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Decision-making on the solution of non-linear dynamical systems of Kannan non-expansive type in Nakano sequence space of fuzzy numbers

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Abstract

In general, we have constructed the operators ideal generated by extended s-fuzzy numbers and a certain space of sequences of fuzzy numbers. An investigation into the conditions sufficient for Nakano sequence space of fuzzy numbers furnished with the definite function to create pre-quasi Banach and closed is carried out. The (R) and the normal structural properties of this space are shown. Fixed points for Kannan contraction and non-expansive mapping have been introduced. Lastly, we explore whether the Kannan contraction mapping has a fixed point in its associated pre-quasi operator ideal. The existence of solutions to non-linear difference equations is illustrated with a few real-world examples and applications.

Keywords: Pre-quasi norm, hydrodynamics, fuzzy numbers, Nakano sequence space, operator ideal, Kannan non-expansive mappings.

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1. Introduction

After Zadeh [29] established the concept of fuzzy sets and fuzzy set operations, many researchers adopted the concept of fuzziness in cybernetics and artificial intelligence as well as in expert systems and fuzzy control. Fuzzy sequence spaces were introduced, and their various features were studied by many workers on sequence spaces and summability theory. Nuray and Savaş [20] defined and studied the Nakano sequences of fuzzy numbers, $\ell^F(\tau)$ equipped with the function h. Operators ideal are very important in fixed point theory, Banach space geometry, normal series theory, approximation theory, and ideal transformations, see [17, 21, 22]. Pre-quasi operator ideals are more extensive than quasi-operator ideals, according to Faried and Bakery [10]. The learning about the variable exponent Lebesgue spaces obtained impetus from the mathematical description of the hydrodynamics of non-Newtonian fluids (see

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[24, 26]). There are numerous uses for electrorheological fluids, which include military science, civil engineering, and orthopedic. There have been many developments in mathematics since the Banach fixed point theorem [8] was first published. While contractions have fixed point actions, Kannan [13] cited an example of a type of mapping that is not continuous. In Reference [11], the only attempt was made to explain Kannan operators in modular vector spaces. For more details on Kannan's fixed point theorems see [2, 4–7, 25]. Given that the demonstration of many fixed point theorems in a given space entails either expanding the space itself or its self-mapping, both of these alternatives are possible. In this paper, we examined a novel general space known as the Nakano sequence space of fuzzy numbers equipped with various pre-quasi functions and its associated mappings' ideal solutions space for several stochastic non-linear and matrix systems of Kannan type, respectively. These spaces' geometric and topological structures connected to a Kannan-type fixed point are presented. We have addressed the requirements on these spaces essential for these dynamical systems to have unique or many solutions in these spaces. It is the goal of this work to introduce the certain space of sequences of fuzzy numbers, in short (cssf), under a certain function to be pre-quasi (cssf). This space and s-numbers have been used to describe the structure of the ideal operators. An investigation into the conditions necessary to create pre-quasi Banach and closed (cssf) $\left(\ell^F(\tau)\right)_h$ furnished with the definite function h is carried out. The (R) and the normal structure-property of this space are shown. Fixed points for Kannan contraction and non-expansive mapping have been introduced. Lastly, we explore whether the Kannan contraction mapping has a fixed point in its associated pre-quasi operator ideal. The existence of solutions to non-linear difference equations is illustrated with a few real-world examples and applications.

2. Definitions and preliminaries

As a reminder, Matloka [16] presented the notion of ordinary convergence of sequences of fuzzy numbers, where he introduced bounded and convergent fuzzy numbers, explored some of their features, and proved that any convergent fuzzy number sequence is bounded. Nanda [19] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. Kumar et al. [15] investigated the limit points and cluster points of sequences of fuzzy numbers. Let Ω be the set of all closed and bounded intervals on the real line $\mathfrak{R}$. For $f = [f_1, f_2]$ and $g = [g_1, g_2]$ in Ω, suppose

$$f \leqslant g \text{ if and only if } f_1 \leqslant g_1 \text{ and } f_2 \leqslant g_2.$$

Define a metric ρ on Ω by

$$\rho(f, g) = \max\{|f_1 - g_1|, |f_2 - g_2|\}.$$

Matloka [16] showed that ρ is a metric on Ω and (Ω, ρ) is a complete metric space. Also, the relation $\leqslant$ is a partial order on Ω.

Definition 2.1. A fuzzy number g is a fuzzy subset of $\mathfrak{R}$, i.e., a mapping $g : \mathfrak{R} \to [0, 1]$, which verifies the following four settings:

(a) g is fuzzy convex, i.e., for $x, y \in \mathfrak{R}$ and $\alpha \in [0, 1]$, $g(\alpha x + (1 - \alpha)y) \geqslant \min\{g(x), g(y)\}$;
(b) g is normal, i.e., there is $y_0 \in \mathfrak{R}$ such that $g(y_0) = 1$;
(c) g is an upper-semi continuous, i.e., for all $\alpha > 0$, $g^{-1}([0, x + \alpha))$ for all $x \in [0, 1]$ is open in the usual topology of $\mathfrak{R}$;
(d) the closure of $g^0 := \{y \in \mathfrak{R} : \ g(y) > 0\}$ is compact.

The β-level set of a fuzzy real number g, $0 < \beta < 1$ indicated by g^β is defined as

$$g^\beta = \{y \in \mathfrak{R} : \ g(y) \geqslant \beta\}.$$

The set of every upper semi-continuous, normal, convex fuzzy number and g^β is compact is denoted by $\mathfrak{R}([0,1])$. The set $\mathfrak{R}$ can be embedded in $\mathfrak{R}([0,1])$, if we define $r \in \mathfrak{R}([0,1])$ by

$$\overline{r}(t) = \begin{cases} 1, & t = r, \\ 0, & t \neq r. \end{cases}$$

The additive identity and multiplicative identity in $\mathfrak{R}[0,1]$ are denoted by $\overline{0}$ and $\overline{1}$, respectively. The arithmetic operations on $\mathfrak{R}[0,1]$ are defined as follows:

$$(f \oplus g)(y) = \sup_{y \in \mathfrak{R}} \min\{f(x), g(y-x)\}, \qquad (f \ominus g)(y) = \sup_{y \in \mathfrak{R}} \min\{f(x), g(x-y)\},$$

$$(f \otimes g)(y) = \sup_{y \in \mathfrak{R}} \min\{f(x), g(\frac{y}{x})\}, \qquad (\frac{f}{g})(y) = \sup_{y \in \mathfrak{R}} \min\{f(xy), g(x)\},$$

$$xf(y) = \begin{cases} f(x^{-1}y), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

The absolute value $|f|$ of $f \in \mathfrak{R}[0,1]$ is defined by

$$|f|(y) = \begin{cases} \max\{f(y), f(-y)\}, & \text{if } y \geqslant 0, \\ 0, & \text{if } y < 0. \end{cases}$$

Suppose $f, g \in \mathfrak{R}[0,1]$ and the β-level sets are $[f]^\beta = [f_1^\beta, f_2^\beta]$, $[g]^\beta = [g_1^\beta, g_2^\beta]$, $\beta \in [0,1]$. A partial ordering for any $f, g \in \mathfrak{R}[0,1]$ is as follows: $f \preceq g$ if and only if $f^\beta \leqslant g^\beta$, for all $\beta \in [0,1]$. Then the above operations can be defined in terms of β-level sets as follows:

$$[f \oplus g]^\beta = [f_1^\beta + g_1^\beta, f_2^\beta + g_2^\beta], \qquad [f \ominus g]^\beta = [f_1^\beta - g_2^\beta, f_2^\beta - g_1^\beta],$$

$$[f \otimes g]^\beta = [\min_{j \in \{1,2\}} f_j^\beta g_j^\beta, \max_{j \in \{1,2\}} f_j^\beta g_j^\beta], \qquad [f^{-1}]^\beta = [(f_2^\beta)^{-1}, (f_1^\beta)^{-1}], f_j^\beta > 0, \text{ for every } \beta \in (0,1],$$

$$[xf]^\beta = \begin{cases} [xf_1^\beta, xf_2^\beta], & x \geqslant 0, \\ [xf_2^\beta, xf_1^\beta], & x < 0. \end{cases}$$

Assume $\overline{\rho} : \mathfrak{R}[0,1] \times \mathfrak{R}[0,1] \to \mathfrak{R}^+ \cup \{0\}$ is defined by $\overline{\rho}(f,g) = \sup_{0 \leqslant \beta \leqslant 1} \rho(f^\beta, g^\beta)$. Recall that:

1. $(\mathfrak{R}[0,1], \overline{\rho})$ is a complete metric space;
2. $\overline{\rho}(f+k, g+k) = \overline{\rho}(f,g)$ for all $f, g, k \in \mathfrak{R}[0,1]$;
3. $\overline{\rho}(f+k, g+l) \leqslant \overline{\rho}(f,g) + \overline{\rho}(k,l)$;
4. $\overline{\rho}(\xi f, \xi g) = |\xi| \overline{\rho}(f,g)$, for all $\xi \in \mathfrak{R}$.

Definition 2.2. A sequence $f = (f_j)$ of fuzzy numbers is said to be

(a) bounded if the set $\{f_j : j \in \mathbb{N}\}$ of fuzzy numbers is bounded, i.e., if a sequence (f_j) is bounded, then there are two fuzzy numbers g, l such that $g \leqslant f_j \leqslant l$;
(b) convergent to a fuzzy real number f_0 if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\overline{\rho}(f_j, f_0) < \varepsilon$, for all $j \geqslant j_0$.

By ℓ_∞ and ℓ_r, we denote the spaces of bounded and r-absolutely summable sequences of real numbers, respectively. Let $\omega(F)$ denote the classes of all sequence spaces of fuzzy real numbers. Suppose $\tau = (\tau_a) \in \mathfrak{R}^{+\mathcal{N}}$, where $\mathfrak{R}^{+\mathcal{N}}$ is the space of positive real sequences. The Nakano sequence space of fuzzy numbers defined and studied in [20] is denoted by: $\ell^F(\tau) = \Big\{\overline{v} = (\overline{v_a}) \in \omega(F) : h(\mu\overline{v}) < \infty \text{ , for some } \mu > 0\Big\}$, when

$h(\overline{v}) = \sum_{a=0}^{\infty}[\overline{\rho}(\overline{v_a},\overline{0})]^{\tau_a}$. The space $(\ell^F(\tau),\|.\|)$, where $\|\overline{v}\| = \inf\left\{\kappa > 0 : h(\frac{\overline{v}}{\kappa}) \leqslant 1\right\}$ and $\tau_a \geqslant 1$, for all $a \in \mathcal{N}$, is a Banach space. If $(\tau_a) \in \ell_\infty$, then

$$\begin{aligned}\ell^F(\tau) &= \left\{\overline{v} = (\overline{v_a}) \in \omega(F) : h(\mu\overline{v}) < \infty \text{ , for some } \mu > 0\right\}\\ &= \left\{\overline{v} = (\overline{v_a}) \in \omega(F) : \inf_a |\mu|^{\tau_a}\sum_{a=0}^{\infty}[\overline{\rho}(\overline{v_a},\overline{0})]^{\tau_a} \leqslant \sum_{a=0}^{\infty}[\overline{\rho}(\mu\overline{v_a},\overline{0})]^{\tau_a} < \infty \text{ , for some } \mu > 0\right\}\\ &= \left\{\overline{v} = (\overline{v_a}) \in \omega(F) : \sum_{a=0}^{\infty}[\overline{\rho}(\overline{v_a},\overline{0})]^{\tau_a} < \infty\right\}\\ &= \left\{\overline{v} = (\overline{v_a}) \in \omega(F) : h(\mu\overline{v}) < \infty \text{ , for any } \mu > 0\right\}.\end{aligned}$$

Lemma 2.3 ([3]). *Suppose* $\tau_a > 0$ *and* $v_a, t_a \in \mathfrak{R}$, *for every* $a \in \mathcal{N}$, *then* $|v_a + t_a|^{\tau_a} \leqslant 2^{K-1}(|v_a|^{\tau_a} + |t_a|^{\tau_a})$, *where* $K = \max\{1, \sup_a \tau_a\}$.

3. Main results

3.1. Some properties of $\ell^F(\tau)$

In this section, we have introduced the certain space of sequences of fuzzy numbers, (cssf), under definite function to be pre-quasi (cssf). We explain the sufficient setting of $\ell^F(\tau)$ equipped with the definite function h to be pre-quasi Banach and closed (cssf). The Fatou property of various pre-quasi norms h on $\ell^F(\tau)$ has been investigated. We have presented this space's uniform convexity (UUC 2), the property (R), and the h-normal structure-property.

Definition 3.1. The linear space $\mathbf{U}$ is said to be a certain space of sequences of fuzzy numbers (cssf), if

(1) $\{\overline{b}_q\}_{q\in\mathcal{N}} \subseteq \mathbf{U}$, where $\overline{b}_q = \{\overline{0},\overline{0},\dots,\overline{1},\overline{0},\overline{0},\dots\}$, while $\overline{1}$ displays at the q^{th} place;
(2) $\mathbf{U}$ is solid, i.e., suppose $\overline{Y} = (\overline{Y_q}) \in \omega(F)$, $\overline{Z} = (\overline{Z_q}) \in \mathbf{U}$ and $|\overline{Y_q}| \leqslant |\overline{Z_q}|$, for all $q \in \mathcal{N}$, then $\overline{Y} \in \mathbf{U}$;
(3) $(\overline{Y_{[\frac{q}{2}]}})_{q=0}^{\infty} \in \mathbf{U}$, where $[\frac{q}{2}]$ marks the integral part of $\frac{q}{2}$, if $(\overline{Y_q})_{q=0}^{\infty} \in \mathbf{U}$.

Definition 3.2. A subclass $\mathbf{U}_h$ of $\mathbf{U}$ is called a pre-modular (cssf), if there is $h \in [0,\infty)^{\mathbf{U}}$ that satisfies the next settings:

(i) if $\overline{Y} \in \mathbf{U}$, $\overline{Y} = \overline{\vartheta} \Leftrightarrow h(\overline{Y}) = 0$ with $h(\overline{Y}) \geqslant 0$, where $\overline{\vartheta} = (\overline{0},\overline{0},\overline{0},\dots)$;
(ii) there is $Q \geqslant 1$, the inequality $h(\alpha\overline{Y}) \leqslant Q|\alpha|h(\overline{Y})$ holds, for every $\overline{Y} \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$;
(iii) there is $P \geqslant 1$, the inequality $h(\overline{Y}+\overline{Z}) \leqslant P(h(\overline{Y}) + h(\overline{Z}))$ holds, for every $\overline{Y},\overline{Z} \in \mathbf{U}$;
(iv) if $|\overline{Y_q}| \leqslant |\overline{Z_q}|$, for every $q \in \mathcal{N}$, one has $h((\overline{Y_q})) \leqslant h((\overline{Z_q}))$;
(v) the inequality $h((\overline{Y_q})) \leqslant h((\overline{Y_{[\frac{q}{2}]}})) \leqslant P_0 h((\overline{Y_q}))$ holds, for some $P_0 \geqslant 1$;
(vi) let E be the space of finite sequences of fuzzy numbers, then the closure of $E = \mathbf{U}_h$;
(vii) there is $\sigma > 0$ with $h(\overline{\alpha},\overline{0},\overline{0},\overline{0},\dots) \geqslant \sigma|\alpha|h(\overline{1},\overline{0},\overline{0},\overline{0},\dots)$, where

$$\overline{\alpha}(y) = \begin{cases} 1, & y = \alpha,\\ 0, & y \neq \alpha.\end{cases}$$

Definition 3.3. Suppose $\mathbf{U}$ is a (cssf). The function $h \in [0,\infty)^{\mathbf{U}}$ is called a pre-quasi norm on $\mathbf{U}$, if it holds the following conditions:

(i) if $\overline{Y} \in \mathbf{U}$, $\overline{Y} = \overline{\vartheta} \Leftrightarrow h(\overline{Y}) = 0$ with $h(\overline{Y}) \geqslant 0$, where $\overline{\vartheta} = (\overline{0},\overline{0},\overline{0},\dots)$;
(ii) there is $Q \geqslant 1$, the inequality $h(\alpha\overline{Y}) \leqslant Q|\alpha|h(\overline{Y})$ satisfies, for every $\overline{Y} \in \mathbf{U}$ and $\alpha \in \mathfrak{R}$;
(iii) there is $P \geqslant 1$, the inequality $h(\overline{Y}+\overline{Z}) \leqslant P(h(\overline{Y}) + h(\overline{Z}))$ holds, for each $\overline{Y},\overline{Z} \in \mathbf{U}$.

Clearly, from the last two definitions, we conclude the following two theorems.

Theorem 3.4. *If* **U** *is a pre-modular (cssf), then it is pre-quasi normed (cssf).*

Theorem 3.5. **U** *is a pre-quasi normed (cssf) if it is quasi-normed (cssf).*

Definition 3.6.

(a) The function h on $\ell^F(\tau)$ is said to be h-convex, if

$$h(\alpha\overline{Y} + (1-\alpha)\overline{Z}) \leqslant \alpha h(\overline{Y}) + (1-\alpha)h(\overline{Z}),$$

for every $\alpha \in [0,1]$ and $\overline{Y}, \overline{Z} \in \ell^F(\tau)$.

(b) $\{\overline{Y_q}\}_{q\in\mathcal{N}} \subseteq \left(\ell^F(\tau)\right)_h$ is h-convergent to $\overline{Y} \in \left(\ell^F(\tau)\right)_h$, if and only if, $\lim_{q\to\infty} h(\overline{Y_q} - \overline{Y}) = 0$. When the h-limit exists, then it is unique.

(c) $\{\overline{Y_q}\}_{q\in\mathcal{N}} \subseteq \left(\ell^F(\tau)\right)_h$ is h-Cauchy, if $\lim_{q,r\to\infty} h(\overline{Y_q} - \overline{Y_r}) = 0$.

(d) $\Gamma \subset \left(\ell^F(\tau)\right)_h$ is h-closed, when for all h-converges $\{\overline{Y_q}\}_{a\in\mathcal{N}} \subset \Gamma$ to $\overline{Y}$, then $\overline{Y} \in \Gamma$.

(e) $\Gamma \subset \left(\ell^F(\tau)\right)_h$ is h-bounded, if $\delta_h(\Gamma) = \sup\left\{h(\overline{Y} - \overline{Z}) : \overline{Y}, \overline{Z} \in \Gamma\right\} < \infty$.

(f) The h-ball of radius $\varepsilon \geqslant 0$ and center $\overline{Y}$, for every $\overline{Y} \in \left(\ell^F(\tau)\right)_h$, is described as:

$$\mathbf{B}_h(\overline{Y}, \varepsilon) = \left\{\overline{Z} \in \left(\ell^F(\tau)\right)_h : h(\overline{Y} - \overline{Z}) \leqslant \varepsilon\right\}.$$

(g) A pre-quasi norm h on $\ell^F(\tau)$ holds the Fatou property, if for every sequence $\{\overline{Z^q}\} \subseteq \left(\ell^F(\tau)\right)_h$ under $\lim_{q\to\infty} h(\overline{Z^q} - \overline{Z}) = 0$ and all $\overline{Y} \in \left(\ell^F(\tau)\right)_h$, one has $h(\overline{Y} - \overline{Z}) \leqslant \sup_r \inf_{q\geqslant r} h(\overline{Y} - \overline{Z^q})$.

Note that the Fatou property implies the h-closedness of the h-balls. We will denote the space of all increasing sequences of real numbers by $\mathbf{I}$.

Theorem 3.7. $\left(\ell^F(\tau)\right)_h$, *where* $h(\overline{Y}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q}, \overline{0})]^{\tau_q}\right]^{\frac{1}{K}}$, *for all* $\overline{Y} \in \ell^F(\tau)$, *is a pre-modular (cssf), when* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$.

Proof.

(i) Evidently, $h(\overline{Y}) \geqslant 0$ and $h(\overline{Y}) = 0 \Leftrightarrow \overline{Y} = \overline{\vartheta}$.

(1-i) Let $\overline{Y}, \overline{Z} \in \ell^F(\tau)$. One has

$$h(\overline{Y} + \overline{Z}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q} + \overline{Z_q}, \overline{0})]^{\tau_q}\right]^{\frac{1}{K}} \leqslant \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q}, \overline{0})]^{\tau_q}\right]^{\frac{1}{K}} + \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Z_q}, \overline{0})]^{\tau_q}\right]^{\frac{1}{K}} = h(\overline{Y}) + h(\overline{Z}) < \infty,$$

then $\overline{Y} + \overline{Z} \in \ell^F(\tau)$.

(iii) One gets $P \geqslant 1$ with $h(\overline{Y} + \overline{Z}) \leqslant P(h(\overline{Y}) + h(\overline{Z}))$, for all $\overline{Y}, \overline{Z} \in \ell^F(\tau)$.

(1-ii) Assume $\alpha \in \mathfrak{R}$ and $\overline{Y} \in \ell^F(\tau)$, we obtain

$$h(\alpha\overline{Y}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\alpha\overline{Y_q}, \overline{0})]^{\tau_q}\right]^{\frac{1}{K}} \leqslant \sup_q |\alpha|^{\frac{\tau_q}{K}} \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q}, \overline{0})]^{\tau_q}\right]^{\frac{1}{K}} \leqslant Q|\alpha|h(\overline{Y}) < \infty.$$

As $\alpha\overline{Y} \in \ell^F(\tau)$, hence, from conditions (1-i) and (1-ii), one has $\ell^F(\tau)$ is linear. Also $\overline{b}_p \in \ell^F(\tau)$, for all $p \in \mathcal{N}$, since $h(\overline{b}_p) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{b}_p, \overline{0})]^{\tau_q}\right]^{\frac{1}{K}} = 1$.

(ii) There is $Q = \max\left\{1, \sup_q |\alpha|^{\frac{\tau_q}{K}-1}\right\} \geqslant 1$ with $h(\alpha\overline{Y}) \leqslant Q|\alpha|h(\overline{Y})$, for all $\overline{Y} \in \ell^F(\tau)$ and $\alpha \in \mathfrak{R}$.

(2) Assume $|\overline{Y_q}| \leqslant |\overline{Z_q}|$, for all $q \in \mathcal{N}$ and $\overline{Z} \in \ell^F(\tau)$. One finds

$$h(\overline{Y}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}} \leqslant \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Z_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}} = h(\overline{Z}) < \infty,$$

then $\overline{Y} \in \ell^F(\tau)$.

(iv) Obviously, from (2).

(3) Let $(\overline{Y_q}) \in \ell^F(\tau)$, we get

$$\begin{aligned} h\left((\overline{Y_{[\frac{q}{2}]}})\right) &= \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_{[\frac{q}{2}]}},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}} = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_{2q}} + \sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_{2q+1}}\right]^{\frac{1}{K}} \leqslant 2^{\frac{1}{K}}\left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}} \\ &= 2^{\frac{1}{K}} h\left((\overline{Y_q})\right), \end{aligned}$$

then $(\overline{Y_{[\frac{q}{2}]}}) \in \ell^F(\tau)$.

(v) From (3), we obtain $P_0 = 2^{\frac{1}{K}} \geqslant 1$.

(vi) Evidently the closure of $E = \ell^F(\tau)$.

(vii) There is $0 < \sigma \leqslant |\alpha|^{\frac{\tau_0}{K}-1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ with $h(\overline{\alpha},\overline{0},\overline{0},\overline{0},\dots) \geqslant \sigma|\alpha|h(\overline{1},\overline{0},\overline{0},\overline{0},\dots)$. □

Theorem 3.8. *If* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$, *then* $\left(\ell^F(\tau)\right)_h$ *is a pre-quasi Banach (cssf), where* $h(\overline{Y}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}}$, *for every* $\overline{Y} \in \ell^F(\tau)$.

Proof. In view of Theorems 3.7 and 3.4, the space $\left(\ell^F(\tau)\right)_h$ is a pre-quasi normed (cssf). Assume $\overline{Y^l} = (\overline{Y^l_q})_{q=0}^{\infty}$ is a Cauchy sequence in $\left(\ell^F(\tau)\right)_h$. Hence, for every $\varepsilon \in (0,1)$, one has $l_0 \in \mathcal{N}$ such that for all $l, m \geqslant l_0$, one gets

$$h(\overline{Y^l} - \overline{Y^m}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y^l_q} - \overline{Y^m_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}} < \varepsilon.$$

That implies $\overline{\rho}(\overline{Y^l_q} - \overline{Y^m_q},\overline{0}) < \varepsilon$. As $(\mathfrak{R}[0,1],\overline{\rho})$ is a complete metric space, then $(\overline{Y^m_q})$ is a Cauchy sequence in $\mathfrak{R}[0,1]$, for fixed $q \in \mathcal{N}$, which implies $\lim_{m\to\infty}\overline{Y^m}_q = \overline{Y^0_q}$, for constant $q \in \mathcal{N}$. Hence $h(\overline{Y^l} - \overline{Y^0}) < \varepsilon$, for every $l \geqslant l_0$. Since $h(\overline{Y^0}) = h(\overline{Y^0} - \overline{Y^l} + \overline{Y^l}) \leqslant h(\overline{Y^l} - \overline{Y^0}) + h(\overline{Y^l}) < \infty$. So $\overline{Y^0} \in \ell^F(\tau)$. □

Theorem 3.9. *Suppose* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$, *then* $\left(\ell^F(\tau)\right)_h$ *is a pre-quasi closed (cssf), where* $h(\overline{Y}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}}$, *for every* $\overline{Y} \in \ell^F(\tau)$.

Proof. In view of Theorems 3.7 and 3.4, the space $\left(\ell^F(\tau)\right)_h$ is a pre-quasi normed (cssf). Assume $\overline{Y^l} = (\overline{Y^l_q})_{q=0}^{\infty} \in \left(\ell^F(\tau)\right)_h$ and $\lim_{l\to\infty} h(\overline{Y^l} - \overline{Y^0}) = 0$, then for all $\varepsilon \in (0,1)$, there is $l_0 \in \mathcal{N}$ such that for all $l \geqslant l_0$, we obtain

$$\varepsilon > h(\overline{Y^l} - \overline{Y^0}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y^l_q} - \overline{Y^0_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}},$$

which implies $\bar{\rho}(\overline{Y^l_q} - \overline{Y^0_q}, \bar{0}) < \varepsilon$. As $(\mathfrak{R}[0,1], \bar{\rho})$ is a complete metric space, therefore, $(\overline{Y^l_q})$ is a convergent sequence in $\mathfrak{R}[0,1]$, for fixed $q \in \mathcal{N}$. So, $\lim_{l\to\infty} \overline{Y^l_q} = \overline{Y^0_q}$, for fixed $q \in \mathcal{N}$. Since $h(\overline{Y^0}) = h(\overline{Y^0} - \overline{Y^l} + \overline{Y^l}) \leqslant h(\overline{Y^l} - \overline{Y^0}) + h(\overline{Y^l}) < \infty$, one has $\overline{Y^0} \in \ell^F(\tau)$. □

Theorem 3.10. *The function* $h(\bar{Y}) = \left[\sum_{q=0}^{\infty}[\bar{\rho}(\overline{Y_q}, \bar{0})]^{\tau_q}\right]^{\frac{1}{K}}$ *holds the Fatou property, when* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$*, for all* $\bar{Y} \in \ell^F(\tau)$.

Proof. Let $\{\overline{Z^r}\} \subseteq \left(\ell^F(\tau)\right)_h$ such that $\lim_{r\to\infty} h(\overline{Z^r} - \bar{Z}) = 0$. Since $\left(\ell^F(\tau)\right)_h$ is a pre-quasi closed space, one has $\bar{Z} \in \left(\ell^F(\tau)\right)_h$. For all $\bar{Y} \in \left(\ell^F(\tau)\right)_h$, one gets

$$\begin{aligned} h(\bar{Y} - \bar{Z}) &= \left[\sum_{q=0}^{\infty}[\bar{\rho}(\overline{Y_q} - \overline{Z_q}, \bar{0})]^{\tau_q}\right]^{\frac{1}{K}} \leqslant \left[\sum_{q=0}^{\infty}[\bar{\rho}(\overline{Y_q} - \overline{Z^r_q}, \bar{0})]^{\tau_q}\right]^{\frac{1}{K}} + \left[\sum_{q=0}^{\infty}[\bar{\rho}(\overline{Z^r_q} - \overline{Z_q}, \bar{0})]^{\tau_q}\right]^{\frac{1}{K}} \\ &\leqslant \sup_m \inf_{r\geqslant m} h(\bar{Y} - \overline{Z^r}). \end{aligned}$$

□

Theorem 3.11. *The function* $h(\bar{Y}) = \sum_{q=0}^{\infty}[\bar{\rho}(\overline{Y_q}, \bar{0})]^{\tau_q}$ *does not hold the Fatou property, for all* $\bar{Y} \in \ell^F(\tau)$*, when* $(\tau_q) \in \ell_\infty$ *and* $\tau_q > 1$*, for all* $q \in \mathcal{N}$.

Proof. Let $\{\overline{Z^r}\} \subseteq \left(\ell^F(\tau)\right)_h$ so that $\lim_{r\to\infty} h(\overline{Z^r} - \bar{Z}) = 0$. Since $\left(\ell^F(\tau)\right)_h$ is a pre-quasi closed space, one gets $\bar{Z} \in \left(\ell^F(\tau)\right)_h$. For every $\bar{Z} \in \left(\ell^F(\tau)\right)_h$, we obtain

$$\begin{aligned} h(\bar{Y} - \bar{Z}) &= \sum_{q=0}^{\infty}[\bar{\rho}(\overline{Y_q} - \overline{Z_q}, \bar{0})]^{\tau_q} \leqslant 2^{\sup_q \tau_q - 1}\left(\sum_{q=0}^{\infty}[\bar{\rho}(\overline{Y_q} - \overline{Z^r_q}, \bar{0})]^{\tau_q} + \sum_{q=0}^{\infty}[\bar{\rho}(\overline{Z^r_q} - \overline{Z_q}, \bar{0})]^{\tau_q}\right) \\ &\leqslant 2^{\sup_q \tau_q - 1} \sup_m \inf_{r\geqslant m} h(\bar{Y} - \overline{Z^r}). \end{aligned}$$

□

Example 3.12. For $(\tau_q) \in [1, \infty)^{\mathcal{N}}$, the function $h(\bar{Y}) = \inf\left\{\alpha > 0 : \sum_{q\in\mathcal{N}}[\bar{\rho}(\frac{\overline{Y_q}}{\alpha}, \bar{0})]^{\tau_q} \leqslant 1\right\}$ is a norm on $\ell^F(\tau)$.

Example 3.13. The function $h(\bar{Y}) = \sqrt[3]{\sum_{q\in\mathcal{N}}[\bar{\rho}(\overline{Y_q}, \bar{0})]^{\frac{3q+2}{q+1}}}$ is a pre-quasi norm (not a norm) on $\ell^F((\frac{3q+2}{q+1})_{q=0}^{\infty})$.

Example 3.14. The function $h(\bar{Y}) = \sum_{q\in\mathcal{N}}[\bar{\rho}(\overline{Y_q}, \bar{0})]^{\frac{3q+2}{q+1}}$ is a pre-quasi norm (not a quasi norm) on $\ell^F((\frac{3q+2}{q+1})_{q=0}^{\infty})$.

Example 3.15. The function $h(\bar{Y}) = \sqrt[d]{\sum_{q\in\mathcal{N}}[\bar{\rho}(\overline{Y_q}, \bar{0})]^d}$ is a pre-quasi norm, quasi norm and not a norm on ℓ^F_d, for $0 < d < 1$.

Definition 3.16.

(1) [14] Suppose $p > 0$ and $q > 0$. Indicate

$$\mathbb{K}_2(p, q) = \left\{(\bar{Y}, \bar{Z}) : \ \bar{Y}, \bar{Z} \in \mathbf{U}_h, \ h(\bar{Y}) \leqslant p, \ h(\bar{Z}) \leqslant p, \ h\left(\frac{\bar{Y} - \bar{Z}}{2}\right) \geqslant pq\right\}.$$

When $\mathbb{K}_2(p, q) \neq \emptyset$, we put

$$K_2(p, q) = \inf\left\{1 - \frac{1}{p} h\left(\frac{\bar{Y} + \bar{Z}}{2}\right) : \ (\bar{Y}, \bar{Z}) \in \mathbb{K}_2(p, q)\right\}.$$

If $\mathbb{K}_2(p, q) = \emptyset$, we put $K_2(p, q) = 1$.

(2) [14] The function h satisfies (UUC 2) if for every $r \geqslant 0$ and $q > 0$, there is $\beta_2(r,q)$ with

$$K_2(p,q) > \beta_2(r,q) > 0\,, \text{ for } p > r.$$

(3) [18] The function h is strictly convex, (SC), if for all $\overline{Y},\overline{Z} \in \mathbf{U}_h$ such that $h(\overline{Y}) = h(\overline{Z})$ and $h\left(\frac{\overline{Y}+\overline{Z}}{2}\right) = \frac{h(\overline{Y})+h(\overline{Z})}{2}$, we get $\overline{Y} = \overline{Z}$.

Lemma 3.17.

(i) [9] *Let* $t \geqslant 2$ *and for all* $f, g \in \mathfrak{R}$, *then*

$$\left|\frac{f+g}{2}\right|^t + \left|\frac{f-g}{2}\right|^t \leqslant \frac{1}{2}\left(|f|^t + |g|^t\right).$$

(ii) [28] *If* $1 < t \leqslant 2$ *and for every* $f, g \in \mathfrak{R}$ *so that* $|f| + |g| \neq 0$, *then*

$$\left|\frac{f+g}{2}\right|^t + \frac{t(t-1)}{2}\left|\frac{f-g}{|f|+|g|}\right|^{2-t}\left|\frac{f-g}{2}\right|^t \leqslant \frac{1}{2}\left(|f|^t + |g|^t\right).$$

In the next part of this section, we will use the function h as $h(\overline{g}) = \left[\sum_{m=0}^{\infty}\left(\overline{\rho}(\overline{g_m},\overline{0})\right)^{\tau_m}\right]^{\frac{1}{K}}$, for every $\overline{g} \in \ell^F(\tau)$.

Theorem 3.18. *Suppose* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 1$, *then* h *is (UUC2).*

Proof. Let the condition be satisfied, $b > 0$ and $a > r \geqslant 0$. Suppose $\overline{f},\overline{g} \in \ell^F(\tau)_h$ so that

$$h(\overline{f}) \leqslant a,\ h(\overline{g}) \leqslant a \text{ and } h\left(\frac{\overline{f}-\overline{g}}{2}\right) \geqslant ab. \tag{3.1}$$

From the definition of h, we have

$$\begin{aligned} ab \leqslant h\left(\frac{\overline{f}-\overline{g}}{2}\right) = \left[\sum_{m=0}^{\infty}\left(\overline{\rho}\left(\frac{\overline{f_m}-\overline{g_m}}{2},\overline{0}\right)\right)^{\tau_m}\right]^{\frac{1}{K}} &\leqslant 2^{-\frac{\tau_0}{K}}\left(\left[\sum_{m=0}^{\infty}\left(\overline{\rho}(\overline{f_m},\overline{0})\right)^{\tau_m}\right]^{\frac{1}{K}} + \left[\sum_{m=0}^{\infty}\left(\overline{\rho}(\overline{g_m},\overline{0})\right)^{\tau_m}\right]^{\frac{1}{K}}\right) \\ &= 2^{-\frac{\tau_0}{K}}(h(\overline{f}) + h(\overline{g})) \leqslant 2a, \end{aligned}$$

this implies $b \leqslant 2$. Consequently, let $Q = \{x \in \mathcal{N} : 1 < \tau_x < 2\}$ and $P = \{x \in \mathcal{N} : \tau_x \geqslant 2\} = \mathcal{N} \setminus Q$. For every $\overline{w} \in \ell^F(\tau)_h$, we get $h^K(\overline{w}) = h_P^K(\overline{w}) + h_Q^K(\overline{w})$. From the setup, one has $h_P\left(\frac{\overline{f}-\overline{g}}{2}\right) \geqslant \frac{ab}{2}$ or $h_Q\left(\frac{\overline{f}-\overline{g}}{2}\right) \geqslant \frac{ab}{2}$. Assume first $h_P\left(\frac{\overline{f}-\overline{g}}{2}\right) \geqslant \frac{ab}{2}$. By using Lemma 3.17, condition (i), we obtain

$$h_P^K\left(\frac{\overline{f}+\overline{g}}{2}\right) + h_P^K\left(\frac{\overline{f}-\overline{g}}{2}\right) \leqslant \frac{h_P^K(\overline{f}) + h_P^K(\overline{g})}{2},$$

which implies

$$h_P^K\left(\frac{\overline{f}+\overline{g}}{2}\right) \leqslant \frac{h_P^K(\overline{f}) + h_P^K(\overline{g})}{2} - \left(\frac{ab}{2}\right)^K. \tag{3.2}$$

As

$$h_Q^K\left(\frac{\overline{f}+\overline{g}}{2}\right) \leqslant \frac{h_Q^K(\overline{f}) + h_Q^K(\overline{g})}{2}, \tag{3.3}$$

by adding inequalities (3.2) and (3.3), and from inequality (3.1), we have

$$h^K\left(\frac{\bar{f}+\bar{g}}{2}\right) \leqslant \frac{h^K(\bar{f})+h^K(\bar{g})}{2} - \left(\frac{ab}{2}\right)^K \leqslant a^K\left(1-\left(\frac{b}{2}\right)^K\right).$$

That gives

$$h\left(\frac{\bar{f}+\bar{g}}{2}\right) \leqslant a\left(1-\left(\frac{b}{2}\right)^K\right)^{\frac{1}{K}}. \tag{3.4}$$

Next, suppose $h_Q\left(\frac{\bar{f}-\bar{g}}{2}\right) \geqslant \frac{ab}{2}$. Set $B = \left(\frac{b}{4}\right)^K$,

$$Q_1 = \left\{m \in Q : \bar{\rho}(\overline{f_m}-\overline{g_m},\bar{0}) \leqslant B(\bar{\rho}(\overline{f_m},\bar{0})+\bar{\rho}(\overline{g_m},\bar{0}))\right\} \text{ and } Q_2 = Q \setminus Q_1.$$

As $B \leqslant 1$ and the power function is convex, so

$$\begin{aligned} h^K_{Q_1}\left(\frac{\bar{f}-\bar{g}}{2}\right) &\leqslant \sum_{m\in Q_1} B^{\tau_m}\left(\bar{\rho}\left(\frac{\overline{f_m}+\overline{g_m}}{2},\bar{0}\right)\right)^{\tau_m} \\ &\leqslant \left(\frac{B}{2}\right)^{\tau_0}(h^K_{Q_1}(\bar{f})+h^K_{Q_1}(\bar{g})) \leqslant \frac{B}{2}(h^K_Q(\bar{f})+h^K_Q(\bar{g})) \leqslant \frac{B}{2}(h^K(\bar{f})+h^K(\bar{g})) \leqslant Ba^K. \end{aligned}$$

Since $h_Q\left(\frac{\bar{f}-\bar{g}}{2}\right) \geqslant \frac{ab}{2}$, we get

$$h^K_{Q_2}\left(\frac{\bar{f}-\bar{g}}{2}\right) = h^K_Q\left(\frac{\bar{f}-\bar{g}}{2}\right) - h^K_{Q_1}\left(\frac{\bar{f}-\bar{g}}{2}\right) \geqslant a^K\left(\left(\frac{b}{2}\right)^K - \left(\frac{b}{4}\right)^K\right).$$

For any $m \in Q_2$, we have

$$\tau_0 - 1 < \tau_0(\tau_0-1) \leqslant \cdots \leqslant \tau_{m-1}(\tau_{m-1}-1) \leqslant \tau_m(\tau_m-1)$$

and

$$B < B^{2-\tau_m} < \left[\frac{\bar{\rho}(\overline{f_m}-\overline{g_m},\bar{0})}{\bar{\rho}(\overline{f_m},\bar{0})+\bar{\rho}(\overline{g_m},\bar{0})}\right]^{2-\tau_m},$$

by Lemma 3.17, condition (ii), we have that

$$\left(\bar{\rho}\left(\frac{\overline{f_m}+\overline{g_m}}{2},\bar{0}\right)\right)^{\tau_m} + \frac{(\tau_0-1)B}{2}\left(\bar{\rho}\left(\frac{\overline{f_m}-\overline{g_m}}{2},\bar{0}\right)\right)^{\tau_m} \leqslant \frac{1}{2}\left((\bar{\rho}(\overline{f_m},\bar{0}))^{\tau_m}+(\bar{\rho}(\overline{g_m},\bar{0}))^{\tau_m}\right).$$

Hence

$$h^K_{Q_2}\left(\frac{\bar{f}+\bar{g}}{2}\right) + \frac{(\tau_0-1)B}{2}h^K_{Q_2}\left(\frac{\bar{f}-\bar{g}}{2}\right) \leqslant \frac{h^K_{Q_2}(\bar{f})+h^K_{Q_2}(\bar{g})}{2},$$

that investigates

$$h^K_{Q_2}\left(\frac{\bar{f}+\bar{g}}{2}\right) \leqslant \frac{h^K_{Q_2}(\bar{f})+h^K_{Q_2}(\bar{g})}{2} - \frac{(\tau_0-1)B}{2}a^K\left(\left(\frac{b}{2}\right)^K - \left(\frac{b}{4}\right)^K\right). \tag{3.5}$$

Since

$$h^K_{Q_1}\left(\frac{\bar{f}+\bar{g}}{2}\right) \leqslant \frac{h^K_{Q_1}(\bar{f})+h^K_{Q_1}(\bar{g})}{2}, \tag{3.6}$$

by adding inequalities (3.5) and (3.6), one has

$$\begin{aligned} h_Q^K\left(\frac{\bar{f}+\bar{g}}{2}\right) &\leqslant \frac{h_Q^K(\bar{f})+h_Q^K(\bar{g})}{2} - \frac{(\tau_0-1)B}{2}a^K\left(\left(\frac{b}{2}\right)^K-\left(\frac{b}{4}\right)^K\right) \\ &\leqslant \frac{h_Q^K(\bar{f})+h_Q^K(\bar{g})}{2} - \frac{(\tau_0-1)}{2}\left(\frac{b}{4}\right)^{2K} a^K\left(2^K-1\right) \\ &\leqslant \frac{h_Q^K(\bar{f})+h_Q^K(\bar{g})}{2} - \frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K} a^K. \end{aligned} \tag{3.7}$$

Since

$$h_P^K\left(\frac{\bar{f}+\bar{g}}{2}\right) \leqslant \frac{h_P^K(\bar{f})+h_P^K(\bar{g})}{2}, \tag{3.8}$$

by adding inequalities (3.7) and (3.8), and from inequality (3.1), we obtain

$$h^K\left(\frac{\bar{f}+\bar{g}}{2}\right) \leqslant \frac{h^K(\bar{f})+h^K(\bar{g})}{2} - \frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K} a^K \leqslant a^K\left[1-\frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right].$$

This implies

$$h\left(\frac{\bar{f}+\bar{g}}{2}\right) \leqslant a\left[1-\frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]^{\frac{1}{K}}. \tag{3.9}$$

It is clear that

$$1<\tau_0\leqslant K<2^K \Rightarrow 0<\frac{\tau_0-1}{2^K-1}<1.$$

By using inequalities (3.4) and (3.9), and Definition 3.16, if we put

$$\beta_2(r,b)=\min\left(1-\left(1-\left(\frac{b}{2}\right)^K\right)^{\frac{1}{K}}, 1-\left[1-\frac{(\tau_0-1)}{2^K-1}\left(\frac{b}{4}\right)^{2K}\right]^{\frac{1}{K}}\right),$$

one has $K_2(a,b)>\beta_2(r,b)>0$, we deduce that h is (UUC2). □

Definition 3.19. The space $\mathbf{U}_h$ holds the property (R) if and only if for all decreasing sequences $\{\Gamma_j\}_{j\in\mathcal{N}}$ of h-closed and h-convex nonempty subsets of $\mathbf{U}_h$ with $\sup_{j\in\mathcal{N}}\mathfrak{K}_h(\overline{Y},\Gamma_j)<\infty$, for some $\overline{Y}\in\mathbf{U}_h$, one has $\bigcap_{j\in\mathcal{N}}\Gamma_j\neq\emptyset$.

By fixing Γ a nonempty h-closed and h-convex subset of $\left(\ell^F(\tau)\right)_h$ we have the following.

Theorem 3.20. *If* $(\tau_q)_{q\in\mathcal{N}}\in\ell_\infty\cap\mathbf{I}$ *with* $\tau_0>1$, *one has*

(i) *suppose* $\overline{Y}\in\left(\ell^F(\tau)\right)_h$ *with*

$$\mathfrak{K}_h(\overline{Y},\Gamma)=\inf\left\{h(\overline{Y}-\overline{Z}):\overline{Z}\in\Gamma\right\}<\infty,$$

there is a unique $\overline{\kappa}\in\Gamma$ *so that* $\mathfrak{K}_h(\overline{Y},\Gamma)=h(\overline{Y}-\overline{\kappa})$;

(ii) $\left(\ell^F(\tau)\right)_h$ *holds the property (R).*

Proof. To prove (i), assume $\overline{Y} \notin \Gamma$ as Γ is h-closed. One has $C := \mathfrak{K}_h(\overline{Y}, \Gamma) > 0$. Hence for all $r \in \mathcal{N}$, one has $\overline{Z_r} \in \Gamma$ with $h(\overline{Y} - \overline{Z_r}) < C(1 + \frac{1}{r})$. If $\{\frac{\overline{Z_r}}{2}\}$ is not h-Cauchy. One gets a subsequence $\{\frac{\overline{Z_{g(r)}}}{2}\}$ and $l_0 > 0$ with $h\left(\frac{\overline{Z_{g(r)}} - \overline{Z_{g(j)}}}{2}\right) \geqslant l_0$, for every $r > j \geqslant 0$. Also, one gets $K_2\left(C(1+\frac{1}{r}), \frac{l_0}{2C}\right) > \alpha := \beta_2\left(C(1+\frac{1}{r}), \frac{l_0}{2C}\right) > 0$, for all $r \in \mathcal{N}$. Since

$$\max\left(h(\overline{Y} - \overline{Z_{g(r)}}), h(\overline{Y} - \overline{Z_{g(j)}})\right) \leqslant C\left(1 + \frac{1}{g(j)}\right)$$

and

$$h\left(\frac{\overline{Z_{g(r)}} - \overline{Z_{g(j)}}}{2}\right) \geqslant l_0 \geqslant C\left(1 + \frac{1}{g(j)}\right)\frac{l_0}{2C},$$

for every $r > j \geqslant 0$, we obtain

$$h\left(\overline{Y} - \frac{\overline{Z_{g(r)}} + \overline{Z_{g(j)}}}{2}\right) \leqslant C\left(1 + \frac{1}{g(j)}\right)(1-\alpha).$$

Then

$$C = \mathfrak{K}_h(\overline{Y}, \Gamma) \leqslant C\left(1 + \frac{1}{g(j)}\right)(1-\alpha),$$

for all $j \in \mathcal{N}$. By putting $j \to \infty$, one has

$$0 < C \leqslant C\left(1 + \frac{1}{g(j)}\right)(1-\alpha) < C,$$

which is a contradiction. So $\{\frac{\overline{Z_r}}{2}\}$ is h-Cauchy. As $\left(\ell^F(\tau)\right)_h$ is h-complete, then $\{\frac{\overline{Z_r}}{2}\}$ h-converges to some $\overline{Z}$. For all $j \in \mathcal{N}$, one gets $\{\frac{\overline{Z_r} + \overline{Z_j}}{2}\}$ h-converges to $\overline{Z} + \frac{\overline{Z_j}}{2}$. Since Γ is h-closed and h-convex, then $\overline{Z} + \frac{\overline{Z_j}}{2} \in \Gamma$. Since $\overline{Z} + \frac{\overline{Z_j}}{2}$ h-converges to $2\overline{Z}$, then $2\overline{Z} \in \Gamma$. Let $\overline{\lambda} = 2\overline{z}$ and from Theorem 3.10, since h holds the Fatou property, one has

$$\begin{aligned}\mathfrak{K}_h(\overline{Y}, \Gamma) \leqslant h(\overline{Y} - \overline{\lambda}) &\leqslant \sup_i \inf_{j \geqslant i} h\left(\overline{Y} - \left(\overline{Z} + \frac{\overline{Z_j}}{2}\right)\right) \\ &\leqslant \sup_i \inf_{j \geqslant i} \sup_i \inf_{r \geqslant i} h\left(\overline{Y} - \frac{\overline{Z_r} + \overline{Z_j}}{2}\right) \\ &\leqslant \frac{1}{2} \sup_i \inf_{r \geqslant i} \sup_i \inf_{r \geqslant i} \left[h(\overline{Y} - \overline{Z_r}) + h(\overline{Y} - \overline{Z_j})\right] = \mathfrak{K}_h(\overline{Y}, \Gamma).\end{aligned}$$

Then $h(\overline{Y} - \overline{\lambda}) = \mathfrak{K}_h(\overline{Y}, \Gamma)$. Since h is (UUC2), so is (SC), this implies the uniqueness of $\overline{\lambda}$. To prove (ii), assume $\overline{Y} \notin \Gamma_{r_0}$, for some $r_0 \in \mathcal{N}$. Since $\left(\mathfrak{K}_h(\overline{Y}, \Gamma_r)\right)_{r \in \mathcal{N}} \in \ell_\infty$ is increasing, put $\lim_{r \to \infty} \mathfrak{K}_h(\overline{Y}, \Gamma_r) = C$, when $C > 0$. Otherwise $\overline{Y} \in \Gamma_r$, for all $r \in \mathcal{N}$. According to (i), there is one point $\overline{Z_r} \in \Gamma_r$ with $\mathfrak{K}_h(\overline{Y}, \Gamma_r) = h(\overline{Y} - \overline{Z_r})$, for every $r \in \mathcal{N}$. A similar proof will prove that $\{\frac{\overline{Z_r}}{2}\}$ h-converges to some $\overline{Z} \in \left(\ell^F(\tau)\right)_h$. As $\{\Gamma_r\}$ are h-convex, decreasing and h-closed, one has $2\overline{Z} \in \cap_{r \in \mathcal{N}} \Gamma_r$. □

Definition 3.21. $\mathbf{U}_h$ holds the h-normal structure-property if and only if for all nonempty h-bounded, h-convex and h-closed subset Γ of $\mathbf{U}_h$ not decreased to one point, one has $\overline{Y} \in \Gamma$ with

$$\sup_{\overline{Z} \in \Gamma} h(\overline{Y} - \overline{Z}) < \delta_h(\Gamma) := \sup\left\{h(\overline{Y} - \overline{Z}) : \overline{Y}, \overline{Z} \in \Gamma\right\} < \infty.$$

Theorem 3.22. *If $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, then $\left(\ell^F(\tau)\right)_h$ holds the h-normal structure-property.*

Proof. Theorem 3.18 gives that h is (UUC2). If Γ is a h-bounded, h-convex and h-closed subset of $\left(\ell^F(\tau)\right)_h$ not decreased to one point, hence, $\delta_h(\Gamma) > 0$. Set $C = \delta_h(\Gamma)$. Suppose $\overline{Y}, \overline{Z} \in \Gamma$ such that $\overline{Y} \neq \overline{Z}$. Hence $h\left(\frac{\overline{Y}-\overline{Z}}{2}\right) = l > 0$. For every $\overline{\alpha} \in \Gamma$, one gets $h(\overline{Y}-\overline{\alpha}) \leqslant C$ and $h(\overline{Z}-\overline{\alpha}) \leqslant C$. As Γ is h-convex, one gets $\frac{\overline{Y}+\overline{Z}}{2} \in \Gamma$. As

$$h\left(\frac{\overline{Y}+\overline{Z}}{2}-\overline{\alpha}\right) = h\left(\frac{(\overline{Y}-\overline{\alpha})+(\overline{Z}-\overline{\alpha})}{2}\right) \leqslant C\left(1-K_2\left(C,\frac{l}{C}\right)\right),$$

for all $\overline{\alpha} \in \Gamma$, then

$$\sup_{\overline{\alpha}\in\Gamma} h\left(\frac{\overline{Y}+\overline{Z}}{2}-\overline{\alpha}\right) \leqslant C\left(1-K_2\left(C,\frac{l}{C}\right)\right) < C = \delta_h(\Gamma).$$

□

3.2. *Kannan contraction mapping on $\ell^F(\tau)$*

In this section, we look at how to configure $\left(\ell^F(\tau)\right)_h$ with different h so that there is only one fixed point of Kannan contraction mapping.

Definition 3.23. An operator $V : \mathbf{U}_h \to \mathbf{U}_h$ is said to be a Kannan h-contraction, if one gets $\alpha \in [0,\frac{1}{2})$ with $h(V\overline{Y}-V\overline{Z}) \leqslant \alpha(h(V\overline{Y}-\overline{Y})+h(V\overline{Z}-\overline{Z}))$, for all $\overline{Y},\overline{Z} \in \mathbf{U}_h$. The operator V is called Kannan h-non-expansive, when $\alpha = \frac{1}{2}$.

An element $\overline{Y} \in \mathbf{U}_h$ is called a fixed point of V, when $V(\overline{Y}) = \overline{Y}$.

Theorem 3.24. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, and $V : \left(\ell^F(\tau)\right)_h \to \left(\ell^F(\tau)\right)_h$ is Kannan h-contraction mapping, where $h(\overline{Y}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}}$, for all $\overline{Y} \in \ell^F(\tau)$, then V has a unique fixed point.*

Proof. If $\overline{Y} \in \ell^F(\tau)$, one has $V^p\overline{Y} \in \ell^F(\tau)$. As V is a Kannan h-contraction mapping, one gets

$$h(V^{l+1}\overline{Y}-V^l\overline{Y}) \leqslant \alpha\left(h(V^{l+1}\overline{Y}-V^l\overline{Y})+h(V^l\overline{Y}-V^{l-1}\overline{Y})\right)$$
$$\Rightarrow$$
$$h(V^{l+1}\overline{Y}-V^l\overline{Y}) \leqslant \frac{\alpha}{1-\alpha}h(V^l\overline{Y}-V^{l-1}\overline{Y}) \leqslant \left(\frac{\alpha}{1-\alpha}\right)^2 h(V^{l-1}\overline{Y}-V^{l-2}\overline{Y}) \leqslant \cdots \leqslant \left(\frac{\alpha}{1-\alpha}\right)^l h(V\overline{Y}-\overline{Y}).$$

So for all $l,m \in \mathcal{N}$ with $m > l$, one gets

$$h(V^l\overline{Y}-V^m\overline{Y}) \leqslant \alpha\left(h(V^l\overline{Y}-V^{l-1}\overline{Y})+h(V^m\overline{Y}-V^{m-1}\overline{Y})\right) \leqslant \alpha\left(\left(\frac{\alpha}{1-\alpha}\right)^{l-1}+\left(\frac{\alpha}{1-\alpha}\right)^{m-1}\right)h(V\overline{Y}-\overline{Y}).$$

Then, $\{V^l\overline{Y}\}$ is a Cauchy sequence in $\left(\ell^F(\tau)\right)_h$. As the space $\left(\ell^F(\tau)\right)_h$ is pre-quasi Banach space, one has $\overline{Z} \in \left(\ell^F(\tau)\right)_h$ with $\lim_{l\to\infty} V^l\overline{Y} = \overline{Z}$. To prove that $V\overline{Z} = \overline{Z}$, since h has the Fatou property, one obtains

$$h(V\overline{Z}-\overline{Z}) \leqslant \sup_i \inf_{l\geqslant i} h(V^{l+1}\overline{Y}-V^l\overline{Y}) \leqslant \sup_i \inf_{l\geqslant i}\left(\frac{\alpha}{1-\alpha}\right)^l h(V\overline{Y}-\overline{Y}) = 0,$$

then $V\overline{Z} = \overline{Z}$. So $\overline{Z}$ is a fixed point of V. To show the uniqueness, let $\overline{Y},\overline{Z} \in \left(\ell^F(\tau)\right)_h$ be two not equal fixed points of V. One has

$$h(\overline{Y}-\overline{Z}) \leqslant h(V\overline{Y}-V\overline{Z}) \leqslant \alpha\left(h(V\overline{Y}-\overline{Y})+h(V\overline{Z}-\overline{Z})\right) = 0.$$

So, $\overline{Y} = \overline{Z}$. □

Corollary 3.25. *If* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 1$, *and* $V : \left(\ell^F(\tau)\right)_h \to \left(\ell^F(\tau)\right)_h$ *is Kannan* h*-contraction mapping, where* $h(\overline{Y}) = \left[\sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q},\overline{0})]^{\tau_q}\right]^{\frac{1}{K}}$, *for all* $\overline{Y} \in \ell^F(\tau)$, *one has* V *has unique fixed point* $\overline{Z}$ *so that* $h(V^l\overline{Y} - \overline{Z}) \leqslant \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h(V\overline{Y} - \overline{Y})$.

Proof. In view of Theorem 3.24, one has a unique fixed point $\overline{Z}$ of V. So

$$h(V^l\overline{Y} - \overline{Z}) = h(V^l\overline{Y} - V\overline{Z}) \leqslant \alpha\left(h(V^l\overline{Y} - V^{l-1}\overline{Y}) + h(V\overline{Z} - \overline{Z})\right) = \alpha\left(\frac{\alpha}{1-\alpha}\right)^{l-1} h(V\overline{Y} - \overline{Y}).$$

□

Example 3.26. Assume $V : \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h \to \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$, where $h(\overline{g}) = \sqrt{\sum_{q=0}^{\infty}\left(\overline{\rho}(\overline{g_q},\overline{0})\right)^{\frac{2q+3}{q+2}}}$, for every $\overline{g} \in \ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})$ and

$$V(\overline{g}) = \begin{cases} \frac{\overline{g}}{4}, & h(\overline{g}) \in [0,1), \\ \frac{\overline{g}}{5}, & h(\overline{g}) \in [1,\infty). \end{cases}$$

As for each $\overline{g_1}, \overline{g_2} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$ with $h(\overline{g_1}), h(\overline{g_2}) \in [0,1)$, one has

$$h(V\overline{g_1} - V\overline{g_2}) = h(\frac{\overline{g_1}}{4} - \frac{\overline{g_2}}{4}) \leqslant \frac{1}{\sqrt[4]{27}}\Big(h(\frac{3\overline{g_1}}{4}) + h(\frac{3\overline{g_2}}{4})\Big) = \frac{1}{\sqrt[4]{27}}\Big(h(V\overline{g_1} - \overline{g_1}) + h(V\overline{g_2} - \overline{g_2})\Big).$$

For all $\overline{g_1}, \overline{g_2} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$ with $h(\overline{g_1}), h(\overline{g_2}) \in [1,\infty)$, one has

$$h(V\overline{g_1} - V\overline{g_2}) = h(\frac{\overline{g_1}}{5} - \frac{\overline{g_2}}{5}) \leqslant \frac{1}{\sqrt[4]{64}}\Big(h(\frac{4\overline{g_1}}{5}) + h(\frac{4\overline{g_2}}{5})\Big) = \frac{1}{\sqrt[4]{64}}\Big(h(V\overline{g_1} - \overline{g_1}) + h(V\overline{g_2} - \overline{g_2})\Big).$$

For all $\overline{g_1}, \overline{g_2} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$ with $h(\overline{g_1}) \in [0,1)$ and $h(\overline{g_2}) \in [1,\infty)$, we get

$$\begin{aligned} h(V\overline{g_1} - V\overline{g_2}) = h(\frac{\overline{g_1}}{4} - \frac{\overline{g_2}}{5}) &\leqslant \frac{1}{\sqrt[4]{27}}h(\frac{3\overline{g_1}}{4}) + \frac{1}{\sqrt[4]{64}}h(\frac{4\overline{g_2}}{5}) \\ &\leqslant \frac{1}{\sqrt[4]{27}}\Big(h(\frac{3\overline{g_1}}{4}) + h(\frac{4\overline{g_2}}{5})\Big) = \frac{1}{\sqrt[4]{27}}\Big(h(V\overline{g_1} - \overline{g_1}) + h(V\overline{g_2} - \overline{g_2})\Big). \end{aligned}$$

Hence, V is Kannan h-contraction. As h satisfies the Fatou property. From Theorem 3.24, one has V holds one fixed point $\overline{\vartheta} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$.

Definition 3.27. Pick up $\mathbf{U}_h$ be a pre-quasi normed (cssf), $V : \mathbf{U}_h \to \mathbf{U}_h$ and $\overline{Z} \in \mathbf{U}_h$. The operator V is called h-sequentially continuous at $\overline{Z}$, if and only if, when $\lim_{q\to\infty} h(\overline{Y_q} - \overline{Z}) = 0$, then $\lim_{q\to\infty} h(V\overline{Y_q} - V\overline{Z}) = 0$.

Example 3.28. Suppose $V : \left(\ell^F((\frac{q+1}{2q+4})_{q=0}^{\infty})\right)_h \to \left(\ell^F((\frac{q+1}{2q+4})_{q=0}^{\infty})\right)_h$, where $h(\overline{Z}) = \left[\sum_{q=0}^{\infty}\left(\overline{\rho}(\overline{Z_q},\overline{0})\right)^{\frac{q+1}{2q+4}}\right]^4$, for every $\overline{Z} \in \ell^F((\frac{q+1}{2q+4})_{q=0}^{\infty})$ and

$$V(\overline{Z}) = \begin{cases} \frac{1}{18}(\overline{b}_0 + \overline{Z}), & \overline{Z_0}(y) \in [0, \frac{1}{17}), \\ \frac{1}{17}\overline{b}_0, & \overline{Z_0}(y) = \frac{1}{17}, \\ \frac{1}{18}\overline{b}_0, & \overline{Z_0}(y) \in (\frac{1}{17}, 1]. \end{cases}$$

V is clearly both h-sequentially continuous and discontinuous at $\frac{1}{17}\overline{b}_0 \in \left(\ell^F((\frac{q+1}{2q+4})_{q=0}^{\infty})\right)_h$.

Example 3.29. Assume V is defined as in Example 3.26. Suppose $\{\overline{Z^{(n)}}\} \subseteq \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$ is such that $\lim_{n\to\infty} h(\overline{Z^{(n)}} - \overline{Z^{(0)}}) = 0$, where $\overline{Z^{(0)}} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$ with $h(\overline{Z^{(0)}}) = 1$. As the pre-quasi norm h is continuous, we have

$$\lim_{n\to\infty} h(V\overline{Z^{(n)}} - V\overline{Z^{(0)}}) = \lim_{n\to\infty} h\Big(\frac{\overline{Z^{(n)}}}{4} - \frac{\overline{Z^{(0)}}}{5}\Big) = h\Big(\frac{\overline{Z^{(0)}}}{20}\Big) > 0.$$

Therefore, V is not h-sequentially continuous at $\overline{Z^{(0)}}$.

Theorem 3.30. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, and $V: \left(\ell^F(\tau)\right)_h \to \left(\ell^F(\tau)\right)_h$, where $h(\overline{Y}) = \sum_{q=0}^{\infty}[\overline{\rho}(\overline{Y_q}, \overline{0})]^{\tau_q}$, for all $\overline{Y} \in \ell^F(\tau)$, suppose*

(1) V is Kannan h-contraction mapping;

(2) V is h-sequentially continuous at $\overline{Z} \in \left(\ell^F(\tau)\right)_h$;

(3) there is $\overline{Y} \in \left(\ell^F(\tau)\right)_h$ with $\{V^l\overline{Y}\}$ has $\{V^{l_j}\overline{Y}\}$ converging to $\overline{Z}$,

then $\overline{Z} \in \left(\ell^F(\tau)\right)_h$ is the only fixed point of V

Proof. Assume $\overline{Z}$ is not a fixed point of V, one has $V\overline{Z} \neq \overline{Z}$. From parts (2) and (3), we get

$$\lim_{l_j\to\infty} h(V^{l_j}\overline{Y} - \overline{Z}) = 0 \text{ and } \lim_{l_j\to\infty} h(V^{l_j+1}\overline{Y} - V\overline{Z}) = 0.$$

As V is Kannan h-contraction, one obtains

$$\begin{aligned} 0 < h(V\overline{Z} - \overline{Z}) &= h\left((V\overline{Z} - V^{l_j+1}\overline{Y}) + (V^{l_j}\overline{Y} - \overline{Z}) + (V^{l_j+1}\overline{Y} - V^{l_j}\overline{Y})\right) \\ &\leqslant 2^{2\sup_i \tau_i - 2} h\left(V^{l_j+1}\overline{Y} - V\overline{Z}\right) + 2^{2\sup_i \tau_i - 2} h\left(V^{l_j}\overline{Y} - \overline{Z}\right) + 2^{\sup_i \tau_i - 1}\alpha\left(\frac{\alpha}{1-\alpha}\right)^{l_j-1} h(V\overline{Y} - \overline{Y}). \end{aligned}$$

As $l_j \to \infty$, one has a contradiction. Then $\overline{Z}$ is a fixed point of V. To show that the uniqueness, let $\overline{Z}, \overline{Y} \in \left(\ell^F(\tau)\right)_h$ be two not equal fixed points of V. One obtains

$$h(\overline{Z} - \overline{Y}) \leqslant h(V\overline{Z} - V\overline{Y}) \leqslant \alpha\left(h(V\overline{Z} - \overline{Z}) + h(V\overline{Y} - \overline{Y})\right) = 0.$$

Hence, $\overline{Z} = \overline{Y}$. □

Example 3.31. Assume V is defined as in Example 3.26. Let $h(\overline{Y}) = \sum_{q\in\mathcal{N}}\left(\overline{\rho}(\overline{Y_q}, \overline{0})\right)^{\frac{2q+3}{q+2}}$, for all $\overline{Y} \in \ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})$. Since for all $\overline{Y_1}, \overline{Y_2} \in \left(\ell^F((\frac{2a+3}{q+2})_{q=0}^{\infty})\right)_h$ with $h(\overline{Y_1}), h(\overline{Y_2}) \in [0, 1)$, one gets

$$h(V\overline{Y_1} - V\overline{Y_2}) = h(\frac{\overline{Y_1}}{4} - \frac{\overline{Y_2}}{4}) \leqslant \frac{2}{\sqrt{27}}\Big(h(\frac{3\overline{Y_1}}{4}) + h(\frac{3\overline{Y_2}}{4})\Big) = \frac{2}{\sqrt{27}}\Big(h(V\overline{Y_1} - \overline{Y_1}) + h(V\overline{Y_2} - \overline{Y_2})\Big),$$

for all $\overline{Y_1}, \overline{Y_2} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$ with $h(\overline{Y_1}), h(\overline{Y_2}) \in [1, \infty)$, one gets

$$h(V\overline{Y_1} - V\overline{Y_2}) = h(\frac{\overline{Y_1}}{5} - \frac{\overline{Y_2}}{5}) \leqslant \frac{1}{4}\Big(h(\frac{4\overline{Y_1}}{5}) + h(\frac{4\overline{Y_2}}{5})\Big) = \frac{1}{4}\Big(h(V\overline{Y_1} - \overline{Y_1}) + h(V\overline{Y_2} - \overline{Y_2})\Big).$$

For all $\overline{Y_1}, \overline{Y_2} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$ with $h(\overline{Y_1}) \in [0, 1)$ and $h(\overline{Y_2}) \in [1, \infty)$, one gets

$$\begin{aligned} h(V\overline{Y_1} - V\overline{Y_2}) = h(\frac{\overline{Y_1}}{4} - \frac{\overline{Y_2}}{5}) &\leqslant \frac{2}{\sqrt{27}} h(\frac{3\overline{Y_1}}{4}) + \frac{1}{4} h(\frac{4\overline{Y_2}}{5}) \\ &\leqslant \frac{2}{\sqrt{27}}\Big(h(\frac{3\overline{Y_1}}{4}) + h(\frac{4\overline{Y_2}}{5})\Big) = \frac{2}{\sqrt{27}}\Big(h(V\overline{Y_1} - \overline{Y_1}) + h(V\overline{Y_2} - \overline{Y_2})\Big). \end{aligned}$$

So V is Kannan h-contraction and $V^p(\overline{Y}) = \begin{cases} \frac{\overline{Y}}{4^p}, & h(\overline{Y}) \in [0,1), \\ \frac{\overline{Y}}{5^p}, & h(\overline{Y}) \in [1,\infty). \end{cases}$ Obviously V is h-sequentially continuous at $\overline{\vartheta} \in \left(\ell^F((\frac{2a+3}{q+2})_{q=0}^{\infty})\right)_h$ and $\{V^p\overline{Y}\}$ holds, $\{V^{l_j}\overline{Y}\}$ converges to $\overline{\vartheta}$. By Theorem 3.30, the point $\overline{\vartheta} \in \left(\ell^F((\frac{2a+3}{q+2})_{q=0}^{\infty})\right)_h$ is the only fixed point of V.

3.3. *Kannan non-expansive mapping on* $\left(\ell^F(\tau)\right)_h$

We introduce the sufficient conditions of $\left(\ell^F(\tau)\right)_h$, where $h(\overline{g}) = \left[\sum_{m=0}^{\infty}\left(\overline{\rho}(\overline{g_m},\overline{0})\right)^{\tau_m}\right]^{\frac{1}{K}}$, for every $\overline{g} \in \ell^F(\tau)$, such that the Kannan non-expansive mapping on it has a fixed point, by fixing Γ a nonempty h-bounded, h-convex and h-closed subset of $\left(\ell^F(\tau)\right)_h$.

Lemma 3.32. *If* $\left(\ell^F(\tau)\right)_h$ *holds the (R) property and the* h*-quasi-normal property, assume* $V : \Gamma \to \Gamma$ *is a Kannan* h*-non-expansive mapping. For* $t > 0$*, let* $G_t = \left\{\overline{Y} \in \Gamma : h(\overline{Y} - V(\overline{Y})) \leqslant t\right\} \neq \emptyset$*. Put*

$$\Gamma_t = \bigcap \left\{\mathbf{B}_h(r,j) : V(G_t) \subset \mathbf{B}_h(r,j)\right\} \cap \Gamma.$$

Then $\Gamma_t \neq \emptyset$*,* h*-convex,* h*-closed subset of* Γ *and* $V(\Gamma_t) \subset \Gamma_t \subset G_t$ *and* $\delta_h(\Gamma_t) \leqslant t$*.*

Proof. Since $V(G_t) \subset \Gamma_t$, then $\Gamma_t \neq \emptyset$. As the h-balls are h-convex and h-closed, then Γ_t is a h-closed and h-convex subset of Γ. To show that $\Gamma_t \subset G_t$, assume $\overline{Y} \in \Gamma_t$. When $h(\overline{Y} - V(\overline{Y})) = 0$, one has $\overline{Y} \in G_t$. Else, assume $h(\overline{Y} - V(\overline{Y})) > 0$. Put

$$r = \sup\left\{h\Big(V(\overline{Z}) - V(\overline{Y})\Big) : \overline{Z} \in G_t\right\}.$$

From the definition of r, one gets $V(G_t) \subset \mathbf{B}_h\Big(V(\overline{Y}), r\Big)$. Therefore, $\Gamma_t \subset \mathbf{B}_h\Big(V(\overline{Y}), r\Big)$, then $h\Big(\overline{Y} - V(\overline{Y})\Big) \leqslant r$. Let $l > 0$. One has $\overline{Z} \in G_t$ with $r - l \leqslant h\Big(V(\overline{Z}) - V(\overline{Y})\Big)$. So

$$\begin{aligned} h\Big(\overline{Y} - V(\overline{Y})\Big) - l \leqslant r - l \leqslant h\Big(V(\overline{Z}) - V(\overline{Y})\Big) &\leqslant \frac{1}{2}\Big(h\Big(\overline{Y} - V(\overline{Y})\Big) + h\Big(\overline{Z} - V(\overline{Z})\Big)\Big) \\ &\leqslant \frac{1}{2}\Big(h\Big(\overline{Y} - V(\overline{Y})\Big) + t\Big). \end{aligned}$$

As l is an arbitrary positive, one obtains $h\Big(\overline{Y} - V(\overline{Y})\Big) \leqslant t$, then $\overline{Y} \in G_t$. Since $V(G_t) \subset \Gamma_t$, one gets $V(\Gamma_t) \subset V(G_t) \subset \Gamma_t$, so Γ_t is V-invariant. To show that $\delta_h(\Gamma_t) \leqslant t$. Since

$$h\Big(V(\overline{Y}) - V(\overline{Z})\Big) \leqslant \frac{1}{2}\Big(h\Big(\overline{Y} - V(\overline{Y})\Big)\Big) + h\Big(\overline{Z} - V(\overline{Z})\Big)\Big),$$

for all $\overline{Y}, \overline{Z} \in G_t$, let $\overline{Y} \in G_t$, then $V(G_t) \subset \mathbf{B}_h\Big(V(\overline{Y}), t\Big)$. The definition of Γ_t gives $\Gamma_t \subset \mathbf{B}_h\Big(V(\overline{Y}), t\Big)$. Therefore, $V(\overline{Y}) \in \bigcap_{t \in \Gamma_t} \mathbf{B}_h(\overline{Z}, t)$. One has $h\Big(\overline{Z} - \overline{Y}\Big) \leqslant t$, for all $\overline{Z}, \overline{Y} \in \Gamma_t$, so $\delta_h(\Gamma_t) \leqslant t$. □

Theorem 3.33. *If* $\left(\ell^F(\tau)\right)_h$ *holds the* h*-quasi-normal property and the (R) property, let* $V : \Gamma \to \Gamma$ *be a Kannan* h*-non-expansive mapping, then* V *has a fixed point.*

Proof. Let $t_0 = \inf\left\{h\Big(\overline{Y} - V(\overline{Y})\Big) : \overline{Y} \in \Gamma\right\}$ and $t_r = t_0 + \frac{1}{r}$, for every $r \geqslant 1$. By the definition of t_0, one gets $G_{t_r} = \left\{\overline{Y} \in \Gamma : h\Big(\overline{Y} - V(\overline{Y})\Big) \leqslant t_r\right\} \neq \emptyset$, for every $r \geqslant 1$. Assume Γ_{t_r} is defined as in Lemma 3.32. Clearly $\{\Gamma_{t_r}\}$ is a decreasing sequence of nonempty h-bounded, h-closed and h-convex subsets of Γ. The property

(R) investigates that $\Gamma_\infty = \bigcap_{r\geqslant 1}\Gamma_{t_r} \neq \emptyset$. Let $\overline{Y} \in \Gamma_\infty$, one has $h\Big(\overline{Y} - V(\overline{Y})\Big) \leqslant t_r$, for all $r \geqslant 1$. Suppose $r \to \infty$, then $h\Big(\overline{Y} - V(\overline{Y})\Big) \leqslant t_0$, so $h\Big(\overline{Y} - V(\overline{Y})\Big) = t_0$. Therefore, $G_{t_0} \neq \emptyset$. Then $t_0 = 0$. Else, $t_0 > 0$ then V fails to have a fixed point. Let Γ_{t_0} be defined in Lemma 3.32. As V fails to have a fixed point and Γ_{t_0} is V-invariant, then Γ_{t_0} has more than one point, so $\delta_h(\Gamma_{t_0}) > 0$. By the h-quasi-normal property, one has $\overline{Y} \in \Gamma_{t_0}$ with

$$h\Big(\overline{Y} - \overline{Z}\Big) < \delta_h(\Gamma_{t_0}) \leqslant t_0,$$

for all $\overline{Z} \in \Gamma_{t_0}$. From Lemma 3.32, we get $\Gamma_{t_0} \subset G_{t_0}$. From definition of Γ_{t_0}, so $V(\overline{Y}) \in G_{t_0} \subset \Gamma_{t_0}$. Then

$$h\Big(\overline{Y} - V(\overline{Y})\Big) < \delta_h(\Gamma_{t_0}) \leqslant t_0,$$

which contradicts the definition of t_0. Then $t_0 = 0$, which gives that any point in G_{t_0} is a fixed point of V. □

According to Theorems 3.20, 3.22, and 3.33, we conclude the following.

Corollary 3.34. *Assume $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 1$, and $V : \Gamma \to \Gamma$ is a Kannan h-non-expansive mapping. Then V has a fixed point.*

Example 3.35. Assume $V : \Gamma \to \Gamma$ with $V(Y) = \begin{cases} \frac{\overline{Y}}{4}, & h(\overline{Y}) \in [0, 1), \\ \frac{\overline{Y}}{5}, & h(\overline{Y}) \in [1, \infty), \end{cases}$ where $\Gamma = \Big\{\overline{Y} \in \Big(\ell^F((\frac{2q+3}{q+2})_{q=0}^\infty)\Big)_h : \overline{Y_0} = \overline{Y_1} = \overline{0}\Big\}$ and $h(\overline{Y}) = \sqrt{\sum_{q\in\mathcal{N}} \big(\overline{\rho}(\overline{Y_q}, \overline{0})\big)^{\frac{2q+3}{q+2}}}$, for every $\overline{Y} \in \Big(\ell^F((\frac{2q+3}{q+2})_{q=0}^\infty)\Big)_h$. By using Example 3.31, V is Kannan h-contraction. So it is Kannan h-non-expansive. By Corollary 3.34, V holds a fixed point $\overline{\vartheta}$ in Γ.

3.4. *Kannan contraction and structure of operators ideal*

The structure of the operators ideal by $\Big(\ell^F(\tau)\Big)_h$ equipped with the definite function h, where $h(\overline{g}) = \Big[\sum_{m=0}^\infty \big(\overline{\rho}(\overline{g_m}, \overline{0})\big)^{\tau_m}\Big]^{\frac{1}{K}}$, for every $\overline{g} \in \ell^F(\tau)$, and s-numbers has been explained. Finally, we examine the idea of Kannan contraction mapping in its associated pre-quasi operator ideal. As well, the existence of a fixed point of Kannan contraction mapping has been introduced. We indicate the space of all bounded, finite rank linear operators from a Banach space Δ into a Banach space Λ by $\mathcal{L}(\Delta, \Lambda)$, and $\mathfrak{F}(\Delta, \Lambda)$ and if $\Delta = \Lambda$, we inscribe $\mathcal{L}(\Delta)$ and $\mathfrak{F}(\Delta)$.

Definition 3.36 ([23])**.** An s-number function is $s : \mathcal{L}(\Delta, \Lambda) \to \mathfrak{R}^{+\mathcal{N}}$, which sorts every $V \in \mathcal{L}(\Delta, \Lambda)$ as $(s_d(V))_{d=0}^\infty$ and verifies the following settings:

(a) $\|V\| = s_0(V) \geqslant s_1(V) \geqslant s_2(V) \geqslant \cdots \geqslant 0$, for all $V \in \mathcal{L}(\Delta, \Lambda)$;
(b) $s_{l+d-1}(V_1 + V_2) \leqslant s_l(V_1) + s_d(V_2)$, for all $V_1, V_2 \in \mathcal{L}(\Delta, \Lambda)$ and $l, d \in \mathcal{N}$;
(c) $s_d(VYW) \leqslant \|V\| s_d(Y) \|W\|$, for all $W \in \mathcal{L}(\Delta_0, \Delta)$, $Y \in \mathcal{L}(\Delta, \Lambda)$ and $V \in \mathcal{L}(\Lambda, \Lambda_0)$, where Δ_0 and Λ_0 are arbitrary Banach spaces;
(d) if $V \in \mathcal{L}(\Delta, \Lambda)$ and $\gamma \in \mathfrak{R}$, then $s_d(\gamma V) = |\gamma| s_d(V)$;
(e) suppose $\text{rank}(V) \leqslant d$, then $s_d(V) = 0$, for each $V \in \mathcal{L}(\Delta, \Lambda)$;
(f) $s_{l\geqslant a}(I_a) = 0$ or $s_{l<a}(I_a) = 1$, where I_a denotes the unit mapping on the a-dimensional Hilbert space ℓ_2^a.

Definition 3.37 ([22])**.** If $\mathcal{L}$ is the class of all bounded linear operators within any two arbitrary Banach spaces, a subclass $\mathcal{U}$ of $\mathcal{L}$ is said to be an operator ideal, if all $\mathcal{U}(\Delta, \Lambda) = \mathcal{U} \cap \mathcal{L}(\Delta, \Lambda)$ verify the following conditions:

(i) $I_\Gamma \in \mathcal{U}$, where Γ denotes Banach space of one dimension;
(ii) the space $\mathcal{U}(\Delta, \Lambda)$ is linear over $\mathfrak{R}$;
(iii) assume $W \in \mathcal{L}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\Delta, \Lambda)$ and $Y \in \mathcal{L}(\Lambda, \Lambda_0)$, then, $YXW \in \mathcal{U}(\Delta_0, \Lambda_0)$.

Notations 3.38.

$$\overline{\mathfrak{E}}_{\mathbf{U}} := \left\{\overline{\mathfrak{E}}_{\mathbf{U}}(\Delta, \Lambda)\right\}, \text{ where } \overline{\mathfrak{E}}_{\mathbf{U}}(\Delta, \Lambda) := \left\{V \in \mathcal{L}(\Delta, \Lambda) : ((\overline{s_d(V)})_{d=0}^{\infty} \in \mathbf{U}\right\},$$

where

$$\overline{s_d(V)}(x) = \begin{cases} 1, & x = s_d(V), \\ 0, & x \neq s_d(V). \end{cases}$$

Theorem 3.39. *Suppose* $\mathbf{U}$ *is a (cssf), then* $\overline{\mathfrak{E}}_{\mathbf{U}}$ *is an operator ideal.*

Proof.
(i) Assume $V \in \mathfrak{F}(\Delta, \Lambda)$ and $\operatorname{rank}(V) = n$ for all $n \in \mathbb{N}$, as $\overline{b}_i \in \mathbf{U}$ for all $i \in \mathbb{N}$ and $\mathbf{U}$ is a linear space, one has $(\overline{s_i(V)})_{i=0}^{\infty} = (\overline{s_0(V)}, \overline{s_1(V)}, \dots, \overline{s_{n-1}(V)}, \overline{0}, \overline{0}, \overline{0}, \dots) = \sum_{i=0}^{n-1} \overline{s_i(V)} \overline{b}_i \in \mathbf{U}$; for that $V \in \overline{\mathfrak{E}}_{\mathbf{U}}(\Delta, \Lambda)$, then $\mathfrak{F}(\Delta, \Lambda) \subseteq \overline{\mathfrak{E}}_E(\Delta, \Lambda)$.

(ii) Suppose $V_1, V_2 \in \overline{\mathfrak{E}}_{\mathbf{U}}(\Delta, \Lambda)$ and $\beta_1, \beta_2 \in \mathfrak{R}$ then by Definition 3.1 condition (3) one has $(\overline{s_{[\frac{i}{2}]}(V_1)})_{i=0}^{\infty} \in \mathbf{U}$ and $(\overline{s_{[\frac{i}{2}]}(V_1)})_{i=0}^{\infty} \in \mathbf{U}$, as $i \geqslant 2[\frac{i}{2}]$, by the definition of s-numbers and $\overline{s_i(V)}$ is a decreasing sequence of fuzzy numbers, one gets $\overline{s_i(\beta_1 V_1 + \beta_2 V_2)} \leqslant \overline{s_{2[\frac{i}{2}]}(\beta_1 V_1 + \beta_2 V_2)} \leqslant \overline{s_{[\frac{i}{2}]}(\beta_1 V_1) + s_{[\frac{i}{2}]}(\beta_2 V_2)} = |\beta_1| \overline{s_{[\frac{i}{2}]}(V_1)} + |\beta_2| \overline{s_{[\frac{i}{2}]}(V_2)}$ for each $i \in \mathbb{N}$. In view of Definition 3.1 condition (2) and $\mathbf{U}$ is a linear space, one obtains $(\overline{s_i(\beta_1 V_1 + \beta_2 V_2)})_{i=0}^{\infty} \in \mathbf{U}$, hence $\beta_1 V_1 + \beta_2 V_2 \in \overline{\mathfrak{E}}_{\mathbf{U}}(\Delta, \Lambda)$.

(iii) Suppose $P \in \mathcal{L}(\Delta_0, \Delta)$, $T \in \overline{\mathfrak{E}}_{\mathbf{U}}(\Delta, \Lambda)$ and $R \in \mathcal{L}(\Lambda, \Lambda_0)$, one has $(\overline{s_i(T)})_{i=0}^{\infty} \in \mathbf{U}$ and as $\overline{s_i(RTP)} \leqslant \|R\| \overline{s_i(T)}\, \|P\|$, by Definition 3.1 conditions (1) and (2) one gets $(\overline{s_i(RTP)})_{i=0}^{\infty} \in \mathbf{U}$, then $RTP \in \overline{\mathfrak{E}}_{\mathbf{U}}(\Delta_0, \Lambda_0)$. □

According to Theorems 3.7 and 3.39, one concludes the following theorem.

Theorem 3.40. *Let* $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$, *one has* $\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}$ *is an operator ideal.*

Definition 3.41 ([10])**.** A function $H \in [0, \infty)^{\mathcal{U}}$ is called a pre-quasi norm on the ideal $\mathcal{U}$ if the next conditions hold:

(1) let $V \in \mathcal{U}(\Delta, \Lambda)$, $H(V) \geqslant 0$ and $H(V) = 0$, if and only if, $V = 0$;
(2) we have $Q \geqslant 1$ so as to $H(\alpha V) \leqslant D|\alpha| H(V)$, for every $V \in \mathcal{U}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$;
(3) we have $P \geqslant 1$ so that $H(V_1 + V_2) \leqslant P[H(V_1) + H(V_2)]$, for each $V_1, V_2 \in \mathcal{U}(\Delta, \Lambda)$;
(4) we have $\sigma \geqslant 1$ for to if $V \in \mathcal{L}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\Delta, \Lambda)$ and $Y \in \mathcal{L}(\Lambda, \Lambda_0)$, then $H(YXV) \leqslant \sigma \|Y\| H(X)\, \|V\|$.

Theorem 3.42 ([10])**.** H *is a pre-quasi norm on the ideal* $\mathcal{U}$ *if* H *is a quasi norm on the ideal* $\mathcal{U}$.

Theorem 3.43. *If* $(\tau_q)_{q \in \mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$, *then the function* H *is a pre-quasi norm on* $\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}$, *with* $H(Z) = h(\overline{s_q(Z)})_{q=0}^{\infty}$, *for all* $Z \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$.

Proof.
(1) When $X \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$, $H(X) = h(\overline{s_q(X)})_{q=0}^{\infty} \geqslant 0$ and $H(X) = h(\overline{s_q(X)})_{q=0}^{\infty} = 0$, if and only if, $\overline{s_q(X)} = \overline{0}$, for all $q \in \mathcal{N}$, if and only if, $X = 0$.

(2) There is $Q \geqslant 1$ with $H(\alpha X) = h(\overline{s_q(\alpha X)})_{q=0}^{\infty} \leqslant Q|\alpha|H(X)$, for all $X \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$ and $\alpha \in \mathfrak{R}$.

(3) One has $PP_0 \geqslant 1$ so that for $X_1, X_2 \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$, one can see

$$
\begin{aligned}
H(X_1 + X_2) = h(\overline{s_q(X_1 + X_2)})_{q=0}^{\infty} &\leqslant P\left(h(\overline{s_{[\frac{q}{2}]}(X_1)})_{q=0}^{\infty} + h(\overline{s_{[\frac{q}{2}]}(X_2)})_{q=0}^{\infty}\right) \\
&\leqslant PP_0\left(h(\overline{s_q(X_1)})_{q=0}^{\infty} + h(\overline{s_q(X_2)})_{q=0}^{\infty}\right).
\end{aligned}
$$

(4) We have $\rho \geqslant 1$, if $X \in \mathcal{L}(\Delta_0, \Delta)$, $Y \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$ and $Z \in \mathcal{L}(\Lambda, \Lambda_0)$, then $H(ZYX) = h(\overline{s_q(ZYX)})_{q=0}^{\infty}$ $\leqslant h(\ \|X\|\ \|Z\|\overline{s_q(Y)})_{q=0}^{\infty} \leqslant \rho\ \|X\|H(Y)\ \|Z\|$. □

In the next theorems, we will use the notation $\left(\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}, H\right)$, where $H(V) = h\left((\overline{s_q(V)})_{q=0}^{\infty}\right)$, for all $V \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}$.

Theorem 3.44. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, one has $\left(\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}, H\right)$ is a pre-quasi Banach operator ideal.*

Proof. Suppose $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$. As $\mathcal{L}(\Delta, \Lambda) \supseteq S_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$, one has

$$
H(V_r - V_a) = h\left((\overline{s_q(V_r - V_a)})_{q=0}^{\infty}\right) \geqslant h\left(\overline{s_0(V_r - V_a)}, \overline{0}, \overline{0}, \overline{0}, \dots\right) \geqslant \sigma\ \|V_r - V_a\|,
$$

hence $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\mathcal{L}(\Delta, \Lambda)$. As $\mathcal{L}(\Delta, \Lambda)$ is a Banach space, so there exists $V \in \mathcal{L}(\Delta, \Lambda)$ so that $\lim_{a\to\infty}\ \|V_a - V\ \| = 0$ and since $(\overline{s_q(V_a)})_{q=0}^{\infty} \in \left(\ell^F(\tau)\right)_h$, for all $a \in \mathcal{N}$ and $\left(\ell^F(\tau)\right)_h$ is a pre-modular (cssf). Hence, one can see

$$
\begin{aligned}
H(V) = h\left((\overline{s_q(V)})_{q=0}^{\infty}\right) &\leqslant h\left((\overline{s_{[\frac{q}{2}]}(V - V_a)}\right)_{q=0}^{\infty}) + h\left((\overline{s_{[\frac{q}{2}]}(V_a)}_{q=0}^{\infty})\right) \\
&\leqslant h\left((\ \|V_a - V\ \|\overline{1})_{q=0}^{\infty}\right) + (2)^{\frac{1}{K}} h\left((\overline{s_q(V_a)})_{q=0}^{\infty}\right) < \varepsilon,
\end{aligned}
$$

we obtain $(\overline{s_q(V)})_{q=0}^{\infty} \in \left(\ell^F(\tau)\right)_h$, hence $V \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$. □

Theorem 3.45. *If $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, one has $\left(\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}, H\right)$ is a pre-quasi closed operator ideal.*

Proof. Suppose $V_a \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$, for all $a \in \mathcal{N}$ and $\lim_{a\to\infty} H(V_a - V) = 0$. Therefore, there is $\sigma > 0$ and as $\mathcal{L}(\Delta, \Lambda) \supseteq S_{\left(\ell^F(\tau)\right)_h}(\Delta, \Lambda)$, one has

$$
H(V_a - V) = h\left((\overline{s_q(V_a - V)})_{q=0}^{\infty}\right) \geqslant h\left(\overline{s_0(V_a - V)}, \overline{0}, \overline{0}, \overline{0}, \dots\right) \geqslant \sigma\|V_a - V\|.
$$

So $(V_a)_{a\in\mathcal{N}}$ is convergent in $\mathcal{L}(\Delta, \Lambda)$, i.e., $\lim_{a\to\infty}\ \|V_a - V\ \| = 0$ and since $(\overline{s_q(V_a)})_{q=0}^{\infty} \in \left(\ell^F(\tau)\right)_h$, for all $q \in \mathcal{N}$ and $\left(\ell^F(\tau)\right)_h$ is a pre-modular (cssf). Hence, one can see

$$
\begin{aligned}
H(V) = h\left((\overline{s_q(V)})_{q=0}^{\infty}\right) &\leqslant h\left((\overline{s_{[\frac{q}{2}]}(V - V_a)}\right)_{q=0}^{\infty}) + h\left((\overline{s_{[\frac{q}{2}]}(V_a)}_{q=0}^{\infty})\right) \\
&\leqslant h\left((\ \|V_a - V\ \|\overline{1})_{q=0}^{\infty}\right) + (2)^{\frac{1}{K}} h\left((\overline{s_q(V_a)})_{q=0}^{\infty}\right) < \varepsilon,
\end{aligned}
$$

we obtain $(\overline{s_q(V)})_{q=0}^{\infty} \in \Big(\ell^F(\tau)\Big)_h$, hence $V \in \overline{\mathfrak{E}}_{\big(\ell^F(\tau)\big)_h}(\Delta, \Lambda)$. □

Definition 3.46. A pre-quasi norm H on the ideal $\overline{\mathfrak{E}}_{\mathbf{U}_h}$ verifies the Fatou property if for every $\{T_q\}_{q\in\mathcal{N}} \subseteq \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda)$ so that $\lim_{q\to\infty} H(T_q - T) = 0$ and $M \in \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda)$, one gets

$$H(M - T) \leqslant \sup_q \inf_{j\geqslant q} H(M - T_j).$$

Theorem 3.47. *Suppose $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, then $\left(\overline{\mathfrak{E}}_{\big(\ell^F(\tau)\big)_h}, H\right)$ does not hold the Fatou property.*

Proof. Assume $\{T_q\}_{q\in\mathcal{N}} \subseteq \overline{\mathfrak{E}}_{\big(\ell^F(\tau)\big)_h}(\Delta, \Lambda)$ with $\lim_{q\to\infty} H(T_q - T) = 0$. Since $\overline{\mathfrak{E}}_{\big(\ell^F(\tau)\big)_h}$ is a pre-quasi closed ideal, then $T \in \overline{\mathfrak{E}}_{\big(\ell^F(\tau)\big)_h}(\Delta, \Lambda)$. So for every $M \in \overline{\mathfrak{E}}_{\big(\ell^F(\tau)\big)_h}(\Delta, \Lambda)$, one has

$$\begin{aligned}
H(M - T) &= \Big[\sum_{q=0}^{\infty}\Big(\overline{\rho}(\overline{s_q(M-T)}, \overline{0})\Big)^{\tau_q}\Big]^{\frac{1}{K}} \\
&\leqslant \Big[\sum_{q=0}^{\infty}\Big(\overline{\rho}(\overline{s_{[\frac{q}{2}]}(M-T_i)}, \overline{0})\Big)^{\tau_q}\Big]^{\frac{1}{K}} + \Big[\sum_{q=0}^{\infty}\Big(\overline{\rho}(\overline{s_{[\frac{q}{2}]}(T_i-T)}, \overline{0})\Big)^{\tau_q}\Big]^{\frac{1}{K}} \\
&\leqslant (2)^{\frac{1}{K}} \sup_r \inf_{j\geqslant r}\Big[\sum_{q=0}^{\infty}\Big(\overline{\rho}(\overline{s_q(M-T_j)}, \overline{0})\Big)^{\tau_q}\Big]^{\frac{1}{K}}.
\end{aligned}$$

□

Definition 3.48. An operator $V : \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda) \to \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda)$ is said to be a Kannan H-contraction, if one has $\alpha \in [0, \frac{1}{2})$ with $H(VT - VM) \leqslant \alpha\Big(H(VT - T) + H(VM - M)\Big)$, for all $T, M \in \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda)$.

Definition 3.49. An operator $V : \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda) \to \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda)$ is said to be H-sequentially continuous at M, where $M \in \overline{\mathfrak{E}}_{\mathbf{U}_h}(\Delta, \Lambda)$, if and only if, $\lim_{r\to\infty} H(T_r - M) = 0 \Rightarrow \lim_{r\to\infty} H(VT_r - VM) = 0$.

Example 3.50. If

$$V : \overline{\mathfrak{E}}_{\big(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\big)_h}(\Delta, \Lambda) \to \overline{\mathfrak{E}}_{\big(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\big)_h}(\Delta, \Lambda),$$

where $H(T) = \sqrt{\sum_{q=0}^{\infty}\Big(\overline{\rho}(\overline{s_q(T)}, \overline{0})\Big)^{\frac{2q+3}{q+2}}}$, for every $T \in \overline{\mathfrak{E}}_{\big(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\big)_h}(\Delta, \Lambda)$ and

$$V(T) = \begin{cases} \frac{T}{6}, & H(T) \in [0, 1), \\ \frac{T}{7}, & H(T) \in [1, \infty), \end{cases}$$

evidently, V is H-sequentially continuous at the zero operator $\Theta \in \overline{\mathfrak{E}}_{\big(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\big)_h}(\Delta, \Lambda)$. Let $\{T^{(j)}\} \subseteq \overline{\mathfrak{E}}_{\big(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\big)_h}(\Delta, \Lambda)$ be such that $\lim_{j\to\infty} H(T^{(j)} - T^{(0)}) = 0$, where $T^{(0)} \in \overline{\mathfrak{E}}_{\big(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\big)_h}(\Delta, \Lambda)$ with $H(T^{(0)}) = 1$. Since the pre-quasi norm H is continuous, one gets

$$\lim_{j\to\infty} H(VT^{(j)} - VT^{(0)}) = \lim_{j\to\infty} H\Big(\frac{T^{(0)}}{6} - \frac{T^{(0)}}{7}\Big) = H\Big(\frac{T^{(0)}}{42}\Big) > 0.$$

Therefore, V is not H-sequentially continuous at $T^{(0)}$.

Theorem 3.51. *Pick up* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$ *and* $V : \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda) \to \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$. *Assume*

(i) V *is Kannan* H*-contraction mapping;*
(ii) V *is* H*-sequentially continuous at an element* $M \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$;
(iii) *there are* $G \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$ *such that the sequence of iterates* $\{V^r G\}$ *has a* $\{V^{r_m} G\}$ *converging to* M.

Then $M \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$ *is the unique fixed point of* V.

Proof. Let M be not a fixed point of V, hence $VM \neq M$. By using parts (ii) and (iii), we get

$$\lim_{r_m\to\infty} H(V^{r_m}G - M) = 0 \text{ and } \lim_{r_m\to\infty} H(V^{r_m+1}G - VM) = 0.$$

Since V is Kannan H-contraction, one obtains

$$\begin{aligned}0 < H(VM - M) &= H\left((VM - V^{r_m+1}G) + (V^{r_m}G - M) + (V^{r_m+1}G - V^{r_m}G)\right)\\ &\leqslant (2)^{\frac{1}{K}} H\left(V^{r_m+1}G - VM\right) + (2)^{\frac{2}{K}} H\left(V^{r_m}G - M\right) + (2)^{\frac{2}{K}}\alpha\left(\frac{\alpha}{1-\alpha}\right)^{r_m-1} H(VG - G).\end{aligned}$$

As $r_m \to \infty$, there is a contradiction. Hence, M is a fixed point of V. To prove the uniqueness of the fixed point M, suppose one has two not equal fixed points $M, J \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$ of V. So, one gets $H(M - J) \leqslant H(VM - VJ) \leqslant \alpha\Big(H(VM - M) + H(VJ - J)\Big) = 0$. Then, $M = J$. □

Example 3.52. Because of Example 3.50, since for all $T_1, T_2 \in \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2q+3}{q+2}\right)_{q=0}^\infty\right)\right)_h}(\Delta,\Lambda)$ with $H(T_1), H(T_2) \in [0,1)$, we have

$$H(VT_1 - VT_2) = H\left(\frac{T_1}{6} - \frac{T_2}{6}\right) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(H\left(\frac{5T_1}{6}\right) + H\left(\frac{5T_2}{6}\right)\Big) = \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(H(VT_1 - T_1) + H(VT_2 - T_2)\Big).$$

For all $T_1, T_2 \in \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2q+3}{q+2}\right)_{q=0}^\infty\right)\right)_h}(\Delta,\Lambda)$ with $H(T_1), H(T_2) \in [1,\infty)$, we have

$$H(VT_1 - VT_2) = H\left(\frac{T_1}{7} - \frac{T_2}{7}\right) \leqslant \frac{\sqrt{2}}{\sqrt[4]{216}}\Big(H\left(\frac{6T_1}{7}\right) + H\left(\frac{6T_2}{7}\right)\Big) = \frac{\sqrt{2}}{\sqrt[4]{216}}\Big(H(VT_1 - T_1) + H(VT_2 - T_2)\Big).$$

For all $T_1, T_2 \in \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2q+3}{q+2}\right)_{q=0}^\infty\right)\right)_h}(\Delta,\Lambda)$ with $H(T_1) \in [0,1)$ and $H(T_2) \in [1,\infty)$, we have

$$H(VT_1 - VT_2) = H\left(\frac{T_1}{6} - \frac{T_2}{7}\right) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}} H\left(\frac{5T_1}{6}\right) + \frac{\sqrt{2}}{\sqrt[4]{216}} H\left(\frac{6T_2}{7}\right) \leqslant \frac{\sqrt{2}}{\sqrt[4]{125}}\Big(H(VT_1 - T_1) + H(VT_2 - T_2)\Big).$$

Hence, V is Kannan H-contraction and $V^r(T) = \begin{cases}\frac{T}{6^r}, & H(T) \in [0,1),\\ \frac{T}{7^r}, & H(T) \in [1,\infty).\end{cases}$ Obviously, V is H-sequentially continuous at $\Theta \in \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2q+3}{q+2}\right)_{q=0}^\infty\right)\right)_h}(\Delta,\Lambda)$ and $\{V^r T\}$ has a subsequence $\{V^{r_m} T\}$ that converges to Θ. By Theorem 3.51, Θ is the only fixed point of V.

4. Applications

In this section, we have introduced a solution in $\ell^F(\tau)$ with different pre-quasi functions and its associated pre-quasi operators ideal to non-linear uncertainty equation of fuzzy functions (4.1) and non-linear uncertainty matrix equation of fuzzy functions (4.6).

Throughout the next part of this article, we will use the two functions h_1 and h_2 as

$$h_1(\overline{Y}) = \Big[\sum_{j=0}^{\infty} \big(\overline{\rho}(\overline{Y_j},\overline{0})\big)^{\tau_j}\Big]^{\frac{1}{K}} \quad\text{and}\quad h_2(\overline{Y}) = \big(h_1(\overline{Y})\big)^K,$$

for every $\overline{Y} \in \ell^F(\tau)$.

Theorem 4.1. *Consider the summable equations:*

$$\overline{Y_q} = \overline{R_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\overline{Y_r}), \tag{4.1}$$

which has been presented by many authors [1, 12, 27], and assume $V : \left(\ell^F(\tau)\right)_{h_1} \to \left(\ell^F(\tau)\right)_{h_1}$, *where* $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$, *defined by*

$$V(\overline{Y_q})_{q\in\mathcal{N}} = \left(\overline{R_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\overline{Y_r})\right)_{q\in\mathcal{N}}. \tag{4.2}$$

The summable equation (4.1) *has a unique solution in* $\left(\ell^F(\tau)\right)_{h_1}$, *if* $D : \mathcal{N}^2 \to \mathfrak{R}$, $m : \mathcal{N}\times\mathfrak{R}[0,1] \to \mathfrak{R}[0,1]$, $\overline{R} : \mathcal{N} \to \mathfrak{R}[0,1]$, $\overline{Z} : \mathcal{N} \to \mathfrak{R}[0,1]$, *one has* $\varepsilon \in \mathfrak{R}$ *with* $\sup_q |\varepsilon|^{\frac{\tau_q}{K}} \in [0,\frac{1}{2})$ *and for all* $q \in \mathcal{N}$, *let*

$$\begin{aligned}
&\left|\sum_{r\in\mathcal{N}} D(q,r)(m(r,\overline{Y_r}) - m(r,\overline{Z_r}))\right|^{\tau_q}\\
&\quad\leqslant |\varepsilon| \left[\left|\overline{R_q} - \overline{Y_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\overline{Y_r})\right|^{\tau_q} + \left|\overline{R_q} - \overline{Z_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\overline{Z_r})\right|^{\tau_q}\right].
\end{aligned}$$

Proof. One has

$$\begin{aligned}
h_1(V\overline{Y} - V\overline{Z}) &= \left[\sum_{q\in\mathcal{N}} \big(\overline{\rho}(V\overline{Y_q} - V\overline{Z_q},\overline{0})\big)^{\tau_q}\right]^{\frac{1}{K}}\\
&= \left[\sum_{q\in\mathcal{N}} \left(\overline{\rho}\left(\sum_{r\in\mathcal{N}} D(q,r)[m(r,\overline{Y_r}) - m(r,\overline{Z_r})],\overline{0}\right)\right)^{\tau_q}\right]^{\frac{1}{K}}\\
&\leqslant \sup_q |\varepsilon|^{\frac{\tau_q}{K}} \left[\sum_{q\in\mathcal{N}} \left(\overline{\rho}\left(\overline{R_q} - \overline{Y_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\overline{Y_r}),\overline{0}\right)\right)^{\tau_q}\right]^{\frac{1}{K}}\\
&\quad + \sup_q |\varepsilon|^{\frac{\tau_q}{K}} \left[\sum_{q\in\mathcal{N}} \left(\overline{\rho}\left(\overline{R_q} - \overline{Z_q} + \sum_{r=0}^{\infty} D(q,r)m(r,\overline{Z_r}),\overline{0}\right)\right)^{\tau_q}\right]^{\frac{1}{K}}\\
&= \sup_q |\varepsilon|^{\frac{\tau_q}{K}} \left(h_1(V\overline{Y} - \overline{Y}) + h_1(V\overline{Z} - \overline{Z})\right).
\end{aligned}$$

By Theorem 3.24, one gets a unique solution of equation(4.1) in $\left(\ell^F(\tau)\right)_{h_1}$. ∎

Example 4.2. Suppose $\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$, where $h(\overline{Y}) = \sqrt{\sum_{q\in\mathcal{N}} \left(\overline{\rho}(\overline{Y_q},\overline{0})\right)^{\frac{2q+3}{q+2}}}$, for all $\overline{Y} \in \ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})$. Consider the summable equations:

$$\overline{Y_q} = \overline{R_q} + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\overline{Y_q}}{q^2+r^2+1}\right)^t, \tag{4.3}$$

with $q \geqslant 2$ and $t > 0$. Suppose $\Gamma = \left\{\overline{Y} \in \left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h : \overline{Y_0} = \overline{Y_1} = \overline{0}\right\}$. Indeed, Γ is a nonempty, h-convex, h-closed and h-bounded subset of $\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$. Let us define $V : \Gamma \to \Gamma$ as

$$V(\overline{Y_q})_{q\geqslant 2} = \left(\overline{R_q} + \sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\overline{Y_q}}{q^2+r^2+1}\right)^t\right)_{q\geqslant 2}.$$

Obviously,

$$\begin{aligned}&\left|\sum_{r=0}^{\infty}(-1)^q\left(\frac{\overline{Y_q}}{q^2+r^2+1}\right)^t\left((-1)^r-(-1)^r\right)\right|^{\frac{2q+3}{q+2}}\\ &\leqslant \frac{1}{\sqrt{2}}\left[\left|\overline{R_q}-\overline{Y_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\overline{Y_q}}{q^2+r^2+1}\right)^t\right|^{\frac{2q+3}{q+2}} + \left|\overline{R_q}-\overline{Z_q}+\sum_{r=0}^{\infty}(-1)^{q+r}\left(\frac{\overline{Z_q}}{q^2+r^2+1}\right)^t\right|^{\frac{2q+3}{q+2}}\right].\end{aligned}$$

By Theorem 4.1 and Corollary 3.34, the summable equations (4.3) have a solution in Γ.

Example 4.3. Suppose $\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_h$, where $h(\overline{Y}) = \sqrt{\sum_{q\in\mathcal{N}} \left(\overline{\rho}(\overline{Y_q},\overline{0})\right)^{\frac{2q+3}{q+2}}}$, for $\overline{Y} \in \ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})$. Consider the non-linear difference equations:

$$\overline{Y_q} = \overline{\sin(3q-1)} + \sum_{l=0}^{\infty} 7^{q+l}\frac{\overline{Y_{q-2}^r}}{\overline{Y_{q-1}^p}+l^2+1}, \tag{4.4}$$

with $r,p > 0$, $\overline{Y_{-2}}(x), \overline{Y_{-1}}(x) > 0$, for all $x \in \mathfrak{R}$, and assume $V : \ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty}) \to \ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})$, defined by

$$V(\overline{Y_q})_{q=0}^{\infty} = \left(\overline{\sin(3q-1)} + \sum_{l=0}^{\infty} 7^{q+l}\frac{\overline{Y_{q-2}^r}}{\overline{Y_{q-1}^p}+l^2+1}\right)_{q=0}^{\infty}. \tag{4.5}$$

Evidently,

$$\begin{aligned}\left|\sum_{l=0}^{\infty} 7^q\frac{\overline{Y_{q-2}^r}}{\overline{Y_{q-1}^p}+l^2+1}\left(7^l-7^l\right)\right|^{\frac{2q+3}{q+2}} &\leqslant \frac{1}{\sqrt{2}}\left|\overline{\sin(3q-1)}-\overline{Y_q}+\sum_{l=0}^{\infty}7^{q+l}\frac{\overline{Y_{q-2}^r}}{\overline{Y_{q-1}^p}+l^2+1}\right|^{\frac{2q+3}{q+2}}\\ &+\frac{1}{\sqrt{2}}\left|\overline{\sin(3q-1)}-\overline{Z_q}+\sum_{l=0}^{\infty}7^{q+l}\frac{\overline{Z_{q-2}^r}}{\overline{Z_{q-1}^p}+l^2+1}\right|^{\frac{2q+3}{q+2}}.\end{aligned}$$

By Theorem 4.1, the non-linear difference equations (4.4) have a unique solution in $\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})$.

Theorem 4.4. *Suppose* $V : \left(\ell^F(\tau)\right)_{h_2} \to \left(\ell^F(\tau)\right)_{h_2}$ *is defined by equation* (4.2) *and* $\tau_0 > 1$. *The non-linear uncertainty equation of fuzzy functions* (4.1) *has a unique solution* $\bar{l} \in \left(\ell^F(\tau)\right)_{h_2}$, *if the next setups are verified.*

(1) *If* $D:\mathcal{N}^2\to\mathfrak{R}$, $m:\mathcal{N}\times\mathfrak{R}[0,1]\to\mathfrak{R}[0,1]$, $\overline{R}:\mathcal{N}\to\mathfrak{R}[0,1]$, $\overline{Z}:\mathcal{N}\to\mathfrak{R}[0,1]$, *one has* $\varepsilon\in\mathfrak{R}$ *with* $2^{K-1}\sup_q|\varepsilon|^{\tau_q}\in[0,\frac{1}{2})$ *and for all* $q\in\mathcal{N}$, *let*

$$\left|\sum_{r\in\mathcal{N}}D(q,r)(m(r,\overline{Y_r})-m(r,\overline{Z_r}))\right|^{\tau_q}$$
$$\leqslant|\varepsilon|\left[\left|\overline{R_q}-\overline{Y_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\overline{Y_r})\right|^{\tau_q}+\left|\overline{R_q}-\overline{Z_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\overline{Z_r})\right|^{\tau_q}\right].$$

(2) V *is* h_2*-sequentially continuous at* $\bar{l}\in\left(\ell^F(\tau)\right)_{h_2}$.

(3) *There is* $\bar{i}\in\left(\ell^F(\tau)\right)_{h_2}$ *with* $\{V^p\bar{i}\}$ *which has* $\{V^{p_j}\bar{i}\}$ *converging to* $\bar{l}$.

Proof. We have

$$\begin{aligned}h_2(V\overline{Y}-V\overline{Z})&=\sum_{q\in\mathcal{N}}\left(\bar{\rho}(V\overline{Y_q}-V\overline{Z_q},\bar{0})\right)^{\tau_q}=\sum_{q\in\mathcal{N}}\left(\bar{\rho}\left(\sum_{r\in\mathcal{N}}D(q,r)[m(r,\overline{Y_r})-m(r,\overline{Z_r})],\bar{0}\right)\right)^{\tau_q}\\&\leqslant 2^{K-1}\sup_q|\varepsilon|^{\tau_q}\sum_{q\in\mathcal{N}}\left(\bar{\rho}\left(\overline{R_q}-\overline{Y_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\overline{Y_r}),\bar{0}\right)\right)^{\tau_q}\\&\quad+2^{K-1}\sup_q|\varepsilon|^{\tau_q}\sum_{q\in\mathcal{N}}\left(\bar{\rho}\left(\overline{R_q}-\overline{Z_q}+\sum_{r=0}^{\infty}D(q,r)m(r,\overline{Z_r}),\bar{0}\right)\right)^{\tau_q}\\&=2^{K-1}\sup_q|\varepsilon|^{\tau_q}\left(h_2(V\overline{Y}-\overline{Y})+h_2(V\overline{Z}-\overline{Z})\right).\end{aligned}$$

By Theorem 3.30, we obtain a unique solution $\bar{l}\in\left(\ell^F(\tau)\right)_{h_2}$ of equation (4.1). □

Example 4.5. Consider $\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_{h_2}$. Suppose the summable equations (4.4). Assume $V:\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_{h_2}\to\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_{h_2}$ is defined by equation (4.5). If V is h_2-sequentially continuous at $\bar{l}\in\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_{h_2}$, and there is $\bar{i}\in\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_{h_2}$ with $\{V^p\bar{i}\}$ which has $\{V^{p_j}\bar{i}\}$ converging to $\bar{l}$, obviously, one has $\varepsilon\in\mathfrak{R}$ with $2^{K-1}\sup_q|\varepsilon|^{\frac{2q+3}{q+2}}\in[0,\frac{1}{2})$ and for every $q\in\mathcal{N}$, we have

$$\left|\sum_{l=0}^{\infty}7^q\frac{\overline{Y_{q-2}^r}}{\overline{Y_{q-1}^p+l^2+1}}\left(7^l-7^l\right)\right|^{\frac{2q+3}{q+2}}$$
$$\leqslant\frac{1}{\sqrt{2}}\left[\left|\overline{\sin(3q-1)}-\overline{Y_q}+\sum_{l=0}^{\infty}7^{q+l}\frac{\overline{Y_{q-2}^r}}{\overline{Y_{q-1}^p+l^2+1}}\right|^{\frac{2q+3}{q+2}}+\left|\overline{\sin(3q-1)}-\overline{Z_q}+\sum_{l=0}^{\infty}7^{q+l}\frac{\overline{Z_{q-2}^r}}{\overline{Z_{q-1}^p+l^2+1}}\right|^{\frac{2q+3}{q+2}}\right].$$

According to Theorem 4.4, the non-linear uncertainty equation of fuzzy functions (4.4) has a unique solution $\bar{l}\in\left(\ell^F((\frac{2q+3}{q+2})_{q=0}^{\infty})\right)_{h_2}$.

We explain in this part a solution to non-linear matrix equations (4.6) at $B\in\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$, the setups of Theorem 3.7 are confirmed, and $H(T)=\left(\sum_{a=0}^{\infty}\left(\bar{\rho}(\overline{s_a(T)},\bar{0})\right)^{\tau_a}\right)^{\frac{1}{K}}$, for all $T\in\overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$.

Suppose the non-linear uncertainty equation of fuzzy functions:

$$\overline{s_a(T)} = \overline{s_a(A)} + \sum_{q=0}^{\infty} \Pi(a,q) f(q, \overline{s_q(T)}), \tag{4.6}$$

and assume $L : \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda) \to \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$ is defined by

$$L(T) = \left(\overline{s_a(A)} + \sum_{q=0}^{\infty} \Pi(a,q) f(q, \overline{s_q(T)}) \right) I. \tag{4.7}$$

Theorem 4.6. *The non-linear uncertainty equation of fuzzy functions (4.6) has a unique solution* $B \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$*, suppose the next setups are satisfied:*

(1) $\Pi : \mathcal{N}^2 \to \mathfrak{R}$, $f : \mathcal{N} \times \mathfrak{R}([0,1]) \to \mathfrak{R}([0,1])$, $A \in \mathcal{L}(\Delta,\Lambda)$, $U \in \mathcal{L}(\Delta,\Lambda)$, *and for all* $a \in \mathcal{N}$, *one has* ε *with* $\sup_a |\varepsilon|^{\frac{\tau_a}{K}} \in [0, 0.5)$ *and*

$$\left| \sum_{q\in\mathcal{N}} \Pi(a,q) \left(f(q,\overline{s_q(T)}) - f(q,\overline{s_q(U)}) \right) \right|$$
$$\leqslant |\varepsilon| \left[\left| \overline{s_a(A)} - \overline{s_a(T)} + \sum_{q\in\mathcal{N}} \Pi(a,q) f(q,\overline{s_q(T)}) \right| + \left| \overline{s_a(A)} - \overline{s_a(U)} + \sum_{q\in\mathcal{N}} \Pi(a,q) f(q,\overline{s_q(U)}) \right| \right];$$

(2) L *is* H*-sequentially continuous at a point* $B \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$;

(3) *one has* $Q \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$ *with* $\{L^p Q\}$ *having a subsequence* $\{L^{p_i} Q\}$ *converging to* B.

Proof. : Assume $L : \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda) \to \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$ is defined by equation (4.7). Hence

$$\begin{aligned}
H(LT - LU) &= \left[\sum_{a=0}^{\infty} \left(\bar{\rho}(\overline{s_a(T)} - \overline{s_a(U)}, \bar{0}) \right)^{\tau_a} \right]^{\frac{1}{K}} \\
&= \left[\sum_{a=0}^{\infty} \left(\bar{\rho} \left(\sum_{q\in\mathcal{N}} \Pi(a,q) \left(f(q,\overline{s_q(T)}) - f(q,\overline{s_q(U)}) \right), \bar{0} \right) \right)^{\tau_a} \right]^{\frac{1}{K}} \\
&\leqslant \sup_a |\varepsilon|^{\frac{\tau_a}{K}} \left[\sum_{a=0}^{\infty} \left(\bar{\rho} \left(\overline{s_a(A)} - \overline{s_a(T)} + \sum_{q\in\mathcal{N}} \Pi(a,q) f(q,\overline{s_q(T)}) \right), \bar{0} \right)^{\tau_a} \right]^{\frac{1}{K}} \\
&\quad + \sup_a |\varepsilon|^{\frac{\tau_a}{K}} \left[\sum_{a=0}^{\infty} \left(\bar{\rho} \left(\overline{s_a(A)} - \overline{s_a(U)} + \sum_{q\in\mathcal{N}} \Pi(a,q) f(q,\overline{s_q(U)}) \right), \bar{0} \right)^{\tau_a} \right]^{\frac{1}{K}} \\
&= \sup_a |\varepsilon|^{\frac{\tau_a}{K}} \left(H(LT - T) + H(LU - U) \right).
\end{aligned}$$

By Theorem 3.51, we have a unique solution of equation (4.6) at $B \in \overline{\mathfrak{E}}_{\left(\ell^F(\tau)\right)_h}(\Delta,\Lambda)$. □

Example 4.7. Consider $\overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2a+3}{a+2}\right)\right)\right)_h}(\Delta,\Lambda)$, where $H(G) = \sqrt{\sum_{a=0}^{\infty}\left(\overline{\rho}(\overline{s_a(G)},\overline{0})\right)^{\frac{2a+3}{a+2}}}$, for every $G \in \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2a+3}{a+2}\right)\right)\right)_h}(\Delta,\Lambda)$. Suppose the non-linear uncertainty equation of fuzzy functions:

$$\overline{s_z(G)} = \overline{\sin z} + \sum_{m=0}^{\infty} \frac{\cos(3m^2)\tanh(mz)\sinh^b|\overline{s_{z-2}(G)}|}{\sec^d|\overline{s_{z-1}(G)}| + \overline{\ln(m+z+1)} + \overline{1}}, \tag{4.8}$$

where $z \geqslant 2$ and $b, d > 0$ and assume $W : \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2a+3}{a+2}\right)\right)\right)_h}(\Delta,\Lambda) \to \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2a+3}{a+2}\right)\right)\right)_h}(\Delta,\Lambda)$ is defined as

$$W(G) = \left(\overline{\sin z} + \sum_{m=0}^{\infty} \frac{\cos(3m^2)\tanh(mz)\sinh^b|\overline{s_{z-2}(G)}|}{\sec^d|\overline{s_{z-1}(G)}| + \overline{\ln(m+z+1)} + \overline{1}}\right) I.$$

Let W be H-sequentially continuous at a point $D \in \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2a+3}{a+2}\right)\right)\right)_h}(\Delta,\Lambda)$, and there is $B \in \overline{\mathfrak{E}}_{\left(\ell^F\left(\left(\frac{2a+3}{a+2}\right)\right)\right)_h}(\Delta,\Lambda)$ so that the sequence of iterates $\{W^p B\}$ has a subsequence $\{W^{p_i} B\}$ converging to D. Obviously,

$$\begin{aligned} &\left|\sum_{m=0}^{\infty} \frac{\tanh(mz)\sinh^b|\overline{s_{z-2}(G)}|}{\sec^d|\overline{s_{z-1}(G)}| + \overline{\ln(m+z+1)} + \overline{1}}\Big(\cos(3m^2) - \cos(3m^2)\Big)\right| \\ &\leqslant \frac{1}{25}\left|\overline{\sin z} - \overline{s_z(G)} + \sum_{m=0}^{\infty} \frac{\cos(3m^2)\tanh(mz)\sinh^b|\overline{s_{z-2}(G)}|}{\sec^d|\overline{s_{z-1}(G)}| + \overline{\ln(m+z+1)} + \overline{1}}\right| \\ &\quad + \frac{1}{25}\left|\overline{\sin z} - \overline{s_z(T)} + \sum_{m=0}^{\infty} \frac{\cos(3m^2)\tanh(mz)\sinh^b|\overline{s_{z-2}(T)}|}{\sec^d|\overline{s_{z-1}(T)}| + \overline{\ln(m+z+1)} + \overline{1}}\right|. \end{aligned}$$

In view of Theorem 4.6, the non-linear uncertainty equation of fuzzy functions (4.8) has one solution D.

5. Conclusion

Rather than simply referring to a "quasi-normed" place, we used the term "pre-quasi-normed." The concept of a fixed point of the Kannan pre-quasi norm contraction mapping in the pre-quasi Banach (cssf). Pre-quasi-normal structure and (R) are supported. The Kannan non-expansive mapping's presence of a fixed point was investigated. A fixed point of Kannan contraction mapping in the pre-quasi Banach operator ideal produced by Nakano (cssf) and s-fuzzy numbers has also been examined. To put our findings to the test, we introduce several numerical experiments. In addition, various effective implementations of the stochastic non-linear dynamical system are discussed. The fixed points of any Kannan contraction and non-expansive mappings on this new fuzzy functions space, it's associated pre-quasi ideal, and a new general space of solutions for many stochastic non-linear dynamical systems are investigated.

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