R} \to \mathbb{R}$ is called P-convex, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) h(y) + h(z), \quad h \geqslant 0,$$

$\forall\, y, z \in K, \zeta \in [0, 1]$.

We present definition of quasi-convex (concave) function from [8].

Definition 1.3. Any function $h : K \subseteq \mathbb{R} \to \mathbb{R}$ is called quasi-convex, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) \max\{h(y), h(z)\},$$

$\forall\, y, z \in K, \zeta \in [0, 1]$.

Here we provide definition of s-convex (concave) functions in the 1^{st} kind as follows (see [19]).

Definition 1.4. Suppose $s \in (0, 1]$. Any function $h : K \subseteq [0, \infty) \to [0, \infty)$ is known as s-convex (concave) in the 1^{st} kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) \zeta^s h(y) + (1-\zeta^s) h(z),$$

$\forall\, y, z \in K, \zeta \in [0, 1]$.

Remark 1.5. In above definition we also include $s = 0$. Moreover if $s = 0$, we acquire quasi-convexity (see Definition 1.3).

For 2^{nd} kind convexity we have following definition from [19].

Definition 1.6. Suppose $s \in (0, 1]$. Any function $h : K \subseteq [0, \infty) \to [0, \infty)$ is known as s-convex (concave) in the 2^{nd} kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) \zeta^s h(y) + (1-\zeta)^s h(z),$$

$\forall\, y, z \in K, \zeta \in [0, 1]$.

Remark 1.7. In the same way, we have slightly improved definition of 2^{nd} kind convexity by including $s = 0$. Moreover if $s = 0$, we easily acquire P-convexity (see Definition 1.2).

The following definition of m-convex (concave) function is extracted from [9].

Definition 1.8. Suppose $m \in [0, 1]$. Any function $h : [0, \infty) \to \mathbb{R}$ is known as m-convex (concave), if

$$h(\zeta y + m(1-\zeta)z) \leqslant (\geqslant) \zeta h(y) + m(1-\zeta) h(z),$$

$\forall\, y, z \in [0, \infty)$, $\zeta \in [0, 1]$.

Remark 1.9. For $m = 1$ the above definition captures the concept of standard convex (concave) functions in the interval K and for $m = 0$ the concept of star-shaped functions.

A new class of (s, r)-convex (concave) functions in the mixed kind is extracted from [6] as following.

Definition 1.10. Suppose $(s, r) \in [0, 1]^2$. Any function $h : K \subseteq [0, \infty) \to [0, \infty)$ is called (s, r)-convex (concave) in the mixed kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) \zeta^{rs} h(y) + (1-\zeta^r)^s h(z),$$

$\forall\, y, z \in K, \zeta \in [0, 1]$.

Definition 1.11 ([5]). Suppose $(\alpha, \beta) \in [0,1]^2$. Any function $h : K \subseteq [0,\infty) \to [0,\infty)$ is called (α,β)-convex (concave) in the 1st kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) \zeta^{\alpha} h(y) + (1-\zeta^{\beta}) h(z),$$

$\forall\, y, z \in K, \zeta \in [0,1]$.

Definition 1.12 ([5]). Suppose $(\alpha, \beta) \in [0,1]^2$. Any function $h : K \subseteq [0,\infty) \to [0,\infty)$ is called (α,β)-convex (concave) in the 2nd kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) \zeta^{\alpha} h(y) + (1-\zeta)^{\beta} h(z),$$

$\forall\, y, z \in K, \zeta \in [0,1]$.

Tenthly and finally we initiate a latest class of functions, which will be known as class of $(\alpha,\beta,\gamma,\delta)$-convex (concave) functions in mixed kind and containing all above classes of functions. This definition is used sequentially in this paper.

Definition 1.13. Suppose $(\alpha,\beta,\gamma,\delta) \in [0,1]^4$. Any function $h : K \subseteq [0,\infty) \to [0,\infty)$ is called $(\alpha,\beta,\gamma,\delta)$-convex (concave) in the mixed kind, if

$$h(\zeta y + (1-\zeta)z) \leqslant (\geqslant) \zeta^{\alpha\gamma} h(y) + (1-\zeta^{\beta\gamma})^{\delta} h(z), \tag{1.1}$$

$\forall\, y, z \in K, \zeta \in [0,1]$.

Remark 1.14. In Definition 1.13, we have the cases below.

(i) If $\beta = \gamma = 1$ and $\delta = \beta$ in (1.1), we acquire (α,β)-convex (concave) function in the 2nd kind.
(ii) If $\delta = \gamma = 1$ in (1.1), we acquire (α,β)-convex (concave) function in the 1st kind.
(iii) If $\gamma = r$, $\alpha = \delta = s$ and $\beta = 1$ in (1.1), where $r, s \in [0,1]$, we acquire (s,r)-convex (concave) function in the mixed kind.
(iv) If $\delta = \alpha = s$ and $\gamma = \beta = 1$ in (1.1), where $s \in [0,1]$, we acquire s-convex (concave) function in the 2nd kind.
(v) If $\beta = \alpha = s$ and $\delta = \gamma = 1$ in (1.1), where $s \in [0,1]$, we acquire s-convex (concave) function in the 1st kind.
(vi) If $\gamma = s$ and $\alpha = \beta = \delta = 1$ in (1.1), where $s \in [0,1]$, we acquire s-convex (concave) function in the 1st kind.
(vii) If $\beta = \delta = 0$, and $\delta = \gamma = 1$ in (1.1), we acquire quasi-convex (concave) function.
(viii) If $\delta = \alpha = 0$ and $\gamma = \beta = 1$ in (1.1), we acquire P-convex (concave) function.
(ix) If $\delta = \gamma = \beta = \alpha = 1$ in (1.1), we acquire ordinary convex (concave) function.

At this stage, our main focus is on Ostrowski's like inequalities, which was acquired by Ostrowski in 1938 (see [20]), it is an important inequality for convex functions, which has been extensively studied in recent decades and stated as follows.

Proposition 1.15. *Suppose* $h : K \to \mathbb{R}$ *is a differentiable mapping in the interior* K^o *of* K, *where* $j, k \in K^o$ *with* $j < k$. *If* $|h'(y)| \leqslant \mathfrak{M}$, $\forall\, y \in [j,k]$, *where* $\mathfrak{M} > 0$ *is constant, then*

$$\left| h(y) - \frac{1}{k-j}\int_j^k h(\tau)\,d\tau \right| \leqslant \mathfrak{M}(k-j)\left[\frac{1}{4} + \frac{\left(y - \frac{j+k}{2}\right)^2}{(k-j)^2}\right].$$

The value $\frac{1}{4}$ *is the best possible constant that can not be replaced by the smallest one.*

Anastassiou extends Ostrowski like inequalities into the fuzzy setting in 2003 [1]. The concepts of fuzzy Riemann integrals were initiated by Congxin and Ming [3].

2. Preliminaries with notations

Under this heading, we remind few basic definitions and notations that would help us in the sequel manner.

Definition 2.1 ([3]). $\rho : \mathbb{R} \to [0,1]$ is known as fuzzy number if satisfies the following properties.

1. ρ is normal (i.e, $\exists$ a $y_0 \in \mathbb{R}$ such that $\rho(y_0) = 1$).
2. ρ is a convex fuzzy set, i.e., $y\zeta + (1-\zeta)z) \geqslant \min\{\rho(y), \rho(z)\}$, $\forall\, y, z \in \mathbb{R}$, $\zeta \in [0,1]$ (ρ is known as a convex fuzzy subset)
3. ρ is upper semi continuous in $\mathbb{R}$, i.e., $\forall\, y_0 \in \mathbb{R}$ and $\forall\, \epsilon > 0$, $\exists$ neighborhood $V(y_0) : \rho(y) \leqslant \rho(y_0) + \epsilon$, $\forall\, y \in V(y_0)$.
4. The set $[\rho]^0 = \overline{\{y \in \mathbb{R} : \rho(y) > 0\}}$ is compact, where $\overline{A}$ represents the closure of A.

$\mathbb{R}^F$ denotes the set of all fuzzy numbers. For $\alpha \in (0,1]$ and $\rho \in \mathbb{R}^F$, $[\rho]^\alpha = \{y \in \mathbb{R} : \rho(y) \geqslant \alpha\}$. Then, from (1) to (4) it follows that the α-level set $[\rho]^\alpha$ is a closed interval, $\forall\, \alpha \in [0,1]$. Moreover, $[\rho]^\alpha = [\rho_-^{(\alpha)}, \rho_+^{(\alpha)}]$, $\forall\, \alpha \in [0,1]$, where $\rho_-^{(\alpha)} \leqslant \rho_+^{(\alpha)}$ and $\rho_-^{(\alpha)}, \rho_+^{(\alpha)} \in \mathbb{R}$, i.e., $\rho_-^{(\alpha)}$ and $\rho_+^{(\alpha)}$ are the endpoints of $[\rho]^\alpha$.

Definition 2.2 ([22]). Let $\rho, \rho \in \mathbb{R}^F$ and $a \in \mathbb{R}$. Then, the addition and scalar multiplication are defined respectively by following equations.

1. $[\rho \oplus \rho]^\alpha = [\rho]^\alpha + [\rho]^\alpha$;
2. $[a \odot \rho]^\alpha = a[\rho]^\alpha$,

$\forall\, \alpha \in [0,1]$, where $[\rho]^\alpha + [\rho]^\alpha$, i.e., common addition of two intervals (as subsets of $\mathbb{R}$) and $a[\rho]^\alpha$, i.e., common usual product between scalar and subset of $\mathbb{R}$.

Proposition 2.3 ([10]). *Let $\rho, \rho \in \mathbb{R}^F$ and $a \in \mathbb{R}$. Then the given properties holds:*

1. $1 \odot \rho = \rho$;
2. $\rho \oplus \rho = \rho \oplus \rho$;
3. $a \odot \rho = \rho \odot a$;
4. $[\rho]^{\alpha_1} \subseteq [\rho]^{\alpha_2}$, *whenever* $0 \leqslant \alpha_2 \leqslant \alpha_1 \leqslant 1$;
5. *for any α_n converging increasingly to* $\alpha \in (0,1]$, $\bigcap_{n=1}^{\infty} [\rho]^{\alpha_n} = [\rho]^\alpha$.

Definition 2.4 ([3]). Suppose $D : \mathbb{R}^F \times \mathbb{R}^F \to \mathbb{R}_+ \cup \{0\}$ is a function, stated as

$$D(\rho, \rho) = \sup_{\alpha \in [0,1]} \max \left\{ \left| \rho_-^{(\alpha)}, \rho_-^{(\alpha)} \right|, \left| \rho_+^{(\alpha)}, \rho_+^{(\alpha)} \right| \right\}$$

for every $\rho, \rho \in \mathbb{R}^F$, then D is metric on $\mathbb{R}^F$.

Proposition 2.5 ([3]). *Let $\rho, \rho, \sigma, e \in \mathbb{R}^F$ and $a \in \mathbb{R}$, we have*

1. $(\mathbb{R}^F, D)$ *is a complete metric space;*
2. $D(\rho \oplus \sigma, \rho \oplus \sigma) = D(\rho, \rho)$;
3. $D(a \odot \rho, a \odot \rho) = |a| D(\rho, \rho)$;
4. $D(\rho \oplus \rho, \sigma \oplus e) = D(\rho, \sigma) + D(\rho, e)$;
5. $D(\rho \oplus \rho, \widetilde{0}) \leqslant D(\rho, \widetilde{0}) + D(\rho, \widetilde{0})$;
6. $D(\rho \oplus \rho, \sigma) \leqslant D(\rho, \sigma) + D(\rho, \widetilde{0})$,

where $\widetilde{0} \in \mathbb{R}^F$ is stated as $\widetilde{0}(y) = 0$, $\forall\, y \in \mathbb{R}$.

Definition 2.6 ([22]). Let $y, z \in \mathbb{R}^F$ if $\exists\, \theta \in \mathbb{R}^F$ such that $y = z \oplus \theta$, then θ is H-difference of y and z represented by $\theta = y \ominus z$.

Definition 2.7 ([22]). Let $T := [y_0, y_0+\gamma] \subseteq \mathbb{R}$, where $\gamma > 0$. A function $h : T \to \mathbb{R}^F$ is H-differentiable at $y \in T$ if $\exists\, h'(y) \in \mathbb{R}^F$, i.e., both limits (w.r.t the metric D)

$$\lim_{g\to 0^+} \frac{h(y+g)\ominus h(y)}{g}, \quad \lim_{g\to 0^+} \frac{h(y)\ominus h(y-g)}{g}$$

exist and equal to $h'(y)$. We say H-derivative of h or h' the derivative at y. If h is H-differentiable at each $y \in T$, we say H-differentiable or h differentiable and it has H-derivetive over T the function h'.

Definition 2.8 ([7]). Let $h : [j,k] \to \mathbb{R}^F$, if for each $0 < \xi$, $\exists\ 0 < \eta$, for any partition $P = \{[\rho,\rho];\vartheta\}$ of $[j,k]$ with norm $\Delta(P) < \eta$, we have

$$D\left(\sum_{P}^{*}(\rho-\rho)\odot h(\vartheta, K)\right) < \xi,$$

then we call h is fuzzy-Riemann integrable to the interval $K \in \mathbb{R}^F$, we write it as

$$K := (FR)\int_j^k h(y)dy.$$

For few current consequences linked with fuzzy-Riemann integrals see [13].

The purpose of paper is to derive fuzzy Ostrowski like inequalities for $(\alpha,\beta,\gamma,\delta)$-convex function in mixed kind and we obtain various results w.r.t the convexity of function as corollaries and remarks.

3. Fuzzy Ostrowski like inequalities for $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind

We require the Lemma below for the proof of our main results.

Lemma 3.1. *Suppose* $h : K \subset \mathbb{R} \to \mathbb{R}^F$ *is differentiable mapping on* K^o, *where* $j,k \in K$ *with* $j < k$. *If* $h' \in C^F[j,k]\bigcap L^F[j,k]$, *then*

$$\begin{aligned}\frac{1}{k-j}\odot(FR)\int_j^k h(u)du \oplus \frac{(y-j)^2}{k-j}\odot(FR)\int_0^1 \zeta\odot h'(\zeta y+(1-\zeta)j)d\zeta \\ = h(y)\oplus\frac{(k-y)^2}{k-j}\odot(FR)\int_0^1 \zeta\odot h'(\zeta y+(1-\zeta)k)d\zeta,\end{aligned}$$

$\forall\, y \in (j,k)$.

Proof. We derive the required result by using similar techniques as in the proof of Lemma 3.1 of [21]. □

Theorem 3.2. *Under all assumptions of Lemma 3.1 and assuming that* $D(h'(y),\widetilde{0})$ *is an* $(\alpha,\beta,\gamma,\delta)$*-convex function on* $[j,k]$ *and* $D(h'(y),\widetilde{0}) \leqslant M$, *then*

$$D\left(h(y), \frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right) \leqslant M\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)I(y), \tag{3.1}$$

$\forall\, y \in (j,k)$ *and* $\beta\gamma > 0$, *where* $I(y) = \frac{(y-j)^2+(k-y)^2}{k-j}$.

Proof. From Lemma 3.1 and using Proposition 2.5, then we have

$$D\left(h(y), \frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)$$

$$\leqslant D\left(\frac{(y-j)^2}{k-j}\odot(FR)\int_0^1 \zeta h'(\zeta y+(1-\zeta)j)d\zeta,\widetilde{0}\right)+D\left(\frac{(k-y)^2}{k-j}\odot(FR)\int_0^1 \zeta h'(\zeta y+(1-\zeta)k)d\zeta,\widetilde{0}\right) \quad (3.2)$$
$$=\frac{(y-j)^2}{k-j}D\left((FR)\int_0^1 \zeta h'(\zeta y+(1-\zeta)j)d\zeta,\widetilde{0}\right)+\frac{(k-y)^2}{k-j}D\left((FR)\int_0^1 \zeta h'(\zeta y+(1-\zeta)k)d\zeta,\widetilde{0}\right)$$
$$\leqslant\frac{(y-j)^2}{k-j}\int_0^1 \zeta D\left(h'(\zeta y+(1-\zeta)j),\widetilde{0}\right)d\zeta+\frac{(k-y)^2}{k-j}\int_0^1 \zeta D\left(h'(\zeta y+(1-\zeta)k),\widetilde{0}\right)d\zeta.$$

As we know that $D(h'(y),\widetilde{0})$ is an $(\alpha,\beta,\gamma,\delta)$-convex functon and $D(h'(y),\widetilde{0})\leqslant M$, we have

$$D\left(h'(\zeta y+(1-\zeta)j),\widetilde{0}\right)\leqslant \zeta^{\alpha\gamma}D\left(h'(y),\widetilde{0}\right)+\left(1-\zeta^{\beta\gamma}\right)^{\delta}D\left(h'(j),\widetilde{0}\right)\leqslant M\left[\zeta^{\alpha\gamma}+\left(1-\zeta^{\beta\gamma}\right)^{\delta}\right], \quad (3.3)$$
$$D\left(h'(\zeta y+(1-\zeta)k),\widetilde{0}\right)\leqslant \zeta^{\alpha\gamma}D\left(h'(y),\widetilde{0}\right)+\left(1-\zeta^{\beta\gamma}\right)^{\delta}D\left(h'(k),\widetilde{0}\right)\leqslant M\left[\zeta^{\alpha\gamma}+\left(1-\zeta^{\beta\gamma}\right)^{\delta}\right]. \quad (3.4)$$

Now using (3.3) and (3.4) in (3.2) we get (3.1). □

Note that in above B is Beta function and it is stated as $B(l,m)=\int_0^1 \zeta^{l-1}(1-\zeta)^{m-1}d\zeta=\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$, since $\Gamma(l)=\int_0^\infty e^{-u}u^{l-1}du$.

Corollary 3.3. *In Theorem 3.2, we have cases below.*

1. *If* $\gamma=\beta=1$ *and* $\delta=\beta$, $\delta\in[0,1]$, *and* $\alpha\in[0,1]$ *in* (3.1), *we acquire the fuzzy Ostrowski inequality for* (α,β)*-convex functions in* 2^{nd} *kind:*
$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant M\left(\frac{1}{2+\alpha}+\frac{1}{(\beta+1)(\beta+2)}\right)I(y).$$
2. *If* $\delta=\gamma=1$, $\beta\in[0,1]$ *and* $\alpha\in[0,1]$, *in* (3.1), *we acquire the fuzzy Ostrowski inequality for* (α,β)*-convex functions in* 1^{st} *kind:*
$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant M\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)I(y).$$
3. *If* $\gamma=r$, $\delta=\alpha=s$ *and* $\beta=1$, *where* $r\in(0,1]$ *and* $s\in[0,1]$ *in* (3.1), *we acquire the fuzzy Ostrowski inequality for* (s,r)*-convex functions in mixed kind:*
$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant M\left(\frac{1}{sr+2}+\frac{B\left(\frac{2}{r},1+s\right)}{r}\right)I(y).$$
4. *If* $\alpha=\delta=s$ *and* $\beta=\gamma=1$, *where* $s\in[0,1]$, *in* (3.1), *we acquire the fuzzy Ostrowski inequality for s-convex functions in* 2^{nd} *kind:*
$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant M\left(\frac{1}{s+1}\right)I(y).$$
5. *If* $\beta=\alpha=s$ *and* $\delta=\gamma=1$, *where* $s\in(0,1]$ *in* (3.1), *we acquire the fuzzy Ostrowski inequality for s-convex functions in* 1^{st} *kind:*
$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant M\left(\frac{1}{s+2}+\frac{B\left(\frac{2}{s},2\right)}{s}\right)I(y).$$

6. *If $\delta = \alpha = 0$ and $\gamma = \beta = 1$ in (3.1), we acquire the fuzzy Ostrowski inequality for* P*-convex functions:*
$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant MI(y).$$
7. *If $\delta = \gamma = \beta = \alpha = 1$, in (3.1), we acquire the fuzzy Ostrowski inequality for convex functions:*
$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M}{2}I(y).$$

Theorem 3.4. *Under all assumptions of Lemma 3.1 and assuming that $[D(h'(y),\widetilde{0})]^q$ is $(\alpha,\beta,\gamma,\delta)$-convex function on $[j,k]$, $1 \leqslant q$ and $D(h'(y),\widetilde{0}) \leqslant M$, then the following inequality holds:*
$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2} + \frac{B\left(\frac{2}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}} I(y), \tag{3.5}$$
$\forall\, y \in (j,k)$ and $\beta\gamma > 0$.

Proof. From (3.2) and applying power mean inequality, we have
$$\begin{aligned}
&D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right)\\
&\leqslant \frac{(y-j)^2}{k-j}\int_0^1 \zeta D\left(h'(\zeta y+(1-\zeta)j),\widetilde{0}\right)d\zeta + \frac{(k-y)^2}{k-j}\int_0^1 \zeta D\left(h'(\zeta y+(1-\zeta)k),\widetilde{0}\right)d\zeta\\
&\leqslant \frac{(y-j)^2}{k-j}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 \zeta\left[D\left(h'(\zeta y+(1-\zeta)j),\widetilde{0}\right)\right]^q d\zeta\right)^{\frac{1}{q}}\\
&\quad + \frac{(k-y)^2}{k-j}\left(\int_0^1 \zeta d\zeta\right)^{1-\frac{1}{q}}\left(\int_0^1 \zeta\left[D\left(h'(\zeta y+(1-\zeta)k),\widetilde{0}\right)\right]^q d\zeta\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.6}$$
As we know that $[D(h'(y),\widetilde{0})]^q$ is the $(\alpha,\beta,\gamma,\delta)$-convex function and $D(h'(y),\widetilde{0}) \leqslant M$, we have
$$\begin{aligned}
\left[D\left(h'(\zeta y+(1-\zeta)j),\widetilde{0}\right)\right]^q &\leqslant \zeta^{\alpha\gamma}\left[D\left(h'(y),\widetilde{0}\right)\right]^q + \left(1-\zeta^{\beta\gamma}\right)^\delta\left[D\left(h'(j),\widetilde{0}\right)\right]^q\\
&\leqslant M^q\left[\zeta^{\alpha\gamma} + \left(1-\zeta^{\beta\gamma}\right)^\delta\right],
\end{aligned} \tag{3.7}$$
$$\begin{aligned}
\left[D\left(h'(\zeta y+(1-\zeta)k),\widetilde{0}\right)\right]^q &\leqslant \zeta^{\alpha\gamma}\left[D\left(h'(y),\widetilde{0}\right)\right]^q + \left(1-\zeta^{\beta\gamma}\right)^\delta\left[D\left(h'(k),\widetilde{0}\right)\right]^q\\
&\leqslant M^q\left[\zeta^{\alpha\gamma} + \left(1-\zeta^{\beta\gamma}\right)^\delta\right].
\end{aligned} \tag{3.8}$$
Now applying (3.7) and (3.8) in (3.6) we get required theorem. □

Corollary 3.5. *In Theorem 3.4, we have the following cases.*

1. *If $q = 1$, we acquire Theorem 3.2.*
2. *If $\gamma = \beta = 1$ and $\delta = \beta$, $\delta \in [0,1]$ and $\alpha \in [0,1]$, in (3.5), we acquire the fuzzy Ostrowski inequality for (α,β)-convex functions in 2^{nd} kind:*
$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{(2+\beta)} + \frac{1}{(1+\beta)(2+\beta)}\right)^{\frac{1}{q}} I(y).$$

3. *If* $\delta=\gamma=1$, $\beta\in(0,1]$, *and* $\alpha\in[0,1]$, *in* (3.5), *we acquire the fuzzy Ostrowski inequality for* (α,β)*-convex functions in* 1^{st} *kind:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{\alpha+2}+\frac{B\left(\frac{2}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}I(y).$$

4. *If* $\gamma=r$, $\delta=\alpha=s$ *and* $\beta=1$, *where* $r\in(0,1]$ *and* $s\in[0,1]$ *in* (3.5), *we acquire the fuzzy Ostrowski inequality for* (s,r)*-convex functions in mixed kind:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{sr+2}+\frac{B\left(\frac{2}{r},1+s\right)}{r}\right)^{\frac{1}{q}}I(y).$$

5. *If* $\alpha=\delta=s$ *and* $\beta=\gamma=1$, *where* $s\in[0,1]$, *in* (3.5), *we acquire the fuzzy Ostrowski inequality for* s*-convex functions in* 2^{nd} *kind:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}I(y).$$

6. *If* $\alpha=\beta=s$ *and* $\gamma=\delta=1$, *where* $s\in(0,1]$ *in* (3.5), *we acquire the fuzzy Ostrowski inequality for* s*-convex functions in* 1^{st} *kind:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{1}{s+2}+\frac{B\left(\frac{2}{s},2\right)}{s}\right)^{\frac{1}{q}}I(y).$$

7. *If* $\delta=\alpha=0$ *and* $\gamma=\beta=1$ *in* (3.5), *we acquire the fuzzy Ostrowski inequality for* P*-convex functions:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{(2)^{1-\frac{1}{q}}}I(y).$$

8. *If* $\delta=\gamma=\beta=\alpha=1$, *in* (3.5), *we acquire the fuzzy Ostrowski inequality for convex functions:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{2}I(y).$$

Remark 3.6. In Theorem 3.4, we have the following cases.

1. If $y=\frac{j+k}{2}$ in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for $(\alpha,\beta,\gamma,\delta)$- convex functions in mixed kind:

$$D\left(h(\frac{j+k}{2}),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M(k-j)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{\alpha\gamma+2}+\frac{B\left(\frac{2}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}}.$$

2. If $y=\frac{j+k}{2}$, $\gamma=\beta=1$ and $\delta=\beta$, $\delta\in[0,1]$ and $\alpha\in[0,1]$ in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for (α,β)-convex functions in 2^{nd} kind:

$$D\left(h(\frac{j+k}{2}),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M(k-j)}{(2)^{2-\frac{1}{q}}}\left(\frac{1}{(\alpha+2)}+\frac{1}{(\beta+1)(\beta+2)}\right)^{\frac{1}{q}}.$$

3. If $y = \frac{j+k}{2}$, $\delta = \gamma = 1$, $\beta \in (0,1]$ and $\alpha \in [0,1]$ in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for (α, β)-convex functions in 1st kind:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR) \int_j^k h(u)du\right) \leqslant \frac{M\,(\rho_b - \rho_a)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{\alpha+2} + \frac{B\left(\frac{2}{\beta}, 2\right)}{\beta}\right)^{\frac{1}{q}}.$$

4. If $y = \frac{j+k}{2}$, $\gamma = r$, $\delta = \alpha = s$, and $\beta = 1$, where $r \in (0,1]$ and $s \in [0,1]$ in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for (s, r)-convex functions in mixed kind:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR) \int_j^k h(u)du\right) \leqslant \frac{M\,(k-j)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{rs+2} + \frac{B\left(\frac{2}{r}, s+1\right)}{r}\right)^{\frac{1}{q}}.$$

5. If $y = \frac{j+k}{2}$, $\alpha = \delta = s$, $\beta = \gamma = 1$, where $s \in [0,1]$, in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for s-convex functions in 2nd kind:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR) \int_j^k h(u)du\right) \leqslant \frac{M\,(k-j)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}}.$$

6. If $y = \frac{j+k}{2}$, $\alpha = \beta = s$ and $\gamma = \delta = 1$, where $s \in (0,1]$ in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for s-convex functions in 1st kind:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR) \int_j^k h(u)du\right) \leqslant \frac{M\,(k-j)}{(2)^{2-\frac{1}{q}}} \left(\frac{1}{s+2} + \frac{B\left(\frac{2}{s}, 2\right)}{s}\right)^{\frac{1}{q}}.$$

7. If $y = \frac{j+k}{2}$, $\delta = \alpha = 0$ and $\gamma = \beta = 1$ in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for P-convex functions:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR) \int_j^k h(u)du\right) \leqslant \frac{M\,(k-j)}{(2)^{2-\frac{1}{q}}}.$$

8. If $y = \frac{j+k}{2}$, $\alpha = \beta = \gamma = \delta = 1$, in (3.5), we acquire the fuzzy Ostrowski mid-point inequality for convex functions:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR) \int_j^k h(u)du\right) \leqslant \frac{M\,(k-j)}{4}.$$

Theorem 3.7. *Under all assumptions of Lemma 3.1 and assuming that $[D(h'(y), \widetilde{0})]^q$ is an $(\alpha, \beta, \gamma, \delta)$-convex function on $[j,k]$, $1 < p, q$ and $D(h'(y), \widetilde{0}) \leqslant M$, then following inequality holds:*

$$D\left(h(y), \frac{1}{k-j} \odot (FR) \int_j^k h(u)du\right) \leqslant \frac{M}{(p+1)^{\frac{1}{p}}} \left(\frac{1}{1+\alpha\gamma} + \frac{B\left(\frac{1}{\beta\gamma}, \delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}} I(y), \tag{3.9}$$

$\forall\, y \in (j,k)$ and $\beta\gamma > 0$, where $q^{-1} + p^{-1} = 1$.

Proof. From (3.2) and by inequality of Hölder, we have

$$\begin{aligned}
&D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \\
&\leqslant \frac{(y-j)^2}{k-j}\int_0^1 \zeta D\left(h'(\zeta y+(1-\zeta)j), \widetilde{0}\right)d\zeta + \frac{(k-y)^2}{k-j}\int_0^1 \zeta D\left(h'(\zeta y+(1-\zeta)k), \widetilde{0}\right)d\zeta \\
&\leqslant \frac{(y-j)^2}{k-j}\left(\int_0^1 \zeta^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 \left[D\left(h'(\zeta y+(1-\zeta)j), \widetilde{0}\right)\right]^q d\zeta\right)^{\frac{1}{q}} \\
&\quad + \frac{(k-y)^2}{k-j}\left(\int_0^1 \zeta^p d\zeta\right)^{\frac{1}{p}}\left(\int_0^1 \left[D\left(h'(\zeta y+(1-\zeta)k), \widetilde{0}\right)\right]^q d\zeta\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.10}$$

As we know that $[D(h'(y), \widetilde{0})]^q$ is an $(\alpha, \beta, \gamma, \delta)$-convex function and $D(h'(y), \widetilde{0}) \leqslant M$, we have

$$\begin{aligned}
\left[D\left(h'(\zeta y+(1-\zeta)j), \widetilde{0}\right)\right]^q &\leqslant \zeta^{\alpha\gamma}\left[D\left(h'(y), \widetilde{0}\right)\right]^q + (1-\zeta^{\beta\gamma})^{\delta}\left[D\left(h'(j), \widetilde{0}\right)\right]^q \\
&\leqslant M^q\left[\zeta^{\alpha\gamma} + (1-\zeta^{\beta\gamma})^{\delta}\right],
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\left[D\left(h'(\zeta y+(1-\zeta)k), \widetilde{0}\right)\right]^q &\leqslant \zeta^{\alpha\gamma}\left[D\left(h'(y), \widetilde{0}\right)\right]^q + (1-\zeta^{\beta\gamma})^{\delta}\left[D\left(h'(k), \widetilde{0}\right)\right]^q \\
&\leqslant M^q\left[\zeta^{\alpha\gamma} + (1-\zeta^{\beta\gamma})^{\delta}\right].
\end{aligned} \tag{3.12}$$

Now applying (3.11) and (3.12) in (3.10) we get (3.9). □

Corollary 3.8. *In Theorem 3.7, we have the following cases.*

1. *If* $\gamma = \beta = 1$ *and* $\delta = \beta$, $\delta \in [0, 1]$ *and* $\alpha \in [0, 1]$, *in (3.9), we acquire the fuzzy Ostrowski inequality for* (α, β)*-convex functions in* 2^{nd} *kind:*

$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M}{(1+p)^{\frac{1}{p}}}\left(\frac{1}{1+\alpha} + \frac{1}{1+\beta}\right)^{\frac{1}{q}} I(y).$$

2. *If* $\delta = \gamma = 1$, $\beta \in (0, 1]$ *and* $\alpha \in [0, 1]$, *in (3.9), we acquire the fuzzy Ostrowski inequality for* (α, β)*-convex functions in* 1^{st} *kind:*

$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1} + \frac{B\left(\frac{1}{\beta}, 2\right)}{\beta}\right)^{\frac{1}{q}} I(y).$$

3. *If* $\gamma = r$, $\delta = \alpha = s$ *and* $\beta = 1$, *where* $r \in (0, 1]$ *and* $s \in [0, 1]$ *in (3.9), we acquire the fuzzy Ostrowski inequality for* (s, r)*-convex functions in mixed kind:*

$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1+sr} + \frac{B\left(\frac{1}{r}, 1+s\right)}{r}\right)^{\frac{1}{q}} I(y).$$

4. *If* $\alpha = \delta = s$ *and* $\gamma = \beta = 1$, *where* $s \in [0, 1]$ *in (3.9), we acquire the fuzzy Ostrowski inequality for s-convex functions in* 2^{nd} *kind:*

$$D\left(h(y), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}} I(y).$$

5. *If $\beta=\alpha=s$ and $\delta=\gamma=1$, where $s\in(0,1]$ in (3.9), we acquire the fuzzy Ostrowski inequality for s-convex functions in 1^{st} kind:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{1+s}+\frac{B\left(\frac{1}{s},2\right)}{s}\right)^{\frac{1}{q}}I(y).$$

6. *If $\delta=\alpha=0$ and $\gamma=\beta=1$ in (3.9), we acquire the fuzzy Ostrowski inequality for P-convex functions:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{(2)^{\frac{1}{q}}M}{(p+1)^{\frac{1}{p}}}I(y).$$

7. *If $\alpha=\beta=\gamma=\delta=1$ in (3.9), we acquire the fuzzy Ostrowski inequality for convex functions:*

$$D\left(h(y),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M}{(p+1)^{\frac{1}{p}}}I(y).$$

Remark 3.9. In Theorem 3.7, we have the following cases.

1. If $y=\frac{j+k}{2}$ in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for $(\alpha,\beta,\gamma,\delta)$-convex functions in mixed kind:

$$D\left(h(\frac{j+k}{2}),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M(k-j)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha\gamma+1}+\frac{B\left(\frac{1}{\beta\gamma},\delta+1\right)}{\beta\gamma}\right)^{\frac{1}{q}}.$$

2. If $y=\frac{j+k}{2}$, $\gamma=\beta=1$ and $\delta=\beta$, $\delta\in[0,1]$ and $\alpha\in[0,1]$, in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for (α,β)-convex functions in 2^{nd} kind:

$$D\left(h(\frac{j+k}{2}),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M(k-j)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+\frac{1}{\beta+1}\right)^{\frac{1}{q}}.$$

3. If $y=\frac{j+k}{2}$, $\delta=\gamma=1$, $\beta\in(0,1]$ and $\alpha\in[0,1]$, in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for (α,β)-convex functions in 1^{st} kind:

$$D\left(h(\frac{j+k}{2}),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M(k-j)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{\alpha+1}+\frac{B\left(\frac{1}{\beta},2\right)}{\beta}\right)^{\frac{1}{q}}.$$

4. If $y=\frac{j+k}{2}$, $\gamma=r$, $\delta=\alpha=s$ and $\beta=1$, where $r\in(0,1]$ and $s\in[0,1]$ in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for (s,r)-convex functions in mixed kind:

$$D\left(h(\frac{j+k}{2}),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{M(k-j)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{sr+1}+\frac{B\left(\frac{1}{r},s+1\right)}{r}\right)^{\frac{1}{q}}.$$

5. If $y=\frac{j+k}{2}$, $\alpha=\delta=s$ and $\beta=\gamma=1$, where $s\in[0,1]$ in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for s-convex functions in 2^{nd} kind:

$$D\left(h(\frac{j+k}{2}),\frac{1}{k-j}\odot(FR)\int_j^k h(u)du\right)\leqslant\frac{(2)^{\frac{1}{q}-1}M(k-j)}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}.$$

6. If $y = \frac{j+k}{2}$, $\alpha = \beta = s$ and $\gamma = \delta = 1$, where $s \in (0,1]$ in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for s-convex functions in 1^{st} kind:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M(k-j)}{2(p+1)^{\frac{1}{p}}}\left(\frac{1}{s+1} + \frac{B\left(\frac{1}{s},2\right)}{s}\right)^{\frac{1}{q}}.$$

7. If $y = \frac{j+k}{2}$, $\delta = \alpha = 0$ and $\gamma = \beta = 1$ in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for P-convex functions:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{(2)^{\frac{1}{q}-1}M(k-j)}{(p+1)^{\frac{1}{p}}}.$$

8. If $y = \frac{j+k}{2}$ and $\alpha = \beta = \gamma = \delta = 1$ in (3.9), we acquire the fuzzy Ostrowski mid-point inequality for convex functions:

$$D\left(h(\frac{j+k}{2}), \frac{1}{k-j} \odot (FR)\int_j^k h(u)du\right) \leqslant \frac{M(k-j)}{2(p+1)^{\frac{1}{p}}}.$$

4. Conclusion

We derived the generalised notion of $(\alpha, \beta, \gamma, \delta)$-convex (concave) functions in mixed kind. This class of functions have many important classes involving class of convex (concave), P-convex (concave), quasi-convex (concave), s-convex (concave) in 1^{st} kind, s-convex (concave) in 2^{nd} kind, (s, r)-convex (concave) in mixed kind, (α, β)-convex (concave) in 1^{st} kind, (α, β)-convex (concave) in 2^{nd} kind. We have established our 1^{st} main consequence in Section 3, the generalisation of Ostrowski inequality via fuzzy Riemann integrals for $(\alpha, \beta, \gamma, \delta)$-convex functions in mixed kind. Further, we used several techniques involving power mean inequality and inequality of Hölder for generalisation of fuzzy Ostrowski inequality. We also obtained various results w.r.t the convexity of function as corollaries and remarks.

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