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Stability of n-dimensional mixed type quadratic and cubic functional equation

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Abstract

Using Hyers' direct method, we introduced and proved generalized Ulam-Hyers stability of n-dimensional mixed-type quadratic and cubic functional equation of the form

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = \left(\frac{n-6}{2}\right)\left(\sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(-x_i)\right) + \binom{n}{2}\sum_{1\leqslant i<j\leqslant n} f(x_i+x_j)$$
$$+\left(\frac{n-8}{2}\right)\sum_{1\leqslant i<j\leqslant n} f\left(-x_i-x_j\right) + \left(\frac{-n^2+4n}{8}\right)\sum_{i=1}^{n} f(2x_i) + \left(\frac{-n^2+8n-8}{8}\right)\sum_{i=1}^{n} f(-2x_i),$$

where

$$x_{ij} = \begin{cases} -x_j, & \text{if } i=j, \\ x_j, & \text{if } i\neq j, \end{cases}$$

in Banach space.

Keywords: Ulam-Hyers stability, quadratic functional equation, cubic functional equation, Banach space.

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1. Introduction

A classic question in functional equation theory is "Is a function that approximately satisfies the functional equation ϵ necessarily close to the exact solution ϵ? If the problem accepts a solution, one can say the equation ϵ is stable. The concept of stability of a functional equation arises when we replace the functional equation with an inequality which acts as a perturbation of the equation. The question of stability is as follows: how does the solution of the inequality differ from the solution of the given

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the functional equation? In the fall of 1940, Ulam [50] gave a major talk at the University of Wisconsin Mathematics Symposium in which he discussed many important open problems. These include the following questions about homomorphic stability.

Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exists $\delta(\epsilon) > 0$ such that if $h : G_1 \to G_2$ satisfies the inequality

$$d\left(h(x * y), h(x) \diamond h(y)\right) < \delta, \quad \forall\, x, y \in G_1,$$

then there is a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$.

If the answer is yes, the homomorphic functional equation is said to be stable. The following year, Hyers [13] answered this question for additive groups in the affirmative, assuming the group is a Banach space. He did a great job answering Ulam's question about the circumstances under which G_1 and G_2 are assumed to be Banach spaces. Hyers' results are expressed as follows.

Theorem 1.1. *Let $f : E_1 \to E_2$ be a function between Banach spaces such that*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon \tag{1.1}$$

for all $x, y \in E_1$ and $\epsilon > 0$ is a constant. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{1.2}$$

exists for each $x \in E_1$ and $A : E_1 \to E_2$ is unique additive mapping satisfying

$$\|f(x) - A(x)\| \leqslant \epsilon$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then the function A is linear.

Given this well-known result, the additive equation of the Cauchy function $f(x+y) = f(x) + f(y)$ is said to be Hyers-Ulam stable on (E_1, E_2) if a certain every function $f : E_1 \to E_2$ of a particular $\epsilon \geqslant 0$ and of all the inequalities (1.1) of $x, y \in E_1$ has an additive function $A : E_1 \to E_2$ makes $f - A$ bounded by E_1. The method in (1.2) provided by Hyers which produces the additive function A will be called the direct method. This method is the most important and powerful tool for studying the stability of various functional equations. Stability results for Hyers functions similar to unlimited Cauchy differences can be displayed. Aoki (1950) [4] first generalized Hyers' theorem for unlimited Cauchy differences with sums of norms $\|x\|^p + \|y\|^p$. Rassias found the results [35] in 1978 and proved a generalization of Hyers' theorem for additive maps. This stability result is called Hyers-Ulam-Rassias stability or Hyers-Ulam-Aoki-Rassias stability of the functional equation.

In 1982, Rassias [36] developed an innovative approach following Rassias' theorem [35], where he substitutes the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \|y\|^q$ with $p + q \neq 1$. Later this stability result was called Ulam-Gavruta-Rassias stability of functional equation. This stability result was later called stability of the equation of the Ulam-Gavruta-Rassias function. At the 27th International symposium on functional equations in 1990, Rassias raised the question whether such theorem in [38] can also prove that the value of p is greater than or equal to 1 . In 1991, Gajda [11] provided an affirmative solution to Rassias' question for p strictly greater than one. In 1994, Rassias' Theorem [35] was further generalized by P. Găvruța [12], substituted the bound $\epsilon\left(\|x\|^p + \|y\|^p\right)$ via the general control function $\phi(x, y)$. This stability result is called the generalized Hyers-Ulam-Rassias stability of the functional equation. In the sprit of Rassias approach, Ravi et al. [46] considered the sum of the sum and the product of two p-norms, called the Rassias Stability. The problem of the stability of various functional and differential equations has been studied in detail by many authors, and there are many interesting results on this problem (see [1, 2, 8, 9, 21–25, 29–31, 33, 34, 37, 46, 49, 51]) and references therein quoted.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is said to be quadratic functional equation because the quadratic function $f(x) = ax^2$ is a solution of the functional equation (1.3).

Rassias [41] introduced a cubic functional equation

$$c(x+2y)+3c(x)=3c(x+y)+c(x-y)+6c(y)$$

and investigated its Ulam stability problem. Also Jun and Kim [16] discussed the generalized Hyers-Ulam-Rassias stability of a cubic functional equation of the form

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.4}$$

More recently, Karthikeyan et al., introduced and established the general Ulam-Hyers stability of mixed-type functional equations in various spaces [5, 26, 27] and cited references therein.

In this paper, the authors established generalized Ulam-Hyers stability of n-dimensional mixed type quadratic and cubic functional equation of the form

$$\begin{aligned}\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) &= \left(\frac{n-6}{2}\right)\left(\sum_{i=1}^{n} f(x_i)+\sum_{i=1}^{n} f(-x_i)\right)+\left(\frac{n}{2}\right)\sum_{1\leqslant i<j\leqslant n} f(x_i+x_j)\\ &\quad+\left(\frac{n-8}{2}\right)\sum_{1\leqslant i<j\leqslant n} f\left(-x_i-x_j\right)+\left(\frac{-n^2+4n}{8}\right)\sum_{i=1}^{n} f(2x_i)\\ &\quad+\left(\frac{-n^2+8n-8}{8}\right)\sum_{i=1}^{n} f(-2x_i),\end{aligned} \tag{1.5}$$

where

$$x_{ij}=\begin{cases}-x_j, & \text{if } i=j,\\ x_j, & \text{if } i\neq j,\end{cases}$$

in Banach space using Hyers' direct method.

In Section 2, the authors discussed the general solution of the functional equation (1.5). The generalized Ulam-Hyers stability of the functional equation (1.5) is presented in Section 3.

2. General solution of the functional equation (1.5)

In this section, the authors investigate the general solution of the mixed type functional equation (1.5). Through out this section let us consider X and Y be real vector spaces.

Theorem 2.1. *Let $f : X \to Y$ be an odd function, it satisfies the functional equation (1.5) for all $x_1, x_2, \ldots, x_n \in X$ if and only if $f : X \to Y$ satisfies the functional equation (1.4) for all $x, y \in X$.*

Proof. Since f is an odd function, one can deduce from (1.5) that

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right)=(n-6)f\left(\sum_{i=1}^{n} x_i\right)+4\sum_{1\leqslant i<j\leqslant n} f(x_i+x_j)-\frac{(n-2)}{2}\sum_{i=1}^{n} f(2x_i)$$

for all $x_1, x_2, \ldots, x_n \in X$. The rest of the proof for this theorem can be derived from Theorem 2.1 in [7]. □

Theorem 2.2. *Let $f : X \to Y$ be an even function, it satisfies the functional equation (1.5) for all $x_1, x_2, \ldots, x_n \in X$ if and only if $f : X \to Y$ satisfies the functional equation*

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$.

Proof. Since f is an even function, one can deduce from (1.5) that

$$\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) = (-n^2+6n-4)\sum_{i=1}^{n} f(x_i) + (n-4)\sum_{1\leqslant i<j\leqslant n} f\left(x_i+x_j\right)$$

for all $x_1, x_2, \dots, x_n \in X$. The rest of the proof for this theorem can be derived from Theorem 2.1 in [6]. □

3. Stability results: Hyers' direct method

In this section, we presented the generalized Ulam-Hyers stability of the functional equation (1.5) using Hyers' direct method.

Through out this section, let X be a normed space and Y be a Banach space. Define a mapping $Df : X \to Y$ by

$$\begin{aligned} D\, f(x_1, x_2, \dots, x_n) = &\sum_{i=1}^{n} f\left(\sum_{j=1}^{n} x_{ij}\right) - \left(\frac{n-6}{2}\right)\left(\sum_{i=1}^{n} f(x_i) + \sum_{i=1}^{n} f(-x_i)\right) \\ &- \left(\frac{n}{2}\right)\sum_{1\leqslant i<j\leqslant n} f(x_i+x_j) - \left(\frac{n-8}{2}\right)\sum_{1\leqslant i<j\leqslant n} f\left(-x_i-x_j\right) \\ &- \left(\frac{-n^2+4n}{8}\right)\sum_{i=1}^{n} f(2x_i) - \left(\frac{-n^2+8n-8}{8}\right)\sum_{i=1}^{n} f(-2x_i) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in X$. Hereafter throughout this paper let us take $\Lambda = n-2$.

3.1. *Stability results: even case*

In this subsection, the authors discuss the Ulam-Hyers stability results of a quadratic functional equation (1.5) using Hyers' direct method in Banach spaces.

Theorem 3.1. *Let $j \in \{-1, 1\}$. Let $\vartheta : X^n \to [0, \infty)$ be a function such that $\sum_{k=0}^{\infty} \frac{\vartheta(\Lambda^{kj}x_1, \Lambda^{kj}x_2, \dots, \Lambda^{kj}x_n)}{\Lambda^{2kj}}$ converges to $\mathbb{R}$ and*

$$\lim_{k\to\infty} \frac{\vartheta\left(\Lambda^{kj}x_1, \Lambda^{kj}x_2, \dots, \Lambda^{kj}x_n\right)}{\Lambda^{2kj}} < \infty \tag{3.1}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and let $f_q : X \to Y$ be an even function that satisfies the inequality

$$\|D\, f_q(x_1, x_2, \dots, x_n)\| \leqslant \vartheta\left(x_1, x_2, \dots, x_n\right) \tag{3.2}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f_q(x) - Q(x)\| \leqslant \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\beta(\Lambda^{ij}x)}{\Lambda^{2ij}}, \tag{3.3}$$

where

$$\beta(\Lambda^{ij}x) = \frac{1}{n\Lambda^2}\vartheta(\Lambda^{ij}x, \Lambda^{ij}x, \dots, \Lambda^{ij}x)$$

for all $x \in X$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{k\to\infty} \frac{f_q(\Lambda^{kj}x)}{\Lambda^{2kj}} \tag{3.4}$$

for all $x \in X$.

Proof. Assume $j=1$. Since f is an even function, replacing $(x_1, x_2, \dots, x_n)$ by $(x, x, \dots, x)$ and dividing by Λ^2, we get

$$\left\| \frac{f_q(\Lambda x)}{\Lambda^2} - f_q(x) \right\| \leqslant \frac{1}{n\Lambda^2} \vartheta(x, x, \dots, x) \tag{3.5}$$

for all $x \in X$. Letting $\beta(x) = \frac{1}{n\Lambda^2}\vartheta(x, x, \dots, x)$ in (3.5), we arrive at

$$\left\| \frac{f_q(\Lambda x)}{\Lambda^2} - f_q(x) \right\| \leqslant \beta(x) \tag{3.6}$$

for all $x \in X$. Replacing x by Λx in (3.6) and dividing by Λ^2, we get

$$\left\| \frac{f_q(\Lambda^2 x)}{\Lambda^4} - \frac{f_q(\Lambda x)}{\Lambda^2} \right\| \leqslant \frac{\beta(\Lambda x)}{\Lambda^2} \tag{3.7}$$

for all $x \in X$. Combining (3.6) and (3.7), we obtain

$$\left\| \frac{f_q(\Lambda^2 x)}{\Lambda^4} - f_q(x) \right\| \leqslant \left[\beta\,(x) + \frac{\beta\,(\Lambda x)}{\Lambda^2} \right]$$

for all $x \in X$. Using induction on a positive integer k, we obtain that

$$\left\| \frac{f_q(\Lambda^k x)}{\Lambda^{2k}} - f_q(x) \right\| \leqslant \sum_{i=0}^{k-1} \frac{\beta\,(\Lambda^i x)}{\Lambda^{2i}} \leqslant \sum_{i=0}^{\infty} \frac{\beta\,(\Lambda^i x)}{\Lambda^{2i}} \tag{3.8}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{f_q(\Lambda^k x)}{\Lambda^{2k}} \right\}$, replacing x by $\Lambda^m x$ and dividing by Λ^{2m} in (3.8), for any $m, k > 0$, we arrive at

$$\left\| \frac{f_q(\Lambda^k \Lambda^m x)}{\Lambda^{2k+2m}} - \frac{f_q(\Lambda^m x)}{\Lambda^{2m}} \right\| = \left\| \frac{f_q(\Lambda^k \Lambda^m x)}{\Lambda^{2k}} - f_q(\Lambda^m x) \right\| \leqslant \sum_{i=0}^{k-1} \frac{\beta\,(\Lambda^i \Lambda^m x)}{\Lambda^{2i+2m}} \leqslant \sum_{i=0}^{\infty} \frac{\beta\,(\Lambda^{2+m} x)}{\Lambda^{2i+2m}} \tag{3.9}$$

for all $x \in X$. Since the right hand side of the inequality (3.9) tends to 0 as $m \to \infty$, the sequence $\left\{ \frac{f_q(\Lambda^k x)}{\Lambda^{2k}} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $Q : X \to Y$ such that

$$Q(x) = \lim_{k \to \infty} \frac{f_q(\Lambda^k x)}{\Lambda^{2k}}, \quad \forall\ x \in X.$$

Letting $k \to \infty$ in (3.8), we see that (3.3) holds for all $x \in X$. Now we need to prove Q satisfies (1.5), replacing $(x_1, x_2, \dots, x_n)$ by $(\Lambda^k x_1, \Lambda^k x_2, \dots, \Lambda^k x_n)$ and dividing by Λ^{2k} in (3.2), we arrive at

$$\frac{1}{\Lambda^{2k}} \left\| Df_q\left(\Lambda^k x_1, \Lambda^k x_2, \dots, \Lambda^k x_n\right) \right\| \leqslant \frac{\vartheta\left(\Lambda^k x_1, \Lambda^k x_2, \dots, \Lambda^k x_n\right)}{\Lambda^{2k}}$$

for all $x_1, x_2, \dots, x_n \in X$. Hence we get

$$\left\| D\,Q\left(\Lambda^k x_1, \Lambda^k x_2, \dots, \Lambda^k x_n\right) \right\| = 0.$$

Hence Q satisfies (1.5) for all $x_1, x_2, \dots, x_n \in X$. In order to prove Q is unique, let $Q'(x)$ be another quadratic mapping satisfying (3.3) and (1.5). Then

$$\begin{aligned} \left\| Q(x) - Q'(x) \right\| &= \frac{1}{\Lambda^{2k}} \left\| Q(\Lambda^k x) - Q'(\Lambda^k x) \right\| \\ &\leqslant \frac{1}{\Lambda^{2k}} \left\{ \left\| Q(\Lambda^k x) - f_q(\Lambda^k x) \right\| + \left\| f_q(\Lambda^k x) - Q'(\Lambda^k x) \right\| \right\} \leqslant 2 \sum_{i=0}^{\infty} \frac{\beta(\Lambda^{k+i} x)}{\Lambda^{2(k+i)}} \to 0 \text{ as } n \to \infty \end{aligned}$$

for all $x \in X$. Hence Q is unique.

For $j=-1$, we can prove the similar stability result. Hence completes the proof. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the stability of (1.5).

Corollary 3.2. *Let λ and s be nonnegative real numbers. If an even function $f : X \to Y$ satisfies the inequality*

$$\|Df_q(x_1, x_2, \dots, x_n)\| \leqslant \begin{cases} \lambda, & \\ \lambda \sum\limits_{i=1}^{n} \|x_i\|^s, & s \neq 2, \\ \lambda \left\{ \prod\limits_{i=1}^{n} \|x_i\|^s \right\}, & s \neq \frac{2}{n}, \\ \lambda \left\{ \prod\limits_{i=1}^{n} \|x_i\|^s + \sum\limits_{i=1}^{n} \|x_i\|^{ns} \right\}, & s \neq \frac{2}{n}, \end{cases}$$

for all $x_1, x_2, \dots, x_n \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f_q(x) - Q(x)\| \leqslant \begin{cases} \dfrac{\lambda}{n\,(\Lambda^2 - 1)}, \\ \dfrac{\lambda\|x\|^s}{|\Lambda^2 - \Lambda^s|}, \\ \dfrac{\lambda\|x\|^{ns}}{n|\Lambda^2 - \Lambda^{ns}|}, \\ \dfrac{\lambda(n+1)\|x\|^{ns}}{n|\Lambda^2 - \Lambda^{ns}|}, \end{cases}$$

for all $x \in X$.

3.2. *Stability results: odd case*

In this subsection, the authors discuss the Ulam-Hyers stability results of a cubic functional equation (1.5) using Hyers' direct method in Banach spaces.

Theorem 3.3. *Let $j \in \{-1, 1\}$. Let $\vartheta : X^n \to [0, \infty)$ be a function such that $\sum\limits_{k=0}^{\infty} \frac{\vartheta(n^{kj}x, n^{kj}x_2, \dots, n^{kj}x_n)}{n^{nj}}$ converges to $\mathbb{R}$ and*

$$\lim_{k \to \infty} \frac{\vartheta\left(n^{kj}x, n^{kj}x_2, \dots, n^{kj}x_n\right)}{n^{kj}} = 0 \tag{3.10}$$

for all $x_1, x_2, x_3, \dots, x_n \in X$ and let $f_c : X \to Y$ be an odd function that satisfies the inequality

$$\|D\ f_c(x_1, x_2, x_3, \dots, x_n)\| \leqslant \vartheta\left(x_1, x_2, \dots, x_n\right)$$

for all $x_1, x_2, x_3, \dots, x_n \in X$. Then there exists a unique cubic function $C : X \to Y$ such that

$$\|f_c(x) - C(x)\| \leqslant \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\varphi(n^{ij}x)}{n^{3ij}} \tag{3.11}$$

for all $x \in X$. The mapping $C(x)$ is defined by

$$C(x) = \lim_{k \to \infty} \frac{f_c(k^{ij}x)}{k^{3ij}} \tag{3.12}$$

for all $x \in X$.

Proof. Assume $j = 1$. Since f is an odd function, replacing $(x_1, x_2, \ldots, x_n)$ by $(x, x, \ldots, x)$ and dividing by $(n-6)n^3$, we get

$$\left\| \frac{f_c(nx)}{n^3} - f_c(x) \right\| \leqslant \frac{1}{(n-6)n^3} \vartheta(x, x, \ldots, x). \tag{3.13}$$

Letting $\varphi(x) = \frac{1}{(n-6)n^3}\vartheta(x, x, \ldots, x)$ in (3.13) we get

$$\left\| \frac{f_c(nx)}{n^3} - f_c(x) \right\| \leqslant \varphi(x) \tag{3.14}$$

for all $x \in X$. Setting x by nx in (3.14) and dividing by n^3, we get

$$\left\| \frac{f_c(n^2x)}{n^6} - \frac{f_c(nx)}{n^3} \right\| \leqslant \frac{\varphi(nx)}{n^3} \tag{3.15}$$

for all $x \in X$. Combining (3.14) and (3.15), we obtain

$$\left\| \frac{f_c(n^2x)}{n^6} - f_c(x) \right\| \leqslant \left[\varphi(x) + \frac{\varphi(nx)}{n^3} \right] \tag{3.16}$$

for all $x \in X$. Using induction on a positive integer k, we obtain that

$$\left\| \frac{f_c(n^kx)}{n^{3k}} - f_c(x) \right\| \leqslant \sum_{i=0}^{k-1} \frac{\varphi(n^ix)}{n^{3i}} \leqslant \sum_{i=0}^{\infty} \frac{\varphi(n^ix)}{n^{3i}} \tag{3.17}$$

for all $x \in X$. In order to prove the convergence of the sequence $\left\{ \frac{f_c(n^kx)}{n^{3k}} \right\}$, replacing x by n^mx and dividing by n^{3m} in (3.16), for any $m, k > 0$, we arrive at

$$\left\| \frac{f_c(n^kn^mv)}{n^{3(k+m)}} - \frac{f_c(n^mx)}{n^{3m}} \right\| = \frac{1}{n^{3m}} \left\| \frac{f_c(n^kn^mx)}{n^{3m}} - f_c(n^mx) \right\| \leqslant \sum_{i=0}^{k-1} \frac{\varphi(n^in^mx)}{n^{3(i+m)}} \leqslant \sum_{i=0}^{\infty} \frac{\varphi(n^{i+m}x)}{n^{3(i+m)}} \tag{3.18}$$

for all $x \in X$. Since the right hand side of the inequality (3.18) tends to 0 as $m \to \infty$, the sequence $\left\{ \frac{f_c(n^kx)}{n^{3k}} \right\}$ is a Cauchy sequence. Since Y is complete, there exists a mapping $C : X \to Y$ such that

$$C(x) = \lim_{k\to\infty} \frac{f_c(n^kx)}{n^{3k}}, \ \forall \ x \in X.$$

Letting $k \to \infty$ in (3.17), we see that (3.11) holds for all $x \in X$. In order to prove C satisfies (1.5) and it is unique the proof is similar to that of Theorem 3.1. □

The following corollary is an immediate consequence of Theorem 3.3 concerning the stability of (1.5).

Corollary 3.4. *Let n and s be nonnegative real numbers. If an odd function $f : X \to Y$ satisfies the inequality*

$$\|Df_c(x_1, x_2, \ldots, x_n)\| \leqslant \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \|x_i\|^s, & s \neq 3, \\ \lambda \prod_{i=1}^{n} \|x_i\|^s, & s \neq \frac{3}{n}, \\ \lambda \left\{ \prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns} \right\}, & s \neq \frac{3}{n}, \end{cases}$$

for all $x_1, x_2, \ldots, x_n$ *in* X*, then there exists a unique cubic function* $C : X \to Y$ *such that*

$$\|f_c(x) - C(x)\| \leqslant \begin{cases} \dfrac{\lambda}{(n-6)|n^3-1|}, \\ \dfrac{n\lambda\|x\|^s}{(n-6)|n^3-n^s|}, \\ \dfrac{\lambda\|x\|^{ns}}{(n-6)|n^3-n^{ns}|}, \\ \dfrac{(n+1)\lambda\|x\|^{ns}}{(n-6)|n^3-n^{ns}|}, \end{cases}$$

for all $x \in X$.

3.3. *Stability results: mixed case*

This subsection deals with the Ulam-Hyers stability results of quadratic-cubic mixed type functional equation (1.5) using Hyers' direct method in Banach spaces.

Theorem 3.5. *Let* $j \in \{-1, 1\}$ *and* $\vartheta : X^n \to [0, \infty)$ *be a function that satisfies* (3.1) *and* (3.10) *for all* $x_1, x_2, \ldots, x_n \in X$*. Let* $f : X \to Y$ *be a function satisfies the inequality*

$$\|D\, f(x_1, x_2, \ldots, x_n)\| \leqslant \vartheta\,(x_1, x_2, \ldots, x_n)$$

for all $x_1, x_2, \ldots, x_n \in X$*. Then there exists a unique quadratic function* $Q : X \to Y$ *and a unique cubic function* $C : X \to Y$ *such that*

$$\|f(x) - C(x) - Q(x)\| \leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\beta(\Lambda^{ij}x)}{\Lambda^{2ij}} + \frac{\beta(-\Lambda^{ij}x)}{\Lambda^{2ij}}\right) + \sum_{i=0}^{\infty}\left(\frac{\varphi(n^{ij}x)}{n^{3ij}} + \frac{\varphi(-n^{ij}x)}{n^{3ij}}\right)\right\}$$

for all $x \in X$*. The mappings* $C(x)$ *and* $Q(x)$ *are respectively defined in* (3.12) *and* (3.4) *for all* $x \in X$.

Proof. Let $f_q(x) = \frac{1}{2}\{f\,(x) + f\,(-x)\}$ for all $x \in X$. Then $f_q\,(0) = 0, f_q\,(x) = f_q\,(-x)$. Hence

$$\begin{aligned} \|Df_q\,(x_1, x_2, \ldots, x_n)\| &= \frac{1}{2}\{\|Df\,(x_1, x_2, \ldots, x_n) + Df\,(-x_1, -x_2, \ldots, -x_n)\|\} \\ &\leqslant \frac{1}{2}\{\|Df\,(x_1, x_2, \ldots, x_n)\| + \|Df\,(-x_1, -x_2, \ldots, -x_n)\|\} \\ &\leqslant \frac{1}{2}\{\vartheta\,(x_1, x_2, \ldots, x_n) + \vartheta\,(-x_1, -x_2, \ldots, -x_n)\} \end{aligned}$$

for all $x \in X$. Hence from Theorem 3.1, there exits a unique quadratic function $Q : X \to Y$ such that

$$\|f_q(x) - Q(x)\| \leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\beta(\Lambda^{ij}x)}{\Lambda^{2ij}} + \frac{\beta(-\Lambda^{ij}x)}{\Lambda^{2ij}}\right)\right\} \tag{3.19}$$

for all $x \in X$. Again $f_c(x) = \frac{1}{2}\{f\,(x) - f\,(-x)\}$ for all $x \in X$. Then $f_c\,(0) = 0, f_c\,(x) = -f_c\,(-x)$. Hence

$$\begin{aligned} \|Df_c\,(x_1, x_2, \ldots, x_n)\| &= \frac{1}{2}\{\|Df\,(x_1, x_2, \ldots, x_n) + Df\,(-x_1, -x_2, \ldots, -x_n)\|\} \\ &\leqslant \frac{1}{2}\{\|Df\,(x_1, x_2, \ldots, x_n)\| + \|Df\,(-x_1, -x_2, \ldots, -x_n)\|\} \\ &\leqslant \frac{1}{2}\{\vartheta\,(x_1, x_2, \ldots, x_n) + \vartheta\,(-x_1, -x_2, \ldots, -x_n)\} \end{aligned}$$

for all $x \in X$. Hence from Theorem 3.3, there exits a unique cubic function $C : X \to Y$ such that

$$\|f_c(x) - C(x)\| \leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\varphi(n^{ij}x)}{n^{3ij}} + \frac{\varphi(-n^{ij}x)}{n^{3ij}}\right)\right\} \tag{3.20}$$

for all $x \in X$. Since $f(x) = f_q(x) + f_c(x)$, then it follows from (3.19) and (3.20) that

$$\begin{aligned}\|f(x) - C(x) - Q(x)\| &= \|f_q(x) + f_c(x) - C(x) - Q(x)\| \\ &\leqslant \|f_q(x) - Q(x)\| + \|f_c(x) - C(x)\| \\ &\leqslant \frac{1}{2}\left\{\sum_{i=0}^{\infty}\left(\frac{\beta(\Lambda^{ij}x)}{\Lambda^{2ij}} + \frac{\beta(-\Lambda^{ij}x)}{\Lambda^{2ij}}\right) + \sum_{i=0}^{\infty}\left(\frac{\varphi(n^{ij}x)}{n^{3ij}} + \frac{\varphi(-n^{ij}x)}{n^{3ij}}\right)\right\}\end{aligned}$$

for all $x \in X$. Hence it completes the proof. □

The following corollary is an immediate consequence of Theorem 3.5 concerning the stability of (1.5).

Corollary 3.6. *Let λ and s be nonnegative real numbers. If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x_1, x_2, \ldots, x_n)\| \leqslant \begin{cases} \lambda, & \\ \lambda \sum_{i=1}^{n} \|x_i\|^s, & s \neq 1 \text{ and } s \neq 3, \\ \lambda \prod_{i=1}^{n} \|x_i\|^s, & s \neq \frac{1}{n} \text{ and } s \neq \frac{3}{n}, \\ \lambda \left\{\prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns}\right\}, & s \neq \frac{1}{n} \text{ and } s \neq \frac{3}{n}, \end{cases}$$

for all $x_1, x_2, \ldots, x_n$ in X, then there exists a unique quadratic function $Q : X \to Y$ and a unique cubic function $C : X \to Y$ such that

$$\|f(x) - C(x) - Q(x)\| \leqslant \begin{cases} \lambda\left(\dfrac{1}{(n-6)|n^3-1|} + \dfrac{1}{n(\Lambda^2-1)}\right), \\ \lambda\left(\dfrac{n}{(n-6)|n^3-n^s|} + \dfrac{1}{|\Lambda^2-\Lambda^s|}\right)\|x\|^s, \\ \lambda\left(\dfrac{1}{|(n-6)|n^3-n^{ns}|} + \dfrac{1}{n|\Lambda^2-\Lambda^s|}\right)\|x\|^{ns}, \\ (n+1)\lambda\left(\dfrac{1}{(n-6)|n^3-n^{ns}|} + \dfrac{1}{n|\Lambda^2-\Lambda^{ns}|}\right)\|x\|^{ns}, \end{cases}$$

for all $x \in X$.

4. Conclusion

This article has proved the Hyers-Ulam, Hyers-Ulam-Rassias, generalized Hyers-Ulam-Rassias, and Rassias stability results of the quadratic functional equation, the cubic functional equation, and the quadratic-cubic mixed type functional equation in Banach space.

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