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Strongly separation axioms in fuzzifying bitopological spaces

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Abstract

In this paper, we study the pairwise open sets in the fuzzifying bitopological spaces and obtain new results. Also, we investigate of separation axioms in the fuzzifying bitopological spaces and obtain important results. Furthermore, we introduce and study new strongly pairwise axioms in the fuzzifying bitopological spaces.

Keywords: Fuzzifying topology, fuzzifying bitopological space, separation axioms.

2020 MSC: 54E55, 54D15, 54D10.

1. Introduction

The study of bitopological spaces was first initiated by Kelley [4] in 1963. In 1991-1993 [7–9], Ying studied the basic concepts of the fuzzifying topological spaces (briefly, FTSs) using the fuzzy logic. In 1993 Shen [6] introduced and studied the separation axioms in the FTSs.

In 2003 Zhang et al. [10], defined the fuzzifying bitopological spaces (briefly, FBTSs) and studied more concepts in this subject. In 2007 Kilicman et al. [5], continued the study of bitopological separation axioms which introduced by Kelley. In 2020, Allam et al. [1], introduced and investigated the separation axioms in the FBTSs.

In 2019-2020 Al-Shami et al [2, 3], studied and investigated different types of separation axioms in soft topology (consider as an extension of fuzzy topology).

This paper is presented as follows. In Section 3, the pairwise open sets in the FBTSs were studied and some of new results between pairwise closure and pairwise interior operators were obtained. In Section 4, some important results regarding a pairwise normality, a weakly pairwise normality and a pairwise regularity in the FBTSs were obtained. In Section 5, the concept of a new strongly pairwise axioms in the FBTSs was introduced and studied. Finally, in Section 6 a conclusion is given.

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2. Preliminaries

Definition 2.1 ([7]). Let X be a universe of discourse. $\zeta \in \Im(P(X))$ is called a fuzzifying topology, if

(1) $\zeta(X) = 1$ and $\zeta(\emptyset) = 1$;
(2) $\forall O, G \in P(X), \zeta(O \cap G) \geqslant \zeta(O) \wedge \zeta(G)$;
(3) for any $\{O_\lambda : \lambda \in \Lambda\}, \zeta(\bigcup_{\lambda \in \Lambda} O_\lambda) \geqslant \bigwedge_{\lambda \in \Lambda} \zeta(O_\lambda)$.

Then (X, ζ) is called a FTS.

Definition 2.2 ([7]). Let (X, ζ) be a FTS. The family of all fuzzifying closed sets "$\mathcal{F}$" is defined as $\mathcal{F}(D) = \zeta(X \sim D)$, where $X \sim D$ is the complement of D.

Definition 2.3 ([7]). Let (X, ζ) be a FTS. The fuzzifying closure (interior) operator "σ" ("ι"), is defined as $\sigma(O)(x) = \bigwedge_{x \notin G \supseteq O} (1 - \mathcal{F}(G)), \left(\iota(O)(x) = \bigvee_{x \in G \subseteq O} \zeta(G)\right)$.

Definition 2.4 ([10]). Let (X, ζ_1) and (X, ζ_2) be two FTSs. A system (X, ζ_1, ζ_2) is called a FBTS.

Definition 2.5 ([1]). Let (X, ζ_1, ζ_2) be a FBTS.

(1) The family of all fuzzifying pairwise open sets "$\mathcal{O}^p$" is defined as $\mathcal{O}^p(Q) = \min(\zeta_1(Q), \zeta_2(Q))$.
(2) The family of all fuzzifying pairwise closed "$\mathcal{F}^p$" is defined as $\mathcal{F}^p(G) = \mathcal{O}^p(X \sim G)$.

Lemma 2.6 ([1]). *Let* (X, ζ_1, ζ_2) *be a FBTS. Then* $\mathcal{F}^p(G) = \min(\mathcal{F}_1(G), \mathcal{F}_2(G))$.

Lemma 2.7 ([1]). *Let* (X, ζ_1, ζ_2) *be a FBTS. Then*

(1) $\models \mathcal{O}^p \longrightarrow \zeta_i$, $i = 1, 2$;
(2) $\models \mathcal{F}^p \longrightarrow \mathcal{F}_i$, $i = 1, 2$.

Definition 2.8 ([1]). Let $\wp$ be the class of all FBTSs. The unary fuzzy predicates $T_N^{(i,j)}$, ${}^{w}T_N^{(i,j)} \in \Im(\wp)$ (pairwise normality axioms and pairwise weakly normality axioms), $T_{R_i}^{(i,j)}, T_{R_j}^{(i,j)} \in \Im(\wp)$, and $T_R^{(i,j)} \in \Im(\wp)$ (pairwise regularity axioms) are defined as follows respectively:

(1) $(X, \zeta_1, \zeta_2) \in T_N^{(i,j)} := \forall O \forall G (O \in \mathcal{F}_i \wedge G \in \mathcal{F}_j \wedge O \cap G = \emptyset \longrightarrow \exists Q \exists D (Q \in \zeta_i \wedge D \in \zeta_j \wedge O \subseteq D \wedge G \subseteq Q \wedge Q \cap D = \emptyset)$;
(2) $(X, \zeta_1, \zeta_2) \in {}^{w}T_N^{(i,j)} := \forall O \forall G (O \in \mathcal{F}^p \wedge G \in \mathcal{F}^p \wedge O \cap G = \emptyset \longrightarrow \exists Q \exists D (Q \in \zeta_i \wedge D \in \zeta_j \wedge O \subseteq D \wedge G \subseteq Q \wedge Q \cap D = \emptyset)$;
(3) $(X, \zeta_1, \zeta_2) \in T_{R_i}^{(i,j)} := \forall x \forall Q \big(x \in X \wedge Q \in \mathcal{F}_i \wedge x \notin Q \longrightarrow \exists O (O \in N_x^i \wedge \sigma_j(O) \cap Q = \emptyset)\big)$;
(4) $(X, \zeta_1, \zeta_2) \in T_{R_j}^{(i,j)} := \forall x \forall Q \big(x \in X \wedge Q \in \mathcal{F}_j \wedge x \notin Q \longrightarrow \exists O (O \in N_x^j \wedge \sigma_i(O) \cap Q = \emptyset)\big)$;
(5) $(X, \zeta_1, \zeta_2) \in T_R^{(i,j)} := (X, \zeta_1, \zeta_2) \in T_{R_i}^{(i,j)} \wedge (X, \zeta_1, \zeta_2) \in T_{R_j}^{(i,j)}$.

3. Pairwise open sets in FBTSs

Theorem 3.1. *Let* (X, ζ_1, ζ_2) *be a FBTS. Then* $(X, \mathcal{O}^p)$ *is the FTS.*

Proof.

(1) Clearly $\mathcal{O}^p(X) = 1$ and $\mathcal{O}^p(\emptyset) = 1$.

(2) Let $Q_1, Q_2 \in \mathcal{O}^p$, then

$$\begin{aligned} \mathcal{O}^p(Q_1 \cap Q_2) &= \zeta_1(Q_1 \cap Q_2) \wedge \zeta_2(Q_1 \cap Q_2) \\ &\geqslant \big(\zeta_1(Q_1) \wedge \zeta_1(Q_2)\big) \wedge \big(\zeta_2(Q_1) \wedge \zeta_2(Q_2)\big) \\ &= \big(\zeta_1(Q_1) \wedge \zeta_2(Q_1)\big) \wedge \big(\zeta_1(Q_2) \wedge \zeta_2(Q_2)\big) = \mathcal{O}^p(Q_1) \wedge \mathcal{O}^p(Q_2). \end{aligned}$$

(3) Let $\{D_i : i \in I\}$ be a family of pairwise open sets (i.e., $D_i \in \mathcal{O}^p$, $\forall i \in I$), then

$$\mathcal{O}^p(\bigcup_{i\in I} D_i) = \zeta_1(\bigcup_{i\in I} D_i) \wedge \zeta_2(\bigcup_{i\in I} D_i) \geqslant (\bigwedge_{i\in I} \zeta_1(D_i)) \wedge (\bigwedge_{i\in I} \zeta_2(D_i)) = \bigwedge_{i\in I} (\zeta_1(D_i) \wedge \zeta_2(D_i)) = \bigwedge_{i\in I} \mathcal{O}^p(D_i).$$

□

Definition 3.2. Let (X, ζ_1, ζ_2) be a FBTS and $O \subseteq X$. The fuzzifying pairwise closure operator "σ^p" is defined as $\sigma^p(O)(x) = \bigwedge_{x\notin D\supseteq O} (1 - \mathcal{F}^p(D))$.

Theorem 3.3. *Let (X, ζ) be a FTS. Then* $\sigma^p(O)(x) = \bigwedge_{x\in Q, Q\cap O=\emptyset} (1 - \min(\zeta_1(Q), \zeta_2(Q)))$.

Proof.

$$\begin{aligned}
\sigma^p(O)(x) = \bigwedge_{x\notin D\supseteq O} (1 - \mathcal{F}^p(D)) &= \bigwedge_{x\notin D\supseteq O} (1 - \min(\mathcal{F}_1(D), \mathcal{F}_2(D))) \\
&= \bigwedge_{x\in X\sim D, X\sim D\cap O=\emptyset} (1 - \min(\zeta_1(X \sim D), \zeta_2(X \sim D))) \\
&= \bigwedge_{x\in Q, Q\cap O=\emptyset} (1 - \min(\zeta_1(Q), \zeta_2(Q))).
\end{aligned}$$

□

Definition 3.4. Let (X, ζ_1, ζ_2) be a FBTS and $O \subseteq X$. The fuzzifying pairwise interior "ι^p" is defined as $\iota^p(O)(x) = \bigvee_{x\in G\subseteq O} \mathcal{O}_p(G)$.

Theorem 3.5. *Let (X, ζ_1, ζ_2) be a FBTS and $O \subseteq X$. Then $\sigma^p(O) = X \sim \iota^p(X \sim O)$.*

Proof. From Theorem 3.3, we have

$$\sigma^p(O)(x) = \bigwedge_{x\in Q, Q\cap O=\emptyset} (1 - \min(\zeta_1(Q), \zeta_2(Q))) = 1 - \bigvee_{x\in Q\subseteq X\sim O} \min(\zeta_1(Q), \zeta_2(Q)) = 1 - \iota^p(X \sim O)(x).$$

□

Theorem 3.6. *Let (X, ζ_1, ζ_2) be a FBTS and $O \subseteq X$. Then*

(1) $\sigma_i(O) \leqslant \sigma^p(O)$, $i = 1, 2$;
(2) $\iota^p(O) \leqslant \iota_i(O)$, $i = 1, 2$.

Proof.

(1) From Lemma 2.7, we have

$$\sigma^p(O)(x) = \bigwedge_{x\notin D\supseteq O} (1 - \mathcal{F}^p(D)) \geqslant \bigwedge_{x\notin D\supseteq O} (1 - \mathcal{F}_1(D)) = \sigma_1(O)(x).$$

By the same way, we can prove $\sigma^p(O)(x) \geqslant \sigma_2(O)(x)$, $\forall x \in X$.

(2) From Theorem 3.5 and (1) above, we have

$$\iota^p(O)(x) = 1 - \sigma^p(X \sim O)(x) \leqslant 1 - \sigma_i(X \sim O)(x), \text{ (where } i = 1, 2) = \iota_i(O)(x).$$

□

From Theorem 3.6, we can have the following theorem.

Theorem 3.7. *Let* (X, ζ_1, ζ_2) *be a FBTS and* $O \subseteq X$. *Then*

(1) $\sigma_1(O) \wedge \sigma_2(O) \leqslant \sigma^p(O)$;
(2) $\iota^p(O) \leqslant \iota_1(O) \wedge \iota_2(O)$.

Proof. It is clear. □

The following example shows that:

(1) $\sigma_1(O) \wedge \sigma_2(O) \ngeqslant \sigma^p(O)$;
(2) $\iota^p(O) \ngeqslant \iota_1(O) \wedge \iota_2(O)$.

Example 3.8. Let $X = \{s, t, w\}$, $B = \{s\}$, $D = \{t, w\}$ and ζ_1, ζ_2 be two fuzzifying topologies on X defined as follow:

$$\zeta_1(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{s\}, \{s, w\}\}, \\ 1/4, & \text{if } O \in \{\{w\}, \{t, w\}\}, \\ 0, & \text{if } O \in \{\{t\}, \{s, t\}\}, \end{cases} \qquad \zeta_2(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{t\}, \{s, w\}\}, \\ 1/4, & \text{if } O \in \{\{s\}, \{s, t\}\}, \\ 0, & \text{if } O \in \{\{w\}, \{t, w\}\}. \end{cases}$$

We can easily note that,

$$\mathcal{F}_1(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{t\}, \{t, w\}\}, \\ 1/4, & \text{if } O \in \{\{s\}, \{s, t\}\}, \\ 0, & \text{if } O \in \{\{w\}, \{s, w\}\}, \end{cases} \qquad \mathcal{F}_2(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{t\}, \{s, w\}\}, \\ 1/4, & \text{if } O \in \{\{w\}, \{t, w\}\}, \\ 0, & \text{if } O \in \{\{s\}, \{s, t\}\}, \end{cases}$$

and

$$\mathcal{O}^p(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{s, w\}\}, \\ 1/4, & \text{if } O = \{s\}, \\ 0, & \text{o.w.}, \end{cases} \qquad \mathcal{F}^p(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{t\}\}, \\ 1/4, & \text{if } O = \{t, w\}, \\ 0, & \text{o.w.}. \end{cases}$$

We have $\sigma_1(B) = \{(s, 1), (t, 3/4), (w, 3/4)\}$, $\sigma_2(B) = \{(s, 1), (t, 0), (w, 1)\}$ and $\sigma^p(B) = \{(s, 1), (t, 1), (w, 1)\}$. Also, $\iota_1(D) = \{(s, 0), (t, 1/4), (w, 1/4)\}$, $\iota_2(D) = \{(s, 0), (t, 1), (w, 0)\}$ and $\iota^p(D) = \{(s, 0), (t, 0), (w, 0)\} = \emptyset$. Therefore $\sigma_1(B) \wedge \sigma_2(B) = \{(s, 1), (t, 0), (w, 3/4)\} \ngeqslant \sigma^p(B)$ and $\iota_1(D) \wedge \iota_2(D) = \{(s, 0), (t, 1/4), (w, 0)\} \nleqslant \iota^p(D)$.

4. The pairwise normality and the pairwise regularity axioms in FBTSs

Theorem 4.1. *Let* (X, ζ_1, ζ_2) *be a FBTS. Then* $\models (X, \zeta_1, \zeta_2) \in T_N^{(i,j)} \longleftrightarrow \forall M \forall G (M \in \zeta_i \wedge G \in \mathcal{F}_j \wedge G \subseteq M \to \exists Q \exists K (Q \in \zeta_i \wedge K \in \mathcal{F}_j \wedge G \subseteq Q \subseteq K \subseteq M)$.

Proof. To simplify, we put $O^c = X \sim O, D^c = X \sim D$,

$$\begin{aligned} [(X, \zeta_1, \zeta_2) \in T_N^{(i,j)}] &= \bigwedge_{O \cap G = \emptyset} \min\Big(1, 1 - \min\big(\mathcal{F}_i(O), \mathcal{F}_j(G)\big) + \bigvee_{Q \cap D = \emptyset, O \subseteq D, G \subseteq Q} \min\big(\zeta_i(Q), \zeta_j(D)\big)\Big) \\ &= \bigwedge_{G \subseteq O^c} \min\Big(1, 1 - \min\big(\zeta_i(O^c), \mathcal{F}_j(G)\big) + \bigvee_{Q \subseteq D^c, D^c \subseteq O^c, G \subseteq Q} \min\big(\zeta_i(Q), \mathcal{F}_j(D^c)\big)\Big) \\ &= \bigwedge_{G \subseteq M} \min\Big(1, 1 - \min\big(\zeta_i(M), \mathcal{F}_j(G)\big) + \bigvee_{Q \subseteq K, K \subseteq M, G \subseteq Q} \min\big(\zeta_i(Q), \mathcal{F}_j(K)\big)\Big) \\ &= \bigwedge_{G \subseteq M} \min\Big(1, 1 - \min\big(\zeta_i(M), \mathcal{F}_j(G)\big) + \bigvee_{G \subseteq Q \subseteq K \subseteq M} \min\big(\zeta_i(Q), \mathcal{F}_j(K)\big)\Big). \end{aligned}$$

□

Remark 4.2. Note that in Theorem 4.1, if $G = M$, then $G = Q = K = M$. Thus $\min\Big(1, 1 - \min\big(\zeta_i(M), \mathcal{F}_j(G)\big) + \min\big(\zeta_i(Q), \mathcal{F}_j(K)\big)\Big) = 1$. Therefore, we can define

$$[(X, \zeta_1, \zeta_2) \in T_N^{(i,j)}] = \bigwedge_{G \subset M} \min\Big(1, 1 - \min\big(\zeta_i(M), \mathcal{F}_j(G)\big) + \bigvee_{G \subseteq Q \subseteq K \subseteq M} \min\big(\zeta_i(Q), \mathcal{F}_j(K)\big)\Big).$$

The following example illustrates the Remark 4.2.

Example 4.3. Let $X = \{s, t, w\}$, and ζ_1, ζ_2 be two fuzzifying topologies on X defined as follows:

$$\zeta_1(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X\}, \\ 3/4, & \text{if } O = \{s, t\}, \\ 0, & \text{if o.w.,} \end{cases} \qquad \zeta_2(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X\}, \\ 1/3, & \text{if } O = \{t, w\}, \\ 0, & \text{if o.w.,} \end{cases}$$

$$\mathcal{F}_1(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X\}, \\ 3/4, & \text{if } O = \{w\}, \\ 0, & \text{if o.w.,} \end{cases} \qquad \mathcal{F}_2(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X\}, \\ 1/3, & \text{if } O = \{s\}, \\ 0, & \text{if o.w..} \end{cases}$$

Taking $G = \{s\}$ and $M = \{s, t\}$, we have

$$\Big[(X, \zeta_1, \zeta_2) \in T_N^{(1,2)}\Big] = \bigwedge_{G \subset M} \min\Big(1, 1 - \min\big(\zeta_1(M), \mathcal{F}_2(G)\big) + \bigvee_{G \subseteq Q \subseteq K \subseteq M} \min\big(\zeta_1(Q), \mathcal{F}_2(K)\big)\Big) = 2/3.$$

Theorem 4.4. *Let* (X, ζ_1, ζ_2) *be a FBTS. Then* $\models (X, \zeta_1, \zeta_2) \in {}^{w}T_N^{(i,j)} \longleftrightarrow \forall M \forall G (M \in \mathcal{O}^p \wedge G \in \mathcal{F}^p \wedge G \subseteq M \to \exists Q \exists K (Q \in \zeta_i \wedge K \in \mathcal{F}_j \wedge G \subseteq Q \subseteq K \subseteq M)$.

Proof. To simplify, we put $O^c = X \sim O, D^c = X \sim D$,

$$\begin{aligned} [(X, \zeta_1, \zeta_2) \in {}^{w}T_N^{(i,j)}] &= \bigwedge_{O \cap G = \emptyset} \min\Big(1, 1 - \min\big(\mathcal{F}^p(O), \mathcal{F}^p(G)\big) + \bigvee_{Q \cap D = \emptyset, O \subseteq D, G \subseteq Q} \min\big(\zeta_i(Q), \zeta_j(D)\big)\Big) \\ &= \bigwedge_{G \subseteq O^c} \min\Big(1, 1 - \min\big(\mathcal{O}^p(O^c), \mathcal{F}^p(G)\big) + \bigvee_{Q \subseteq D^c, D^c \subseteq O^c, G \subseteq Q} \min\big(\zeta_i(Q), \mathcal{F}_j(D^c)\big)\Big) \\ &= \bigwedge_{G \subseteq M} \min\Big(1, 1 - \min\big(\mathcal{O}^p(M), \mathcal{F}^p(G)\big) + \bigvee_{Q \subseteq K, K \subseteq M, G \subseteq Q} \min\big(\zeta_i(Q), \mathcal{F}_j(K)\big)\Big) \\ &= \bigwedge_{G \subseteq M} \min\Big(1, 1 - \min\big(\mathcal{O}^p(M), \mathcal{F}^p(G)\big) + \bigvee_{G \subseteq Q \subseteq K \subseteq M} \min\big(\zeta_i(Q), \mathcal{F}_j(K)\big)\Big). \end{aligned}$$

□

Remark 4.5. From Remark 4.2, we have

$$\Big[(X, \zeta_1, \zeta_2) \in {}^{w}T_N^{(i,j)}\Big] = \bigwedge_{G \subset M} \min\Big(1, 1 - \min\big(\mathcal{O}^p(M), \mathcal{F}^p(G)\big) + \bigvee_{G \subseteq Q \subseteq K \subseteq M} \min\big(\zeta_i(Q), \mathcal{F}_j(K)\big)\Big).$$

Theorem 4.6. *Let* (X, ζ_1, ζ_2) *be a FBTS. Then*

$$\models (X, \zeta_1, \zeta_2) \in T_{R_i}^{(i,j)} \longleftrightarrow \forall x \forall H \big(x \in X \wedge H \in \zeta_i \wedge x \in H \to \exists O (O \in \zeta_i \wedge x \in O \subseteq \sigma_j(O) \subseteq H)\big).$$

Proof.

$$\Big[(X, \zeta_1, \zeta_2) \in T_{R_i}^{(i,j)}\Big] = \bigwedge_{x \notin Q} \min\Big(1, 1 - \mathcal{F}_i(Q) + \bigvee_{O \in P(X)} \min\big(N_x^i(O), \bigwedge_{y \in Q} (1 - \sigma_j(O)(y))\big)\Big)$$

$$
\begin{aligned}
&= \bigwedge_{x\in X\sim Q} \min\Big(1, 1-\zeta_i(X\sim Q) + \bigvee_{O\in P(X)} \min\big(N^i_x(O), \bigwedge_{y\notin X\sim Q} (1-\sigma_j(O)(y))\big)\Big)\\
&= \bigwedge_{x\in H} \min\Big(1, 1-\zeta_i(H) + \bigvee_{O\in P(X)} \min\big(\bigvee_{x\in G\subseteq O} \zeta_i(G), \bigwedge_{y\notin H} (1-\sigma_j(O)(y))\big)\Big)\\
&= \bigwedge_{x\in H} \min\Big(1, 1-\zeta_i(H) + \bigvee_{O\in P(X)} \bigvee_{x\in G\subseteq O} \min\big(\zeta_i(G), \bigwedge_{y\notin H} (1-\sigma_j(O)(y))\big)\Big)\\
&= \bigwedge_{x\in H} \min\Big(1, 1-\zeta_i(H) + \bigvee_{O\in P(X), x\in G\subseteq O} \min\big(\zeta_i(G), \bigwedge_{y\notin H} (1-\sigma_j(O)(y))\big)\Big)\\
&= \bigwedge_{x\in H} \min\Big(1, 1-\zeta_i(H) + \bigvee_{O\in P(X), x\in O} \min\big(\zeta_i(O), \bigwedge_{y\notin H} (1-\sigma_j(O)(y))\big)\Big).
\end{aligned}
$$

□

Theorem 4.7. *Let (X, ζ_1, ζ_2) be a FBTS. Then*

$$\models (X, \zeta_1, \zeta_2) \in T^{(i,j)}_{R_j} \longleftrightarrow \forall x \forall H\big(x \in X \wedge H \in \zeta_j \wedge x \in H \to \exists O(O \in \zeta_j \wedge x \in O \subseteq \sigma_i(O) \subseteq H)\big).$$

Proof. It is similar to the proof of Theorem 4.6. □

5. New strongly separation axioms in FBTSs

Now, a new strongly form of pairwise regular FBTS will be defined and studied.

Definition 5.1. Let $\wp$ be the class of all FBTSs. The unary fuzzy predicates ${}^sT^{(i,j)}_{R_i}$, ${}^sT^{(i,j)}_{R_j} \in \Im(\wp)$ and ${}^sT^{(i,j)}_{R} \in \Im(\wp)$ (pairwise strongly regularity axioms) are defined as follows, respectively,

(1) $(X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R_i} := \forall x \forall H\big(x \in X \wedge H \in \zeta_i \wedge x \in H \to \exists O(O \in \zeta_i \wedge x \in O \subseteq \sigma^p(O) \subseteq H)\big)$;

(2) $(X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R_j} := \forall x \forall H\big(x \in X \wedge H \in \zeta_j \wedge x \in H \to \exists O(O \in \zeta_j \wedge x \in O \subseteq \sigma^p(O) \subseteq H)\big)$;

(3) $(X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R} := (X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R_i} \wedge (X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R_j}$.

Theorem 5.2. *Let (X, ζ_1, ζ_2) be a FBTS. Then*

(1) $\models (X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R_i} \longrightarrow (X, \zeta_1, \zeta_2) \in T^{(i,j)}_{R_i}$;

(2) $\models (X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R_j} \longrightarrow (X, \zeta_1, \zeta_2) \in T^{(i,j)}_{R_j}$;

(3) $\models (X, \zeta_1, \zeta_2) \in {}^sT^{(i,j)}_{R} \longrightarrow (X, \zeta_1, \zeta_2) \in T^{(i,j)}_{R}$.

Proof. From Theorem 3.6, it is obvious. □

The following example shows that in general the reverse of Theorem 5.2 need not be true.

Example 5.3. Let $X = \{s, t, w\}$ and ζ_1, ζ_2 be two fuzzifying topologies defined as follows:

$$\zeta_1(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{s,t\}\},\\ 1/4, & \text{if } O = \{w\},\\ 0, & \text{o.w.}, \end{cases} \qquad \zeta_2(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X\},\\ 1/3, & \text{if } O \in \{\{w\}, \{s,t\}\},\\ 0, & \text{o.w.}. \end{cases}$$

Note that,

$$\mathcal{F}_1(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X, \{w\}\},\\ 1/4, & \text{if } O = \{s,t\},\\ 0, & \text{o.w.}, \end{cases} \qquad \mathcal{F}_2(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X\},\\ 1/3, & \text{if } O \in \{\{w\}, \{s,t\}\},\\ 0, & \text{o.w.}. \end{cases}$$

Thus

$$\mathcal{F}^p(O) = \begin{cases} 1, & \text{if } O \in \{\emptyset, X\}, \\ 1/3, & \text{if } O = \{w\} \\ 1/4, & \text{if } O = \{s,t\}, \\ 0, & \text{o.w..} \end{cases}$$

Since $[(X,\zeta_1,\zeta_2) \in T^{(1,2)}_{R_i}] = \bigwedge_{x\in H} \min\left(1, 1-\zeta_i(H) + \bigvee_{O\in P(X), x\in O} \min\left(\zeta_i(O), \bigwedge_{y\notin H}(1-\sigma_j(O)(y))\right)\right)$, we have two cases to determine H as follows.

Case 1: $H = \{w\}$, $x = w$, and $O = \{w\}$.

Case 2:

(1) $H = \{s,t\}$, $x = s$, and $O = \{s,t\}$;
(2) $H = \{s,t\}$, $x = t$, and $O = \{s,t\}$.

Note that, $\sigma_1(\{w\}) = \{(s,0),(t,0),(w,1)\}$, $\sigma_1(\{s,t\}) = \{(s,1),(t,1),(w,3/4)\}$, $\sigma_2(\{w\}) = \{(s,2/3),(t,2/3),(w,1)\}$, $\sigma_2(\{s,t\}) = \{(s,1),(t,1),(w,2/3)\}$, and $\sigma^p(\{w\}) = \{(s,2/3),(t,2/3),(w,1)\}$, $\sigma^p(\{s,t\}) = \{(s,1),(t,1),(w,3/4)\}$. Thus $[(X,\zeta_1,\zeta_2) \in T^{(1,2)}_{R_1}] = 1/3$, $[(X,\zeta_1,\zeta_2) \in T^{(1,2)}_{R_2}] = 11/12$. Also, we have $[(X,\zeta_1,\zeta_2) \in {}^sT^{(1,2)}_{R_1}] = 1/4$, $[(X,\zeta_1,\zeta_2) \in {}^sT^{(1,2)}_{R_2}] = 11/12$. Therefore $[(X,\zeta_1,\zeta_2) \in T^{(1,2)}_{R}] = 1/3 \not\leq 1/4 = [(X,\zeta_1,\zeta_2) \in {}^sT^{(1,2)}_{R}]$.

6. Conclusion

In the present paper, we introduced a generalization of pairwise separation axioms in the fuzzifying bitopological spaces, also, we introduced and studied new strongly pairwise axioms in the fuzzifying bitopological spaces. In the future, we hope study these axioms in the fuzzifying soft bitopological spaces and its applications.

References

[1] A. A. E.-M. Allam, A. M. Zahran, A. K. Mousa, H. M. Binshahnah, *On separation axioms in fuzzifying bitopological spaces*, Int. J. Fuzzy Log. Intell. Syst., **20** (2020), 77–86. 1, 2.5, 2.6, 2.7, 2.8
[2] T. M. Al-Shami, M. E. El-Shafei, *Two types of separation axioms on supra soft separation spaces*, Demonstr. Math., **52** (2019), 147–165. 1
[3] T. M. Al-Shami, M. E. El-Shafei, *Two new forms of ordered soft separation axioms*, Demonstr. Math., **53** (2020), 8–26. 1
[4] J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc., **13** (1963), 71–89. 1
[5] A. Kilicman, Z. Salleh, *On pairwise Lindelöf bitopological spaces*, Topology Appl., **154** (2007), 1600–1607. 1
[6] J. Z. Shen, *Separation axiom in fuzzifying topology*, Fuzzy Sets and Systems, **57** (1993), 111–123. 1
[7] M. S. Ying, *A new approach for fuzzy topology. I*, Fuzzy Sets and Systems, **39** (1991), 302–321. 1, 2.1, 2.2, 2.3
[8] M. S. Ying, *A new approach for fuzzy topology. (II)*, Fuzzy Sets and Systems, **47** (1992), 221–232.
[9] M. S. Ying, *A new approach for fuzzy topology. III*, Fuzzy Sets and Systems, **55** (1993), 193–207. 1
[10] G. J. Zhang, M. J. Liu, *On properties of $\theta_{(i,j)}$-open sets in fuzzifying bitopological space*, J. Fuzzy Math., **11** (2003), 165–178. 1, 2.4