J. Math. Computer Sci., 31 (2023), 188–196

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

On ω-almost-regularity and ω-semi-regularity in topological spaces

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Abstract

In this paper, we give many characterizations of ω-almost-regular topological spaces and show that ω-almost-regularity lies strictly between regularity and almost-regularity. Also, we give several sufficient conditions for the equivalence between "ω-almost-regularity" and "regularity", and between "ω-almost-regularity" and "almost-regularity." Moreover, we show that ω-almost-regularity is hereditary for certain classes of subspaces. Furthermore, we show that the product of two ω-almost-regular topological spaces is ω-almost-regular. In addition to these, we define ω-semi-regularity as a new topological property. With the help of examples, we study several relationships regarding ω-semi-regularity, in particular, we show that ω-semi-regularity is strictly weaker than each of ω-regularity and semi-regularity and that ω-regular Hausdorff topological spaces are ω-Urysohn.

Keywords: ω-openness, $R\omega$-openness, regularity; almost regularity, ω-regularity, semi-regularity.

2020 MSC: 54D15, 54B10.

1. Introduction and preliminaries

Topology is one of the most active fields in mathematics today. It is traditionally regarded as one of the three major branches of pure mathematics (together with algebra and analysis). Topology has recently become an important component of applied mathematics, with many mathematicians and scientists using topological notions to model and comprehend real-world structures and processes. General topology is the field of topology that deals with the fundamental set-theoretic concepts and constructs of topology. Most other fields of topology, such as differential topology, geometric topology, and algebraic topology, are built on it.

Generalized open sets and closed sets serve a significant role in general topology and are presently the focus of study for many topologists across the world. Indeed, a major subject in general topology and real analysis is the use of extended open sets to provide widely modified forms of continuity, separation axioms, compactness, connectedness, and so on. For example, Levine [18] introduced generalized closed (g-closed) sets in a topological space to extend many of the fundamental features of closed sets to a larger family. For example, it has been demonstrated that g-closed subsets inherit compactness, normalcy, and completeness in a uniform space.

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doi: 10.22436/jmcs.031.02.05

Received: 2023-01-22 Revised: 2023-03-07 Accepted: 2023-03-10

As a generalizations of closed sets and open sets, Hdeib [16] defined ω-closed sets and ω-open sets in which he proved that ω-closed sets inherits Lindelofness, and he characterized Lindelofness via ω-open sets. The work of Hdeib is continued by several researchers [2–15, 17, 20–22, 28, 29].

The concept of "ω-almost-regular" topological spaces was introduced in [9]. In this paper, we give many characterizations of ω-almost-regular topological spaces and show that ω-almost-regularity lies strictly between regularity and almost-regularity. Also, we give several sufficient conditions for the equivalence between "ω-almost-regularity" and "regularity", and between "ω-almost-regularity" and "almost-regularity." Moreover, we show that ω-almost-regularity is hereditary for certain classes of subspaces. Furthermore, we show that the product of two ω-almost-regular topological spaces is ω-almost-regular. In addition to these, we define ω-semi-regularity as a new topological property. With the help of examples, we study several relationships regarding ω-semi-regularity, in particular, we show that ω-semi-regularity is strictly weaker than each of ω-regularity and semi-regularity and that ω-regular Hausdorff topological spaces are ω-Urysohn.

Throughout this paper, TS will denote a topological space. Let (H, Θ) be a TS and let $D \subseteq H$. In this paper, the closure of D in (H, Θ) and the interior of in (H, Θ) will be denoted by $Cl_\Theta(D)$ and $Int_\Theta(D)$, respectively. Also, the family of closed subsets of (H, Θ) will be denoted by Θ^c.

The arrangement of this article is as follows. In Section 2, we characterize ω-almost-regular topological spaces in a variety of ways and demonstrate that ω-almost-regularity lies strictly between regularity and almost-regularity. We also provide a number of necessary conditions for the equivalence between "ω-almost-regularity" and "regularity," as well as between "ω-almost-regularity" and "almost-regularity." Moreover, we demonstrate that for specific classes of subspaces, ω-almost-regularity is hereditary. In addition, we demonstrate that the product of two ω-almost-regular topological spaces is ω-almost-regular. In Section 3, we introduce a new topological property called ω-semi-regularity. With the help of examples, we study several relationships involving ω-semi-regularity. In particular, we demonstrate that ω-semi-regularity is strictly weaker than each of ω-regularity and semi-regularity and that ω-regular Hausdorff topological spaces are ω-Urysohn. In Section 4, we summarize the main contributions and suggest some future work.

The following definitions will be used in the sequel.

Definition 1.1 ([16])**.** Let (H, Θ) be a TS and $D \subseteq H$. A point $h \in H$ is a condensation point of D if for each $E \in \Theta$ with $h \in E$, the set $E \cap D$ is uncountable. D is called an ω-closed set in (H, Θ) if it contains all its condensation points. Complements ω-closed sets in (H, Θ) are called ω-open sets in (H, Θ). The family of all ω-open sets in (H, Θ) will be denoted by Θ_ω.

Definition 1.2. Let (H, Θ) be a TS and $D \subseteq H$. Then

(a) [26] D is called a regular open set in (H, Θ) if $Int_\Theta(Cl_\Theta(D)) = D$, complements of regular open sets in (H, Θ) are called regular closed sets, the collection of all regular open sets in (H, Θ) (resp. regular closed sets in (H, Θ)) will be denoted by $RO(H, \Theta)$ (resp. $RC(H, \Theta)$);
(b) [20] D is called an $R\omega$-open set in (H, Θ) if $Int_\Theta(Cl_{\Theta_\omega}(D)) = D$, complements of $R\omega$-open sets in (H, Θ) are called $R\omega$-closed sets, the collection of all $R\omega$-open sets in (H, Θ) (resp. $R\omega$-closed sets in (H, Θ)) will be denoted by $R\omega O(H, \Theta)$ (resp. $R\omega C(H, \Theta)$).

Definition 1.3. Let (H, Θ) be a TS and $D \subseteq H$. Then

(a) [27] the set $\{h \in H : \text{ for each } E \in \Theta \text{ with } h \in E, Int_\Theta(Cl_\Theta(E)) \cap D \neq \emptyset\}$ is called the δ-closure of D in (H, Θ) is denoted by $Cl_\delta^\Theta(D)$; D is called δ-closed in (H, Θ) if $Cl_\delta^\Theta(D) = D$; D is called δ-open in (H, Θ) if $H - D$ is δ-closed in (H, Θ), the collection of all δ-open sets in (H, Θ) is denoted by Θ_δ;
(b) [21] the set $\{h \in H : \text{ for each } E \in \Theta \text{ with } h \in E, Int_\Theta(Cl_{\Theta_\omega}(E)) \cap D \neq \emptyset\}$ is called the δ_ω-closure of D in (H, Θ) is denoted by $Cl_{\delta_\omega}^\Theta(D)$; D is called δ_ω-open in (H, Θ) if $H - D$ is δ_ω-closed in (H, Θ), the collection of all δ_ω-open sets in (H, Θ) is denoted by Θ_{δ_ω}.

It is known that Θ_δ and Θ_{δ_ω} are topologies with $\Theta_\delta \subseteq \Theta_{\delta_\omega} \subseteq \Theta$.

Definition 1.4. A TS (H, Θ) is said to be

(a) [24] almost-regular if for each $h \in H$ and each $W \in RC(H, \Theta)$ such that $h \notin W$, there are $E, F \in \Theta$ such that $h \in E$, $W \subseteq F$, and $E \cap F = \emptyset$;
(b) [19] semi-regular if $RO(H, \Theta)$ forms a base for Θ;
(c) [1] ω-regular if for each $h \in H$ and each $W \in \Theta^c$ such that $h \notin W$, there are $E \in \Theta$ and $F \in \Theta_\omega$ such that $h \in E$, $W \subseteq F$, and $E \cap F = \emptyset$;
(d) [21] ω-almost-regular if for each $h \in H$ and each $W \in R\omega C(H, \Theta)$ such that $h \notin W$, there are $E \in \Theta$ and $E, F \in \Theta$ such that $h \in E$, $W \subseteq F$, and $E \cap F = \emptyset$;
(e) [21] ω-Urysohn if for any $x, y \in H$ such that $x \neq y$, there are $E, F \in \Theta$ such that $x \in E$, $y \in F$, and $Cl_{\Theta_\omega}(E) \cap Cl_{\Theta_\omega}(F) = \emptyset$;
(f) [23] locally countable if for each $h \in H$, there is $E \in \Theta$ such that $h \in E$ and E is countable;
(g) [14] anti-locally countable if each $E \in \Theta - \{\emptyset\}$ is uncountable.

2. ω-almost-regularity

In this section, we characterize ω-almost-regular topological spaces in a variety of ways and demonstrate that ω-almost-regularity lies strictly between regularity and almost-regularity. We also provide a number of necessary conditions for the equivalence between "ω-almost-regularity" and "regularity," as well as between "ω-almost-regularity" and "almost-regularity." Moreover, we demonstrate that for specific classes of subspaces, ω-almost-regularity is hereditary. In addition, we demonstrate that the product of two ω-almost-regular topological spaces is ω-almost-regular.

Theorem 2.1. *For any TS (H, Θ), the following are equivalent:*

(a) *(H, Θ) is ω-almost-regular;*
(b) *for each $h \in H$ and each $E \in R\omega O(H, \Theta)$ such that $h \in E$, there is $F \in \Theta$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$;*
(c) *for each $h \in H$ and each $E \in R\omega O(H, \Theta)$ such that $h \in E$, there is $F \in RO(H, \Theta)$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$;*
(d) *for each $h \in H$ and each $E \in R\omega O(H, \Theta)$ such that $h \in E$, there is $F \in R\omega O(H, \Theta)$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$;*
(e) *for each $h \in H$ and each $E \in \Theta$ such that $h \in E$, there is $F \in R\omega O(H, \Theta)$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq Int_\Theta(Cl_{\Theta_\omega}(E))$;*
(f) *for each $h \in H$ and each $E \in \Theta$ such that $h \in E$, there is $F \in \Theta$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq Int_\Theta(Cl_{\Theta_\omega}(E))$;*
(g) *for each $h \in H$ and each $W \in R\omega C(H, \Theta)$ such that $h \notin W$, there are $E, F \in \Theta$ such that $h \in E$, $W \subseteq F$, and $Cl_\Theta(E) \cap Cl_\Theta(F) = \emptyset$;*
(h) *for each $W \in R\omega C(H, \Theta)$, $W = \cap\{Cl_\Theta(T) : T \in \Theta$ and $W \subseteq T\}$;*
(i) *for each $W \in R\omega C(H, \Theta)$, $W = \cap\{X : X \in \Theta^c$ and $W \subseteq Int_\Theta(X)\}$;*
(j) *for each $A \subseteq H$ and each $B \in R\omega O(H, \Theta)$ such that $A \cap B \neq \emptyset$, there is $E \in \Theta$ such that $A \cap E \neq \emptyset$ and $Cl_\Theta(E) \subseteq B$;*
(k) *for each non-empty subset $A \subseteq H$ and each $B \in R\omega C(H, \Theta)$ such that $A \cap B = \emptyset$, there are $E, F \in \Theta$ such that $A \cap E \neq \emptyset$ and $B \subseteq F$.*

Proof.

(a) $\longrightarrow$ (b): Let $h \in H$ and let $E \in R\omega O(H, \Theta)$ such that $h \in E$. Then $h \notin H - E \in R\omega C(H, \Theta)$ and by (a), there are $F, T \in \Theta$ such that $h \in F$, $H - E \subseteq T$, and $F \cap T = \emptyset$. So, we have $h \in F \subseteq H - T \subseteq E$ with $H - T \in \Theta^c$, and hence $h \in F \subseteq Cl_\Theta(F) \subseteq H - T \subseteq E$. This ends the proof.

(b) $\longrightarrow$ (c): Let $h \in H$ and let $E \in R\omega O(H, \Theta)$ such that $h \in E$. Then by (b), there is $S \in \Theta$ such that $h \in S \subseteq Cl_\Theta(S) \subseteq E$. Put $F = Int_\Theta(Cl_\Theta(S))$. Then $F \in RO(H, \Theta)$. Since $F \subseteq Cl_\Theta(S) \subseteq E$, then $Cl_\Theta(F) \subseteq Cl_\Theta(S) \subseteq E$. This ends the proof.

(c) $\longrightarrow$ (d): Let $h \in H$ and let $E \in R\omega O(H,\Theta)$ such that $h \in E$. Then by (c), there is $F \in RO(H,\Theta)$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$. Since $F \in RO(H,\Theta)$ and by Theorem 3.1 of [20], $RO(H,\Theta) \subseteq R\omega O(H,\Theta)$, then $F \in R\omega O(H,\Theta)$. This ends the proof.

(d) $\longrightarrow$ (e): Let $h \in H$ and let $E \in \Theta$ such that $h \in E$. Since by Theorem 3.1 of [20], $Int_\Theta(Cl_{\Theta_\omega}(E)) \in R\omega O(H,\Theta)$, then by (d), there is $F \in R\omega O(H,\Theta)$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq Int_\Theta(Cl_{\Theta_\omega}(E))$. This ends the proof.

(e) $\longrightarrow$ (f): Let $h \in H$ and let $E \in \Theta$ such that $h \in E$. Then by (e), there is $F \in R\omega O(H,\Theta)$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq Int_\Theta(Cl_{\Theta_\omega}(E))$. Since $R\omega O(H,\Theta) \subseteq \Theta$, then $F \in \Theta$. This ends the proof.

(f) $\longrightarrow$ (g): Let $h \in H$ and $W \in R\omega C(H,\Theta)$ such that $h \notin W$. Then $h \in H - W \in R\omega O(H,\Theta) \subseteq \Theta$. Thus, by (f), there is $F \in \Theta$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq Int_\Theta(Cl_{\Theta_\omega}(H-W)) = H - W$. Again by (f), there is $S \in \Theta$ such that $h \in S \subseteq Cl_\Theta(S) \subseteq Int_\Theta(Cl_{\Theta_\omega}(F)) \subseteq Cl_\Theta(F) \subseteq H - W$. Let $T = H - Cl_\Theta(F)$. Then $S, T \in \Theta$ and $h \in S$. Since $Cl_\Theta(F) \subseteq H - W$, then $W \subseteq H - Cl_\Theta(F) = T$.

Claim. $Cl_\Theta(S) \cap Cl_\Theta(T) = \emptyset$.

Proof of Claim. Suppose to the contrary that there is $r \in Cl_\Theta(S) \cap Cl_\Theta(T) = \emptyset$. Since $r \in Cl_\Theta(T)$ and $r \in Cl_\Theta(S) \subseteq Int_\Theta(Cl_{\Theta_\omega}(F)) \in \Theta$, then $Int_\Theta(Cl_{\Theta_\omega}(F)) \cap T \neq \emptyset$. Since $Int_\Theta(Cl_{\Theta_\omega}(F)) \subseteq Cl_\Theta(F)$, then $Cl_\Theta(F) \cap T = Cl_\Theta(F) \cap (H - Cl_\Theta(F)) \neq \emptyset$, a contradiction. The Claim ends the proof.

(g) $\longrightarrow$ (h): Let $W \in R\omega C(H,\Theta)$. Then for each $r \in H - W$, there are $S_r, T_r \in \Theta$ such that $r \in S_r$, $W \subseteq T_r$, and $Cl_\Theta(S_r) \cap Cl_\Theta(T_r) = \emptyset$. Thus, $W \subseteq T_r$ and $r \notin Cl_\Theta(T_r)$.

Claim. $W = \cap\{Cl_\Theta(T_r) : r \in H - W\}$.

Proof of Claim. For each $r \in H - W$, we have $W \subseteq T_r \subseteq Cl_\Theta(T_r)$ and thus, $W \subseteq \cap\{Cl_\Theta(T_r) : r \in H - W\}$. To see that $\cap\{Cl_\Theta(T_r) : r \in H - W\} \subseteq W$, let $r \in H - W$, then $r \notin Cl_\Theta(T_r)$, so, $r \notin \cap\{Cl_\Theta(T_r) : r \in H - W\}$.

By the above Claim, we conclude that $W \subseteq \cap\{Cl_\Theta(T) : T \in \Theta$ with $W \subseteq T\} \subseteq \cap\{Cl_\Theta(T_r) : r \in H - W\} = W$. This ends the proof.

(h) $\longrightarrow$ (i): Obvious.

(i) $\longrightarrow$ (j): Let $A \subseteq H$ and $B \in R\omega O(H,\Theta)$ such that $A \cap B \neq \emptyset$. Choose $r \in A \cap B$. Since $B \in R\omega O(H,\Theta)$, then $H - B \in R\omega C(H,\Theta)$ and by (i), $H - B = \cap\{X : X \in \Theta^c$ with $H - B \subseteq Int_\Theta(X)\}$. Since $r \in B$, then $r \notin \cap\{X : X \in \Theta^c$ and $H - B \subseteq Int_\Theta(X)\}$ and so there is $X \in \Theta^c$ such that $H - B \subseteq Int_\Theta(X)$ and $r \notin X$. Let $E = H - X$. Then $E \in \Theta$, $E \subseteq H - Int_\Theta(X) \subseteq B$, and $r \in E \cap A$. Since $H - Int_\Theta(X) \in \Theta^c$ and $E \subseteq H - Int_\Theta(X) \subseteq B$, then $Cl_\Theta(E) \subseteq B$. This ends the proof.

(j) $\longrightarrow$ (k): Let A be a non-empty subset of H and $B \in R\omega C(H,\Theta)$ such that $A \cap B = \emptyset$. Then $H - B \in R\omega O(H,\Theta)$ such that $A \cap (H - B) = A \neq \emptyset$. So, by (j), there is $E \in \Theta$ such that $A \cap E \neq \emptyset$ and $Cl_\Theta(E) \subseteq H - B$. Let $F = H - Cl_\Theta(E)$. Then $F \in \Theta$, $B \subseteq F$, and $E \cap (H - Cl_\Theta(E)) = \emptyset$.

(k) $\longrightarrow$ (a): Let $h \in H$ and $W \in R\omega C(H,\Theta)$ such that $h \notin W$. Then $\{h\} \cap W = \emptyset$ and by (k), there are $E, F \in \Theta$ such that $\{h\} \cap E \neq \emptyset$, $W \subseteq F$, and $E \cap F = \emptyset$. Since $\{h\} \cap E \neq \emptyset$, then $h \in E$. This ends the proof. □

Theorem 2.2. *Every regular TS is ω-almost-regular.*

Proof. Let (H,Θ) be regular. Let $h \in H$ and $E \in R\omega O(H,\Theta)$ such that $h \in E$. Since $R\omega O(H,\Theta) \subseteq \Theta$, then $E \in \Theta$. Since (H,Θ) is regular, then there is $F \in \Theta$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$. Thus, by Theorem 2.1 (b), (H,Θ) is ω-almost-regular. □

Theorem 2.3. *Every ω-almost-regular TS is almost-regular.*

Proof. Let (H,Θ) be ω-almost-regular. Let $h \in H$ and $E \in RO(H,\Theta)$ such that $h \in E$. Since $RO(H,\Theta) \subseteq R\omega O(H,\Theta)$, then $E \in R\omega O(H,\Theta)$. Since (H,Θ) is ω-almost-regular, then by Theorem 2.1 (b), there is $F \in \Theta$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$. Thus, by Theorem 2.2 of [24], (H,Θ) is almost-regular. □

From the above theorems, we have following implications, however, Examples 2.7 and 2.8 given below show that the converses of these implications are not true.

$$\text{regularity} \longrightarrow \omega\text{-almost-regularity} \longrightarrow \text{almost-regularity}$$

Lemma 2.4.

(a) *For any locally countable TS* (H, Θ), $R\omega O(H, \Theta) = \Theta$.
(b) *For any anti-locally countable TS* (H, Θ), $RO(H, \Theta) = R\omega O(H, \Theta)$.
(c) *For any TS* (H, Θ), $R\omega O(H, \Theta_\omega) = RO(H, \Theta_\omega)$.

Proof.

(a) Let (H, Θ) be a locally countable TS. We will show that $\Theta \subseteq R\omega O(H, \Theta)$. Let $F \in \Theta$. Since (H, Θ) is locally countable, then Θ_ω is the discrete topology on H and so $Cl_{\Theta_\omega}(F) = F$. Thus, $Int_\Theta(Cl_{\Theta_\omega}(F)) = Int_\Theta(F) = F$. Hence, $F \in R\omega O(H, \Theta)$.

(b) Let (H, Θ) be an anti-locally countable TS. We will show that $R\omega O(H, \Theta) \subseteq RO(H, \Theta)$. Let $F \in R\omega O(H, \Theta)$. Then $Int_\Theta(Cl_{\Theta_\omega}(F)) = F$. Since (H, Θ) is anti-locally countable, then as proved in [1], $Cl_{\Theta_\omega}(F) = Cl_\Theta(F)$. Thus, $Int_\Theta(Cl_\Theta(F)) = Int_\Theta(Cl_{\Theta_\omega}(F)) = F$. Hence, $F \in RO(H, \Theta)$.

(c) Let (H, Θ) be a TS. We will show that $R\omega O(H, \Theta_\omega) \subseteq RO(H, \Theta_\omega)$. Let $F \in R\omega O(H, \Theta_\omega)$. Then $F = Int_{\Theta_\omega}\left(Cl_{(\Theta_\omega)_\omega}(F)\right)$. Since by Proposition 3.1 of [14], $(\Theta_\omega)_\omega = \Theta_\omega$, then $F = Int_{\Theta_\omega}\left(Cl_{(\Theta_\omega)_\omega}(F)\right) = Int_{\Theta_\omega}(Cl_{\Theta_\omega}(F))$. Hence, $F \in R\omega O(H, \Theta_\omega)$. □

Theorem 2.5. *Every locally countable ω-almost-regular TS is regular.*

Proof. Let (H, Θ) be locally countable and ω-almost-regular. Let $h \in H$ and $E \in \Theta$ such that $h \in E$. Since (H, Θ) is locally countable, then by Lemma 2.4 (a), $E \in R\omega O(H, \Theta)$. Since (H, Θ) is ω-almost-regular, then by Theorem 2.1 (a), there is $F \in \Theta$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$. Therefore, (H, Θ) is regular. □

Theorem 2.6. *Every anti-locally countable almost-regular TS is ω-almost-regular.*

Proof. Let (H, Θ) be anti-locally countable and almost-regular. Let $h \in H$ and $E \in R\omega O(H, \Theta)$ such that $h \in E$. Since (H, Θ) is anti-locally countable, then by Lemma 2.4 (b), $E \in RO(H, \Theta)$. Since (H, Θ) is almost-regular, then by Theorem 2.2 of [24], there is $F \in \Theta$ such that $h \in F \subseteq Cl_\Theta(F) \subseteq E$. Therefore, by Theorem 2.1 (a), (H, Θ) is ω-almost-regular. □

The condition 'locally countable' in Theorem 2.5 cannot be dropped:

Example 2.7. Let $H = \mathbb{R}$ and Θ be the cofinite topology on $\mathbb{R}$. Then (H, Θ) is not regular. On the other hand, since (H, Θ) is anti-locally countable, then by Lemma 2.4 (b), $R\omega O(H, \Theta) = RO(H, \Theta) = \{\emptyset, H\}$, and hence, (H, Θ) is ω-almost-regular.

The condition 'anti-locally countable' in Theorem 2.6 cannot be dropped.

Example 2.8. Let $H = \mathbb{N}$ and Θ be the cofinite topology on H. Then (H, Θ) is not regular. So, by Theorem 2.5, (H, Θ) is not ω-almost-regular. On the other hand, since $RO(H, \Theta) = \{\emptyset, H\}$, then (H, Θ) is almost-regular.

Theorem 2.9. *For any TS (H, Θ), (H, Θ_ω) is almost-regular if and only if (H, Θ_ω) is ω-almost-regular.*

Proof. Follows from the definitions and Lemma 2.4 (c). □

Lemma 2.10. *Let (H, Θ) be a TS. If X is a dense subset of (H, Θ_ω), then for any subset $A \subseteq X$,* $Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(A)) = Int_\Theta(Cl_{\Theta_\omega}(A)) \cap X$.

Proof. Suppose that X is a dense subset of (H, Θ_ω) and let $A \subseteq X$. To see that $Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(A)) \subseteq Int_\Theta(Cl_{\Theta_\omega}(A)) \cap X$, let $r \in Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(A))$. Since $Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(A)) \in \Theta_X$, then there is $E \in \Theta$ such that $Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(A)) = E \cap X$. Thus, we have $r \in E \cap X \subseteq Cl_{(\Theta_\omega)_X}(A) = (Cl_{\Theta_\omega}(A)) \cap X$.

Claim. $E \subseteq Cl_{\Theta_\omega}(A)$.

Proof of Claim. Suppose to the contrary that $E \cap (H - Cl_{\Theta_\omega}(A)) \neq \emptyset$. Since $H - Cl_{\Theta_\omega}(A) \in \Theta_\omega$ and $E \in \Theta \subseteq \Theta_\omega$, then $E \cap (H - Cl_{\Theta_\omega}(A)) \in \Theta_\omega$. Since X is a dense subset of (H, Θ_ω), then $E \cap (H - Cl_{\Theta_\omega}(A)) \cap X \neq \emptyset$. Choose $y \in E \cap (H - Cl_{\Theta_\omega}(A)) \cap X$. Then we have $y \in H - Cl_{\Theta_\omega}(A)$ and $y \in E \cap X \subseteq (Cl_{\Theta_\omega}(A)) \cap X \subseteq Cl_{\Theta_\omega}(A)$, a contradiction.

Thus, by the above Claim, we must have $r \in E \subseteq Cl_{\Theta_\omega}(A)$ and hence $r \in Int_\Theta(Cl_{\Theta_\omega}(A)$. Therefore, $r \in Int_\Theta(Cl_{\Theta_\omega}(A)) \cap X$. To see that $Int_\Theta(Cl_{\Theta_\omega}(A)) \cap X \subseteq Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(A))$, let $r \in Int_\Theta(Cl_{\Theta_\omega}(A)) \cap X$. Since $r \in Int_\Theta(Cl_{\Theta_\omega}(A)) \in \Theta$, then there is $E \in \Theta$ such that $r \in E \subseteq Cl_{\Theta_\omega}(A)$ and so $r \in E \cap X \subseteq Cl_{\Theta_\omega}(A) \cap X = Cl_{(\Theta_\omega)_X}(A)$. Since $U \cap X \in \Theta_\omega$, then $r \in Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(A))$. □

Theorem 2.11. *If (H, Θ) is an ω-almost-regular TS and X is a dense subspace of (H, Θ_ω), then (X, Θ_X) is ω-almost-regular.*

Proof. Let $G \in RO(X, \Theta_X)$ and let $r \in G$. Since $G \in RO(X, \Theta_X)$, then $Int_{\Theta_X}(Cl_{(\Theta_X)_\omega}(G)) = G$. Since by Proposition 2.7 of [14] $(\Theta_\omega)_X = (\Theta_X)_\omega$, then $Int_{\Theta_X}(Cl_{(\Theta_\omega)_X}(G)) = G$. So, by Lemma 2.10, $G = Int_\Theta(Cl_{\Theta_\omega}(G)) \cap X$. Thus, we have $r \in Int_\Theta(Cl_{\Theta_\omega}(G)) \in R\omega O(H, \Theta)$. Since (H, Θ) is ω-almost-regular, then by Theorem 2.1 (b), there is $F \in \Theta$ such that $r \in F \subseteq Cl_\Theta(F) \subseteq Int_\Theta(Cl_{\Theta_\omega}(G))$. Therefore, we have $r \in F \cap X \in \Theta_X$ and $Cl_{\Theta_X}(F) = Cl_\Theta(F) \cap X \subseteq Int_\Theta(Cl_{\Theta_\omega}(G)) \cap X = G$. It follows that (X, Θ_X) is ω-almost-regular. □

Lemma 2.12. *Let (H, Θ) be a TS and let $X \in R\omega O(H, \Theta) - \{\emptyset\}$, then $R\omega O(X, \Theta_X) \subseteq R\omega O(H, \Theta)$.*

Proof. Let $X \in R\omega O(H, \Theta) - \{\emptyset\}$ and let $A \in R\omega O(X, \Theta_X)$. Then $A = Int_{\Theta_X}(Cl_{(\Theta_X)_\omega}(A))$. Since by Proposition 2.7 of [14], $(\Theta_\omega)_X = (\Theta_X)_\omega$, then $Cl_{(\Theta_X)_\omega}(A) = Cl_{(\Theta_\omega)_X}(A) = Cl_{\Theta_\omega}(A) \cap X$. Since $r \in R\omega O(H, \Theta) \subseteq \Theta$, then $Int_{\Theta_X}(Cl_{(\Theta_X)_\omega}(A)) = Int_\Theta((Cl_{(\Theta_X)_\omega}(A)))$. Thus, $A = Int_\Theta(Cl_{\Theta_\omega}(A) \cap X) = Int_\Theta(Cl_{\Theta_\omega}(A)) \cap Int_\Theta(X) = Int_\Theta(Cl_{\Theta_\omega}(A)) \cap X$. Since $A \subseteq X$, then $Int_\Theta(Cl_{\Theta_\omega}(A)) \subseteq Int_\Theta(Cl_{\Theta_\omega}(X)) = X$ and thus, $Int_\Theta(Cl_{\Theta_\omega}(A)) \cap X = Int_\Theta(Cl_{\Theta_\omega}(A))$. It follows that $A = Int_\Theta(Cl_{\Theta_\omega}(A))$. Hence, $A \in R\omega O(H, \Theta)$. □

Theorem 2.13. *If (H, Θ) is an ω-almost-regular TS and $X \in R\omega O(H, \Theta) - \{\emptyset\}$, then (X, Θ_X) is ω-almost-regular.*

Proof. Let $G \in RO(X, \Theta_X)$ and let $r \in G$. Since $G \in RO(X, \Theta_X)$, then Lemma 2.12, $G \in R\omega O(H, \Theta)$. Since (H, Θ) is ω-almost-regular, then by Theorem 2.1 (b), there is $F \in \Theta$ such that $r \in F \subseteq Cl_\Theta(F) \subseteq G$. Therefore, we have $r \in F \cap X \in \Theta_X$ and $Cl_{\Theta_X}(F) = Cl_\Theta(F) \cap X \subseteq G$. It follows that (X, Θ_X) is ω-almost-regular. □

Theorem 2.14. *The product of two ω-almost-regular TSs is ω-almost-regular.*

Proof. Let (X, Θ) and (Z, λ) be two ω-almost-regular TSs. Let $(x, y) \in X \times Z$ and let $G \in R\omega O(X \times Z, \Theta \times \lambda)$ such that $(x, y) \in G$. Then $G \in (\Theta \times \lambda)_{\delta_\omega}$ and by Theorem 2.17 of [9], $G \in \Theta_{\delta_\omega} \times \lambda_{\delta_\omega}$. Thus, there are $E \in \Theta_{\delta_\omega}$ and $F \in \lambda_{\delta_\omega}$ such that $(x, y) \in E \times F \subseteq G$. Choose $A \in R\omega O(X, \Theta)$ and $B \in R\omega O(Z, \lambda)$ such that $(x, y) \in A \times B \subseteq E \times F \subseteq G$. So, by Theorem 2.1 (b), there are $S \in \Theta$ and $T \in \lambda$ such that $x \in S \subseteq Cl_\Theta(S) \subseteq A$ and $y \in T \subseteq Cl_\lambda(T) \subseteq B$. Therefore, we have $S \times T \in \Theta \times \lambda$ and $(x, y) \in S \times T \subseteq Cl_{\Theta \times \lambda}(S \times T) = Cl_\Theta(S) \times Cl_\lambda(T) \subseteq A \times B \subseteq E \times F \subseteq G$. It follows that $(X \times Z, \Theta \times \lambda)$ is ω-almost-regular. □

3. ω-semi-regularity

In this section, we introduce a new topological property called ω-semi-regularity. With the help of examples, we study several relationships involving ω-semi-regularity. In particular, we demonstrate that ω-semi-regularity is strictly weaker than each of ω-regularity and semi-regularity and that ω-regular Hausdorff topological spaces are ω-Urysohn.

Definition 3.1. A TS (H, Θ) is said to be ω-semi-regular if $R\omega O(H, \Theta)$ forms abase for Θ.

Theorem 3.2. *For any TS (H, Θ), the following are equivalent.*

(a) *(H, Θ) is ω-semi-regular.*
(b) *For each $E \in \Theta - \{\emptyset\}$ and each $r \in E$, there is $F \in \Theta$ such that $r \in F \subseteq Int_{\Theta}(Cl_{\Theta_{\omega}}(F)) \subseteq E$.*
(c) *$\Theta_{\delta_{\omega}} = \Theta$.*

Proof.
(a) $\longrightarrow$ (b): Let $E \in \Theta - \{\emptyset\}$ and let $r \in E$. By (a), there is $F \in R\omega O(H, \Theta)$ such that $r \in F = Int_{\Theta}(Cl_{\Theta_{\omega}}(F)) \subseteq E$.

(b) $\longrightarrow$ (c): By Theorem 2.6 of [9], we have $\Theta_{\delta_{\omega}} \subseteq \Theta$. To see that $\Theta \subseteq \Theta_{\delta_{\omega}}$, let $E \in \Theta - \{\emptyset\}$, then for each $r \in E$, there is $F_r \in \Theta$ such that $r \in F_r \subseteq Int_{\Theta}(Cl_{\Theta_{\omega}}(F_r)) \subseteq E$. Put $F = F = \cup_{r \in E} Int_{\Theta}(Cl_{\Theta_{\omega}}(F_r))$. Since for each $r \in E$, $Int_{\Theta}(Cl_{\Theta_{\omega}}(F_r)) \in R\omega O(H, \Theta) \subseteq \Theta_{\delta_{\omega}}$, then $F \in \Theta_{\delta_{\omega}}$.

(c) $\longrightarrow$ (a): Since $R\omega O(H, \Theta)$ is a base for $\Theta_{\delta_{\omega}}$ and by (c), $\Theta_{\delta_{\omega}} = \Theta$, then $R\omega O(H, \Theta)$ is a base for Θ. Hence, (H, Θ) is ω-semi-regular. □

Theorem 3.3. *Every semi-regular TS is ω-semi-regular.*

Proof. Let (H, Θ) be semi-regular. Then $\Theta_{\delta} = \Theta$. Thus, by Theorem 2.6 of [9], we have $\Theta = \Theta_{\delta} \subseteq \Theta_{\delta_{\omega}} \subseteq \Theta$, and hence $\Theta_{\delta_{\omega}} = \Theta$. Therefore, by Theorem 3.2, (H, Θ) is ω-semi-regular. □

Theorem 3.4. *Every anti-locally countable ω-semi-regular TS is semi-regular.*

Proof. Let (H, Θ) be anti-locally countable and ω-semi-regular. Since (H, Θ) is ω-semi-regular, then $R\omega O(H, \Theta)$ is a base for Θ. Since (H, Θ) is anti-locally countable, then by Lemma 2.4 (b), $RO(H, \Theta) = R\omega O(H, \Theta)$. Thus, $RO(H, \Theta)$ is a base for Θ. Hence, (H, Θ) is semi-regular. □

Theorem 3.5. *Every ω-regular TS is ω-semi-regular.*

Proof. Let (H, Θ) be ω-regular. Let $E \in \Theta - \{\emptyset\}$ and let $r \in E$. Then there is $F \in \Theta$ such that $r \in F \subseteq Cl_{\Theta_{\omega}}(F) \subseteq E$. Hence, we have $r \in F = Int_{\Theta}(F) \subseteq Int_{\Theta}(Cl_{\Theta_{\omega}}(F)) \subseteq Int_{\Theta}(E) = E$. Therefore, by Theorem 3.2, (H, Θ) be ω-semi-regular. □

From the above theorems, we have following implications, however, Examples 3.8 and 3.9 given below show that the converses of these implications are not true.

$$\begin{array}{ccc} \text{semi-regularity} & \longrightarrow & \omega\text{-semi-regularity} \\ & & \uparrow \\ & & \omega\text{-regularity} \end{array}$$

Theorem 3.6. *Every locally countable TS is ω-semi-regular.*

Proof. Let (H, Θ) be locally countable. Then by Lemma 2.4 (a), $R\omega O(H, \Theta) = \Theta$. Since obviously Θ is a base for Θ, then $R\omega O(H, \Theta)$ is a base for Θ. Therefore, (H, Θ) is ω-semi-regular. □

Theorem 3.7. *Let (H, Θ) be a TS. If (H, Θ_{ω}) is ω-semi-regular, then (H, Θ_{ω}) is semi-regular.*

Proof. Suppose that (H, Θ_{ω}) is ω-semi-regular. Then $R\omega O(H, \Theta_{\omega})$ is a base for Θ_{ω}. Since by Lemma 2.4 (c), $R\omega O(H, \Theta_{\omega}) = RO(H, \Theta_{\omega})$, then $RO(H, \Theta_{\omega})$ is a base for Θ_{ω}. therefore, (H, Θ_{ω}) is semi-regular. □

The following example shows that Theorem 3.3 need not be true in general.

Example 3.8. Let (H, Θ) be as in Example 2.8. Since $RO(H, \Theta) = \{\emptyset, H\}$, then $RO(H, \Theta)$ is not a base for Θ and hence, (H, Θ) is not semi-regular. On the other hand, by Theorem 2.6, (H, Θ) is ω-semi-regular.

The following example shows that Theorem 3.5 need not be true in general.

Example 3.9. Let (H, Θ) be the simplified Arens square (Example 81 of [25]). It is known that (H, Θ) is semi-regular but not regular. Also, clearly that (H, Θ) is anti-locally countable. Therefore, by Theorem 3.19 of [6] and Theorem 3.3, (H, Θ) is ω-semi-regular but not ω-regular.

Theorem 3.10. *A TS (H, Θ) is regular if and only if it is ω-semi-regular and ω-almost-regular.*

Proof.

Necessity. Follows from Corollary 3.15 of [9] and Theorem 3.3.

Sufficiency. Suppose that (H, Θ) is ω-semi-regular and ω-almost-regular. Let $E \in \Theta - \{\emptyset\}$ and let $r \in E$. Since (H, Θ) is ω-semi-regular, then there is $F \in R\omega O(H, \Theta)$ such that $r \in F \subseteq E$. Since (H, Θ) is ω-almost-regular, then by Theorem 2.1 (b), there is $S \in \Theta$ such that $r \in S \subseteq Cl_{\Theta}(S) \subseteq F \subseteq E$. Therefore, (H, Θ) is regular. □

Theorem 3.11. *Every Urysohn TS is ω-Urysohn.*

Proof. Let (H, Θ) be Urysohn and let $x, y \in H$ such that $x \neq y$. Then there are $E, F \in \Theta$ such that $x \in E$, $y \in F$, and $Cl_{\Theta}(E) \cap Cl_{\Theta}(F) = \emptyset$. Since $Cl_{\Theta_{\omega}}(E) \cap Cl_{\Theta_{\omega}}(F) \subseteq Cl_{\Theta}(E) \cap Cl_{\Theta}(F) = \emptyset$, then $Cl_{\Theta_{\omega}}(E) \cap Cl_{\Theta_{\omega}}(F) = \emptyset$. Hence, (H, Θ) is ω-Urysohn. □

Theorem 3.12. *Every anti-locally countable ω-Urysohn TS is Urysohn.*

Proof. Let (H, Θ) be anti-locally countable and ω-Urysohn. Let $x, y \in H$ such that $x \neq y$. Since (H, Θ) is ω-Urysohn, then there are $E, F \in \Theta$ such that $x \in E$, $y \in F$, and $Cl_{\Theta_{\omega}}(E) \cap Cl_{\Theta_{\omega}}(F) = \emptyset$. Since (H, Θ) is anti-locally countable, then $Cl_{\Theta}(E) \cap Cl_{\Theta}(F) = Cl_{\Theta_{\omega}}(E) \cap Cl_{\Theta_{\omega}}(F) = \emptyset$. Hence, (H, Θ) is Urysohn. □

Theorem 3.13. *Every ω-Urysohn TS is Hausdorff.*

Proof. Let (H, Θ) be ω-Urysohn and let $x, y \in H$ such that $x \neq y$. Then there are $E, F \in \Theta$ such that $x \in E$, $y \in F$, and $Cl_{\Theta_{\omega}}(E) \cap Cl_{\Theta_{\omega}}(F) = \emptyset$. Since $E \cap F \subseteq Cl_{\Theta_{\omega}}(E) \cap Cl_{\Theta_{\omega}}(F) = \emptyset$, then $E \cap F = \emptyset$. Hence, (H, Θ) is Hausdorff. □

Theorem 3.14. *Every locally countable Hausdorff TS is ω-Urysohn.*

Proof. Let (H, Θ) be locally countable and Hausdorff. Let $x, y \in H$ such that $x \neq y$. Since (H, Θ) is Hausdorff, then there are $E, F \in \Theta$ such that $x \in E$, $y \in F$, and $E \cap F = \emptyset$. Since (H, Θ) is locally countable, then $Cl_{\Theta_{\omega}}(E) \cap Cl_{\Theta_{\omega}}(F) = E \cap F = \emptyset$. Hence, (H, Θ) is ω-Urysohn. □

The following example shows that the converse of Theorem 3.11 need not be true in general.

Example 3.15. Let (H, Θ) be the Irrational Slope Topology (Example 75 of [25]). It is known that (H, Θ) is Hausdorff but not Urysohn. Since (H, Θ) is locally countable, then by Theorem 3.14, (H, Θ) is ω-Urysohn.

The following example shows that Theorem 3.13 need not be true in general.

Example 3.16. Let (H, Θ) be the Simplified Arens Square (Example 81 of [25]). It is known that (H, Θ) is Hausdorff but not Urysohn. Since (H, Θ) is anti-locally countable, then by Theorem 3.12, (H, Θ) is not ω-Urysohn.

Theorem 3.17. *Every ω-regular Hausdorff TS is ω-Urysohn.*

Proof. Let (H, Θ) be ω-regular and Hausdorff. Let $x, y \in H$ such that $x \neq y$. Since (H, Θ) is Hausdorff, then there are $E, F \in \Theta$ such that $x \in E$, $y \in F$, and $E \cap F = \emptyset$. Since (H, Θ) be ω-regular, then there are $S, T \in \Theta$ such that $x \in S \subseteq Cl_{\Theta_{\omega}}(S) \subseteq E$ and $y \in T \subseteq Cl_{\Theta_{\omega}}(T) \subseteq F$. Therefore, we have $x \in S \in \Theta$, $y \in T \in \Theta$ and $Cl_{\Theta_{\omega}}(S) \cap Cl_{\Theta_{\omega}}(T) \subseteq E \cap F = \emptyset$ and hence (H, Θ) is ω-Urysohn. □

Theorem 3.18. *If (X, Θ) and (Z, λ) are two ω-semi-regular TSs such that the product $(X \times Z, \Theta \times \lambda)$ is ω-semi-regular, then both of (X, Θ) and (Z, λ) are ω-semi-regular.*

Proof. Since $(X \times Z, \Theta \times \lambda)$ is ω-semi-regular, then $(\Theta \times \lambda)_{\delta_\omega} = \Theta \times \lambda$. So, by Theorem 2.17 of [9], $\Theta \times \lambda \subseteq \Theta_{\delta_\omega} \times \lambda_{\delta_\omega}$. Therefore, $\Theta = \Theta_{\delta_\omega}$ and $\lambda = \lambda_{\delta_\omega}$. Hence, (X, Θ) and (Z, λ) are ω-semi-regular. □

4. Conclusion

As two weaker forms of regularity, ω-almost-regularity and ω-semi-regular have been investigated. Several characterizations of them have been introduced (Theorems 2.1, 3.2). Subspaces and product theorems have been given (Theorems 2.11, 2.13, 2.14, 3.18). Also, it is proved that ω-almost-regularity lies strictly between regularity and almost-regularity (Theorems 2.2, 2.3 and Examples 2.7, 2.8) and that ω-semi-regularity is strictly weaker than each of ω-regularity and semi-regularity (Theorems 3.3, 3.5 and Examples 3.8, 3.9). Moreover, a decomposition theorem of regularity via ω-almost-regularity and ω-semi-regular has been introduced (Theorem 3.10). In the upcoming work, we plan: 1) to define and study ω-almost-normality; 2) to extend these concepts to include fuzzy topological spaces.

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