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Generalized π-weak closed sets and some applications on weak structures

Rodyna A. Hosny[a], Tareq M. Al-Shami[b,c], Radwan Abu-Gdairi[d], Abdelwaheb Mhemdi[e,*]

[a]Department of Mathematics, Zagazig University, Zagazig, Egypt.

[b]Department of Mathematics, Sana'a University, P.O. Box 1247, Sana'a, Yemen.

[c]Department of Engineering Mathematics & Physics, Faculty of Engineering & Technology, Future University, Egypt.

[d]Department of Mathematics, Faculty of Science, Zarqa University, P.O. Box 13110 Zarqa, Jordan.

[e]Department of Mathematics, College of Sciences and Humanities in Aflaj, Prince Sattam bin Abdulaziz University, Riyadh, Saudi Arabia.

Abstract

This article highlights the concept of generalized π-weak closed sets (briefly, $g\pi(w)$-closed) in weak structures, that possess a lot of applications in information systems. We study their master properties and show the interrelationships between them and some types of sets with the help of counterexamples. Then, we applied $g\pi(w)$-closed sets to define new types of the concepts of separation axioms, continuous functions, closed w-graph and strongly closed w-graph. We give some characterizations of these concepts and discuss main features. Moreover, we provide some examples to show some topological properties of these concepts that are losing in the frame of weak structures.

Keywords: Weak structure, closed weak graph, strongly closed weak graph, separation axioms.

2020 MSC: 54A05, 54C08, 54J05.

1. Introduction

This paper looks at some topological concepts via the environment of generalized topology by introducing the notions of separation axioms, continuous functions, and graphs. Now, the development of topologies including its generalizations, the motivations of the current research, and the contributions of the present manuscript are illustrated as follows.

1.1. A brief review in topology

Topology is a recent branch of mathematics known as a new kind of geometry that doesn't measure the distance between points but instead looks at how close they are to each other. In other words, it makes it clear what objects are thought to be close to each other. In general, it was built a topology on a

*Corresponding author
Email address: mhemdiabd@gmail.com (Abdelwaheb Mhemdi)

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nonempty set X as a subset of its power set P(X) that is containing the empty and universal set as well as it is closed under finite intersection and arbitrary union.

Topological space is an utmost important tool to handle and address many real-life problems. In rough set theory, it has been successfully applied the interior and closure topological operators to describe the lower and upper rough approximation operators, see, for example, [8, 10, 33]. This positive and interesting interaction allows us to describe the data of information systems by abstract concepts and then move their theoretical properties to practical frameworks. Moreover, it was as established software programs to characterize geographic information systems (GISs) using some formulas inspired by abstract topological concepts, for more details we refer the readers to [36] and the references mentioned therein.

1.2. A brief review in the extensions of topology

There was generalized topological spaces by different methods, which we can classify into three main methods: 1) strengthen or weaken one or more of the three terms of topology which produces different structures such as Alexendroff topology [1], supra-topology [7, 21, 28], infra topology [17], and generalized topology [18]; 2) combine topology with another mathematical structure which produces some structures like ordered topology [6, 11], ideal topology and bitopology; and 3) constructing using one of the generalizations of crisp sets such as fuzzy topology and soft topology [9, 13].

In this article, we focus on the principle of weak structures that is initiated by Császár [19]. This structure is obtained from the first type of the above-mentioned techniques. It is defined on a nonempty set X as a subfamily of the power set P(X) of X containing the empty set. Császár introduced several concepts: such as $\alpha(w), \pi(w), \sigma(w), \beta(w)$ and $\rho(w)$ with the aid of i_w, c_w operators. After that, many of those interested in the idea studied many concepts in the weak structures, see [30, 37–39]. Al-Omari and Noiri [3] introduced $(w, \acute{w})$-continuous function between two weak structures spaces (X, w), $(\acute{X}, \acute{w})$. Kim and Min [23] studied some topological notions via weak structures such as compactness and continuity. Recently, it has been exploited the concept of "weak structure" to investigate some practical issues in different fields. Also, Császár [18] introduced the structure of generalized topology concept by deleting only the intersection condition. In analogy with topology, he studied the concepts of interior and closure operators, continuity and separation axioms on generalized topology. Tyagi and Chauhan [35] introduced the concept of generalized closed sets via the structures of generalized topology.

1.3. Motivations and layout of this work

We notice that study into the basic ideas and concepts of weak structures has not yet been given the attention it needs. So, there are still a lot of interesting ideas in this system that need to be put into words and talked about. We make a new contribution to this field by looking at some notions that can be used to study weak structures.

There are several reasons why it makes sense to study topological ideas in the context of weak structures. First, this setting suffices to preserve some topological characteristics and properties under conditions that do not require a topology; for example, the interior and closure points of a subset are, respectively, still open and closed subsets; a closed subset of a compact space is compact, etc. Second, studying topological ideas via weak structures produces a richer variety of concepts, especially over a finite set; for instance, the only T_1-topology defined on a finite set is the discrete topology (which is a trivial case, and hence meaningless in application areas), whereas there are several sorts of weak structures that produce T_1-spaces. Also, the weak structures show how easy and varied it is to build examples that meet T_i-spaces in weak structures compared to their counterparts in classical topology, especially those related to strong types of separation axioms. Third, weak structures is a good way to explain many real-world problems. This can be seen by looking at rough set approximation operators, which are made by topological approaches. It's important to note that the theory of rough sets was used to analyze information systems using the abstract and theoretical extensions of topology such as supra topology [12], infra topology [14], and weak structure [2, 20]. Other applications for topological extensions can be found in [2, 20, 24, 25, 31, 32, 34].

In this article, we are going to acquaint the ideas of generalized π-weak closed and π-weak open sets as a generalization of gw-closed sets in weak structures and study some of their behaviors, which is the frame of many applications such as computer sciences, medical sciences, and information systems. By using this concept, the specifications of w-regular and w-normal spaces are inspected. Further, between weak structure spaces some kinds of weak continuous functions such $(w, \pi(\acute{w}))$-continuous (resp. $(w, \acute{w})\pi$-continuous, $g(w, \acute{w})\pi$-continuous and almost $(w, \acute{w})\pi$-continuous) functions are given. Finally, a notion of $(w, \acute{w})$-closed in the product space and the functions with closed w-graph, strongly closed w-graph are acquired with some of their principle advantages.

2. Preliminaries

Now, we recall some concepts and results that help us to understand the content of this article. Weak structure space (X, w) will be written henceforth as $\mathbb{WSS}$.

Definition 2.1 ([19])**.** Let N be a subset of a $\mathbb{WSS}$ (X, w). We define $c_w(N)$ as the intersection of all w-closed sets containing N, and $i_w(N)$ as the union of all w-open sets that are contained in N.

Definition 2.2. A subset N of a $\mathbb{WSS}$ (X, w) is called:

(i) regular w-closed [39] if $N = c_w i_w(N)$;
(ii) $\pi(w)$-closed ([19]) if $c_w i_w(N) \subset N$;
(iii) w-nowhere dense [30] if $i_w c_w(N) = \emptyset$;
(iv) gw-closed [39] if $c_w(N) \subset O$, whenever $N \subset O$ and O is w-open;
(v) strongly generalized w-closed [39] if $c_w(N) \subset O$, whenever $N \subset O$ and O is gw-open;
(vi) mildly w-closed [39] if $c_w i_w(N) \subset O$, whenever $N \subset O$ and O is gw-open.

Theorem 2.1 ([39])**.** *Let (X, w) be a $\mathbb{WSS}$. Then every w-closed set is gw-closed.*

Definition 2.3 ([39])**.** A $\mathbb{WSS}$ (X, w) is called w-normal, if for each two disjoint w-closed sets F and H, there exist two disjoint w-open sets U and V such that $F \subseteq U$, $H \subseteq V$.

Definition 2.4 ([3])**.** Let w and $\acute{w}$ be weak structures on X and $\acute{X}$, respectively. A function $f : (X, w) \longrightarrow (\acute{X}, \acute{w})$ is called $(w, \acute{w})$-continuous if for each $x \in X$ and each $V \in \acute{w}$ containing $f(x)$, there exists $U \in w$ containing x such that $f(U) \subseteq V$.

Definition 2.5 ([38])**.** A function f: $(X, w) \longrightarrow (\acute{X}, \acute{w})$ is called almost w-continuous at $x \in X$ if for every $\acute{w}$-open set N containing $f(x)$, there is a w-open set M including x such that $f(M) \subseteq i_w c_w(N)$. A function f is called almost w-continuous if it is almost w-continuous at each $x \in X$.

3. Generalized π-weak closed sets and their basic features

In weak structure spaces, most of the fundamental concepts and facts in ordinary topology are defined analogously. It is expected that many considerable results in ordinary topology will not be carried over and some of interesting properties will be missing or weakened. Thus any statement which is true in weak structure spaces is true in ordinary topological spaces. However, in order to attain desirable and interesting conclusions, additional conditions must be imposed. In this section, we are going to acquaint the ideas of generalized π-weak closed and generalized π-weak open sets in weak structure spaces and study some of their behaviors.

Definition 3.1. A subset N of a $\mathbb{WSS}$ (X, w) is said to be:

(i) a generalized π-weak closed (briefly, $g\pi(w)$-closed) if $c_{\pi(w)}(N) \subset O$ whenever $N \subset O$ and O is w-open, (where $c_{\pi(w)}(N) = N \cup c_w i_w(N)$);
(ii) a generalized π-weak open ($g\pi(w)$-open) set if $(X \setminus N)$ is $g\pi(w)$-closed.

The family of all $g\pi(w)$-closed (resp. $g\pi(w)$-open) sets in a $\mathbb{WSS}$ (X, w) will be denoted by $G\pi(w)C(X)$ (resp. $G\pi(w)O(X)$).

Remark 3.1. If (X, w) is a $\mathbb{WSS}$ such that $X = \cup\{N \subset X \mid N \in w\}$, then $c_w(\emptyset) = \emptyset$.

The proof of the coming theorem is obvious and then omitted.

Theorem 3.1. *Let* N *be a subset of a* $\mathbb{WSS}$ (X, w). *Then*

(i) N *is* $g\pi(w)$*-closed iff* $c_w i_w(N) \subset O$ *whenever* $N \subset O$ *and* O *is* w*-open;*
(ii) N *is* $g\pi(w)$*-open iff* $T \subset i_w c_w(N)$ *whenever* $T \subset N$ *and* T *is* w*-closed.*

Proposition 3.1. *Let* N *be a* w*-nowhere dense subset of a* $\mathbb{WSS}$ (X, w). *Then*

(i) *if* N *is* $g\pi(w)$*-open, then* N *does not contain any* w*-closed set;*
(ii) *if* $c_w(\emptyset) = \emptyset$, *then every* w*-nowhere dense set is* $g\pi(w)$*-closed.*

Proof.

(i) Follows directly from definition of w-nowhere dense and Theorem 3.1.
(ii) Let O be a w-open set with $N \subset O$. If N is a w-nowhere dense subset of X, then $i_w c_w(N) = \emptyset$. Therefore, $c_w i_w(N) = c_w(\emptyset) = \emptyset \subset O$. Hence, N is $g\pi(w)$-closed.

□

Theorem 3.2. *Let* (X, w) *be a* $\mathbb{WSS}$. *Then*

(i) *strongly generalized* w*-closed set* $\Longrightarrow$ *mildly* w*-closed set* $\Longrightarrow$ $g\pi(w)$*-closed set;*
(ii) *strongly generalized* w*-closed set* $\Longrightarrow$ gw*-closed set* $\Longrightarrow$ $g\pi(w)$*-closed set.*

Proof. Straightforward. □

Theorem 3.3. *In a* $\mathbb{WSS}$ (X, w), *the following statements hold:*

(i) *every* $\pi(w)$*-closed set is* $g\pi(w)$*-closed;*
(ii) *every regular* w*-closed set is* $g\pi(w)$*-closed;*
(iii) *every* gw*-closed set is* $g\pi(w)$*-closed;*
(iv) *if* $i_w(N) \in G\pi(w)C(X)$, *then* $N \in G\pi(w)C(X)$.

Proof. We only prove (i) and (iii). The other cases are made similarly.

(i): Let O be a w-open set with $N \subset O$. Since N is a $\pi(w)$-closed set, then $c_w i_w(N) \subset N$. Therefore, N is $g\pi(w)$-closed.

(iii): Let N be a gw-closed set, then $c_w(N) \subset O$ whenever $N \subset O$ and O is a w-open set in X. Therefore, $c_w i_w(N) \subset O$. Hence N is $g\pi(w)$-closed. □

The converse of Theorem 3.3 need not be true as this may be seen from the following two examples.

Example 3.1. Let $w = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ be a weak structure on $X = \{a, b, c\}$. One may notice that $\{a, b\}$ is $g\pi(w)$-closed set, but it is not $\pi(w)$-closed. Also, it is not regular w-closed set. Hence, the converse of the results (i) and (ii) need not be true.

Example 3.2. Consider the $\mathbb{WSS}$ (X, w), where $X = \{a, b, c, d\}$, $w = \{\emptyset, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}$. If $N = \{a\}$, then it is $g\pi(w)$-closed but not gw-closed and $i_w(N) \notin G\pi(w)C(X)$. Then the converse of the results (iii) and (iv) does not come true.

Corollary 3.1. *In a* $\mathbb{WSS}$ (X, w), *every* w*-closed set is* $g\pi(w)$*-closed.*

Proof. It follows from Theorem 2.1 and Theorem 3.3 (iii). □

Corollary 3.2. *In a* $\mathbb{WSS}$ (X, w), *then* w*-open* $\Longrightarrow$ gw*-open* $\Longrightarrow$ $g\pi(w)$*-open.*

Proof. It follows from Theorem 3.3 and Corollary 3.1. □

Theorems 3.4, 3.8, and 3.9 display the appropriate conditions that must be satisfied in weak structure spaces to modify $g\pi(w)$-closed sets to be $\pi(w)$-closed.

Theorem 3.4. *If* N *is a* $g\pi(w)$*-closed and* w*-open set in* (X, w), *then* N *is regular* w*-closed. Furthermore,* N *is* $\pi(w)$*-closed.*

Proof. Since N is a $g\pi(w)$-closed and w-open set, then $c_w i_w(N) \subset N$ (i.e., N is $\pi(w)$-closed) and $N = i_w(N) \subset c_w i_w(N)$. Hence, N is regular w-closed. □

Proposition 3.2. *In a* $\mathbb{WSS}$ (X, w), *each subset of* X *is* $g\pi(w)$*-closed if each* w*-open set is* w*-closed.*

Proof. Suppose N be any subset of X. Let O be a w-open set with $N \subset O$, then $c_w i_w(N) \subset c_w i_w(O) = c_w(O)$. Since each w-open set is w-closed, then $c_w i_w(N) \subset O$ and so N is a $g\pi(w)$-closed set. □

The converse of Proposition 3.2 need not be true in general. The next example supports our claim.

Example 3.3. Let $X = \{a, b, c\}$ and $w = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. One may note that every subset of X is $g\pi(w)$-closed, but $N = \{c\}$ is a w-open set in X and it is not w-closed.

Theorem 3.5. *Let* (X, w) *be a* $\mathbb{WSS}$. *Then every subset of* X *is* $g\pi(w)$*-closed iff for every* w*-open set* O *of* X, $c_w(O) = O$.

Proof. The necessary part is obvious. To prove the sufficient part, let N be any subset of X such that $N \subset O$ and O be a w-open set. Then $c_w i_w(N) \subset c_w i_w(O) = O$. So N is $g\pi(w)$-closed. □

Corollary 3.3. *Let* (X, w) *be a* $\mathbb{WSS}$. *If a subset* N *of* X *is both* w*-open and* $g\pi(w)$*-closed, then it is both regular* w*-open and regular* w*-closed.*

Proof. Since N is both w-open and $g\pi(w)$-closed set, it follows from Theorem 3.5 that $N = c_w(N)$. Consequently, $i_w c_w(N) = i_w(N) = N$ and $c_w i_w(N) = c_w(N) = N$. Hence, N is both regular w-open and regular w-closed set. □

Remark 3.2. The union and intersection of any two $g\pi(w)$-closed sets need not to be $g\pi(w)$-closed in general, as shown by the next example.

Example 3.4. Let $X = \{a, b, c, d\}$ and $w = \{\emptyset, \{a\}, \{b, c\}, \{c, d\}, \{a, b, d\}\}$. Then (X, w) is a $\mathbb{WSS}$.

(i) The two sets $N = \{a, b\}$ and $B = \{b, d\}$ are $g\pi(w)$-closed, but their union $\{a, b, d\}$ is not a $g\pi(w)$-closed set.
(ii) The two sets $N = \{a, b, c\}$ and $B = \{b, c, d\}$ are two $g\pi(w)$-closed, but their intersection $\{b, c\}$ is not a $g\pi(w)$-closed set.

One may notice that $\emptyset$ is not $g\pi(w)$-closed set as seen in the following example.

Example 3.5. Let $X = \{a, b, c\}$, $w = \{\emptyset, \{a\}, \{c\}\}$. Then (X, w) is a $\mathbb{WSS}$. Now, $c_w i_w(\emptyset) = \{b\}$, so that, it is not a $g\pi(w)$-closed set which means that X is not a $g\pi(w)$-open set.

Remark 3.3. $GSWO(X)$ is a weak structure on X finer than w. If $c_w(\emptyset) = \emptyset$, then $GSWO(X)$ is a minimal structure on X (see [27]).

Definition 3.2. Let (X, w) be a $\mathbb{WSS}$ and $N \subset X$. Then the intersection of all $g\pi(w)$-closed sets containing N is called the $g\pi(w)$-closure of N and denoted by $c_{g\pi(w)}(N)$. Also, the union of all $g\pi(w)$-open contained in N is called the $g\pi(w)$-interior of N and denoted by $i_{g\pi(w)}(N)$.

Proposition 3.3. *Let* (X, w) *be a* $\mathbb{WSS}$ *and* $N \subset X$. *Then* $c_{g\pi(w)}(N) = \{x \in X \mid N \cap O \neq \emptyset$ *for all* $g\pi(w)$*-open sets* O *containing* $x\}$.

Proof. Obvious. □

It is clear that $c_{g\pi(w)}(N) \subset c_w(N)$ for every subset N of a $\mathbb{WSS}$ (X, w).

Theorem 3.6. *Let* (X, w) *be a* $\mathbb{WSS}$. *If* N *is a* $g\pi(w)$*-closed set and* $N \subset B \subset c_w i_w(N)$, *then* B *is* $g\pi(w)$*-closed.*

Proof. Let O be a w-open set with $B \subset O$. Since $N \subset B$ and N is a $g\pi(w)$-closed set, then $c_w i_w(N) \subset O$. Since $c_w i_w(B) \subset c_w i_w c_w i_w(N) = c_w i_w(N) \subset O$, B is $g\pi(w)$-closed. □

Corollary 3.4. *Let* (X, w) *be a* $\mathbb{WSS}$ *and* N *be a* $g\pi(w)$*-open set. Then*

(i) *if* $i_w c_w(N) \subset B \subset N$, *then* B *is* $g\pi(w)$*-open;*
(ii) *if* N *is* w*-nowhere dense, then every subset of* N *is* $g\pi(w)$*-open.*

Proposition 3.4. *Let* (X, w) *be a* $\mathbb{WSS}$. *Then the following statements hold.*

(i) *If* $O, V \in w$ *and* $O \cap V = \emptyset$, *then* $c_{\pi(w)}(O) \cap V = \emptyset$ *and* $O \cap c_{\pi(w)}(V) = \emptyset$.
(ii) $N \cup (X \setminus c_w i_w(N))$ *is* w*-dense set for any nonempty set* N *of* X.

Proof.

(i) Let $O, V \in w$ and $O \cap V = \emptyset$. Then $O \subset (X \setminus V)$ and so $c_w(O) \subset c_w(X \setminus V) = (X \setminus V)$. Therefore, $c_w(O) \cap V = \emptyset$ and $O \cap c_w(V) = \emptyset$. Since $c_{\pi(w)}(A) \subset c_w(A)$ for any set A, we obtain $c_{\pi(w)}(O) \cap V = \emptyset$ and $O \cap c_{\pi(w)}(V) = \emptyset$.

(ii) Let $x \in X$ and $x \notin c_w[N \cup (X \setminus c_w i_w(N))]$. Then there exists a w-open set O such that $x \in O$ and $O \cap [N \cup (X \setminus c_w i_w(N))] = \emptyset$. Then $O \cap N = \emptyset$ and $x \in O \subset c_w i_w(N)$. Now, $x \in O \subset c_w(N)$ implies that $O \cap N \neq \emptyset$, a contradiction. Consequently, $c_w[N \cup (X \setminus c_w i_w(N))] = X$, i.e., $N \cup (X \setminus c_w i_w(N))$ is a w-dense set. □

Proposition 3.5. *If* N *is a* $g\pi(w)$*-open set in a* $\mathbb{WSS}$ (X, w), *then* $O = X$ *whenever* O *is a* w*-open set and* $i_w c_w(N) \cup (X \setminus N) \subset O$.

Proof. Assume that O is a w-open set and $i_w c_w(N) \cup (X \setminus N) \subset O$. Then $i_w c_w(N) \subset O$ and $(X \setminus N) \subset O$. Since N is a $g\pi(w)$-open set, then $(X \setminus i_w c_w(N)) \subset O$. On the other hand, $i_w c_w(N) \cup (X \setminus i_w c_w(N)) \subset O$. Hence $O = X$. □

Corollary 3.5. *If a subset* N *of a* $\mathbb{WSS}$ (X, w) *is* $g\pi(w)$*-closed, then* $N \cup (X \setminus c_w i_w(N))$ *is* $g\pi(w)$*-closed.*

Proof. Let O be a w-open set which containing $N \cup (X \setminus c_w i_w(N))$, then $N \subset O$ and $(X \setminus c_w i_w(N)) \subset O$. Since N is a $g\pi(w)$-closed set, so $c_w i_w(N) \subset O$. Hence $c_w i_w(N) \cup (X \setminus c_w i_w(N)) \subset O$ and so $O = X$. Consequently, $c_w i_w(N \cup (X \setminus c_w i_w(N))) \subset O$, i.e., $N \cup (X \setminus c_w i_w(N))$ is $g\pi(w)$-closed. □

To prove the converse need not be true, we consider the following example.

Example 3.6. Let $X = \{a, b, c\}$, $w = \{\emptyset, \{a\}, \{c\}, \{a, b\}\}$. If $N = \{a\}$, then $N \cup (X \setminus c_w i_w(N)) = \{a, c\}$ which is $g\pi(w)$-closed set, but N is not a $g\pi(w)$-closed.

Corollary 3.6. *Let* (X, w) *be a* $\mathbb{WSS}$. *If* N *is a* $g\pi(w)$*-closed set, then* $c_{\pi(w)}(N) \setminus N$ *contains only null* w*-closed set.*

Remark 3.4. The converse of the Corollary 3.6 need not be true. Consider a $\mathbb{WSS}$ (X, w) as in Example 3.6. If $N = \{a\}$, then $c_w i_w(N) \setminus N = \{b\}$ does not contain any non-null w-closed, but N is not a $g\pi(w)$-closed set in X.

Theorem 3.7. *If a* $\mathbb{WSS}$ (X, w) *satisfies the following condition*

(C): for any w-closed set T and any subset M of X, $c_w i_w(M) \cap T$ is w-closed,

then a subset N of X is $g\pi(w)$-closed iff $c_{\pi(w)}(N) \setminus N$ contains only null w-closed set.

Proof. Let N be a $g\pi(w)$-closed set. By Corollary 3.6, $c_{\pi(w)}(N) \setminus N$ contains only null w-closed set. Conversely, suppose the condition (C) holds. Let O be a w-open set which containing N and $c_w i_w(N) \cap (X \setminus O) \neq \emptyset$. Then $c_w i_w(N) \cap (X \setminus O)$ is w-closed set and $[c_w i_w(N) \setminus O] \subset [c_{\pi(w)}(N) \setminus N]$. This is a contradiction the assumption. So N is $g\pi(w)$-closed set. □

Theorem 3.8. *If N is a $g\pi(w)$-closed subset of a* $\mathbb{WSS}$ *(X, w) such that $c_{\pi(w)}(N) \setminus N$ is w-closed, then N is $\pi(w)$-closed.*

Proof. Let N be a $g\pi(w)$-closed subset such that $c_{\pi(w)}(N) \setminus N$ is w-closed. Then, $c_{\pi(w)}(N) \setminus N$ is a w-closed subset of itself. By Corollary 3.6, $c_{\pi(w)}(N) \setminus N = \emptyset$ and hence $c_w i_w(N) \subset N$ showing N is a $\pi(w)$-closed set. □

Corollary 3.7. *Let (X, w) be a* $\mathbb{WSS}$. *If N is a $g\pi(w)$-closed set with $N \subset B \subset c_w i_w(N)$, then $c_{\pi(w)}(B) \setminus B$ is $g\pi(w)$-closed set contains no non-null w-closed.*

Proof. It follows from Theorem 3.6 and Corollary 3.6. □

Proposition 3.6. *Let (X, w) be a* $\mathbb{WSS}$. *Then a set N is a $g\pi(w)$-closed if $c_{\pi(w)}(\{x\}) \cap N \neq \emptyset$ for each $x \in c_{\pi(w)}(N)$.*

Proof. Suppose O is a w-open set and $N \subset O$. Let $x \in c_{\pi(w)}(N)$, then we have two cases..

Case I: $\{x\} \cap N \neq \emptyset$. Hence, $x \in O$ and $c_{\pi(w)}(N) \subset O$.

Case II: $c_w i_w\{x\} \cap N \neq \emptyset$, i.e., there exists $y \in c_w i_w\{x\} \cap N$, so $y \in N \subset O$. Thus $i_w\{x\} \cap O \neq \emptyset$ and hence $\{x\} \cap O \neq \emptyset$. Therefore $x \in O$ and $c_{\pi(w)}(N) \subset O$, which implies that N is $g\pi(w)$-closed. □

Next example shows that the converse does not necessarily have to be true in general.

Example 3.7. Consider Example 3.5, one may notice that $N = \{a, c\}$ is $g\pi(w)$-closed, $c_{\pi(w)}(N) = X$ and $c_{\pi(w)}(\{b\}) = \{b\}$. Hence $c_{\pi(w)}(\{b\}) \cap N = \emptyset$.

Proposition 3.7. *Let (X, w) be a* $\mathbb{WSS}$. *Then $\{x\}$ is w-nowhere dense or $\pi(w)$-open set, for every $x \in X$.*

Proof. Suppose $\{x\}$ is not w-nowhere dense. Then $i_w c_w\{x\} \neq \emptyset$. Hence $x \in i_w c_w\{x\}$ and so $\{x\} \subset i_w c_w\{x\}$, i.e., $\{x\}$ is $\pi(w)$-open. If $\{x\}$ is not $\pi(w)$-open, then $\{x\} \not\subset i_w c_w\{x\}$. Therefore $i_w c_w\{x\} = \emptyset$. Hence $\{x\}$ is w-nowhere dense. □

Theorem 3.9. *Let (X, w) be a* $\mathbb{WSS}$. *If each singleton is either w-closed or w-open, then every $g\pi(w)$-closed subset of X is $\pi(w)$-closed.*

Proof. Let N be a $g\pi(w)$-closed set of X. Then we have two cases. Firstly, suppose that $\{x\}$ is a w-closed set. Let $x \notin N$ and $x \in c_w i_w(N)$, then $\{x\} \subset (X \setminus N)$, which implies $N \subset (X \setminus \{x\})$. Since N is a $g\pi(w)$-closed set and $(X \setminus \{x\})$ is a w-open set, then $c_w i_w(N) \subset (X \setminus \{x\})$ and hence $\{x\} \subset (X \setminus c_w i_w(N))$. Therefore $\{x\} \subset c_w i_w(N) \cap (X \setminus c_w i_w(N)) = \emptyset$, which is a contradiction. Thus $x \in N$ and hence N is $\pi(w)$-closed. Secondly, suppose $\{x\}$ is a w-open set. If $x \in c_w i_w(N)$, then for each w-open set G with $x \in G$, we have $\emptyset \neq G \cap i_w(N) \subset G \cap N$. Hence $\{x\} \cap N \neq \emptyset$ and hence $x \in N$. Thus N is $\pi(w)$-closed. □

4. Separation axioms via closeness of weak structures

Zahran et al. [39] introduced the concepts of $T_{\frac{1}{2}}$-, T_1-, normal, almost normal and weakly normal spaces by using the concepts of gw-closed, sgw-closed and mgw-closed sets in weak structure spaces. In the current part, separation axioms on a $\mathbb{WSS}$ via $g\pi(w)$-closed sets are introduced with some of their properties.

Definition 4.1. A $\mathbb{WSS}$ (X, w) is called

(i) $wT^{\pi}_{\frac{1}{2}}$ if every $g\pi(w)$-closed set is w-closed;

(ii) wT_2 (w-Hausdorff) if for any two distinct points $x_1, x_2 \in X$, there exist disjoint w-open sets O_1, O_2 such that $x_1 \in O_1$ and $x_2 \in O_2$;

(iii) $w\grave{T}_2$ if for any two distinct points $x_1, x_2 \in X$, there exist w-open sets O_1, O_2 such that $x_1 \in O_1, x_2 \in O_2$ and $c_w(O_1) \cap c_w(O_2) = \emptyset$.

Obviously, every $w\grave{T}_2$ space is wT_2.

Recall that a topological space (X, τ) is $T_{\frac{1}{2}}$ if each singleton is either closed or open. In a weak structure space (X, w), the following example shows that, if each singleton is w-open or w-closed, then (X, w) need not be $wT^{\pi}_{\frac{1}{2}}$.

Example 4.1. Let $X = \{a, b, c, d\}$ and $w = \{\emptyset, \{b\}, \{c\}, \{a, d\}, \{a, b, c\}, \{b, c, d\}\}$. One may notice that each singleton is w-open or w-closed. But there exists $N = \{a, b\}$ which is $g\pi(w)$-closed set and N is not w-closed. So (X, w) is not $wT^{\pi}_{\frac{1}{2}}$.

Theorem 4.1. *If (X, w) is a $wT^{\pi}_{\frac{1}{2}}$ space and $c_w(\emptyset) = \emptyset$, then every w-nowhere dense singleton is w-closed.*

Proof. Let $\{x\}$ be a w-nowhere dense set and $c_w(\emptyset) = \emptyset$, then from Proposition 3.1 (ii), $\{x\}$ is $g\pi(w)$-closed. Since (X, w) is $wT^{\pi}_{\frac{1}{2}}$, so $\{x\}$ is w-closed. □

Theorem 4.2. *Every singleton is $g\pi(w)$-closed, if (X, w) satisfies the following condition*

(S): $x \in c_w i_w(\{y\}) \Longrightarrow y \in c_w i_w(\{x\})$, *for each* $x, y \in X$.

Proof. Let $x \in X$ and $\{x\} \subset O \in w$. Suppose that $c_w i_w(\{x\}) \cap (X \setminus O) \neq \emptyset$, then there exist $y \in c_w i_w(\{x\}) \cap (X \setminus O)$. So $y \in c_w i_w(\{x\})$ and $y \in (X \setminus O)$. Since (X, w) satisfies condition (S), hence $x \in c_w i_w(\{y\})$. Since $c_w i_w(\{y\})$ is the intersection of all w-closed sets containing $i_w(\{y\})$ and $(X \setminus O)$ is w-closed set containing $i_w(\{y\})$, then $c_w i_w(\{y\}) \subset (X \setminus O)$. It follows that $x \notin O$. It is a contradiction. Consequently, $c_w i_w(\{x\}) \subset O$, i.e., $\{x\}$ is $g\pi(w)$-closed, for each $x \in X$. □

Definition 4.2. A $\mathbb{WSS}$ (X, w) is said to be w-regular if for each $x \in X$ and w-closed set T such that $x \notin T$, there exist disjoint w-open sets O and V such that $x \in O$ and $T \subset V$.

Theorem 4.3. *If (X, w) is a $wT^{\pi}_{\frac{1}{2}}$ space, then the following statements are equivalent:*

(i) *(X, w) is w-regular;*

(ii) *for each w-closed set S and $x \notin S$, there exist disjoint $g\pi(w)$-open set O and w-open set V such that $x \in O$ and $S \subset V$;*

(iii) *for each $N \subset X$ and each w-closed set S with $N \cap S = \emptyset$, there exist disjoint $g\pi(w)$-open set O and w-open set V such that $N \cap O \neq \emptyset$ and $S \subset V$.*

Proof.

(i)$\Longrightarrow$(ii) In view of Corollary 3.2, the implication holds.

(ii)$\Longrightarrow$(i) If (X, w) is $wT^{\pi}_{\frac{1}{2}}$, then every $g\pi(w)$-open set of X is w-open. Based on the above, the required has been proven

(ii)$\Longleftrightarrow$(iii) Obvious. □

Theorem 4.4. *Let (X, w) be a $wT^{\pi}_{\frac{1}{2}}$ space. Consider the following statements:*

(i) *(X, w) is w-regular;*
(ii) *for each $x \in X$ and each w-open set B with $x \in B$, there exists a $g\pi(w)$-open set O such that $x \in O \subset c_w(O) \subset B$.*

Then the implication (i)$\Longrightarrow$(ii) *holds. If $c_w(O)$ is w-closed for every $g\pi(w)$-open set O of X, then the statements are equivalent.*

Proof.

(i)$\Longrightarrow$(ii) Obvious.

(ii)$\Longrightarrow$(i) Let S be a w-closed set and $x \notin S$, then $B = (X \setminus S)$ is w-open and $x \in B$. Thus by (ii) there exists a $g\pi(w)$-open set O such that $x \in O \subset c_w(O) \subset B$. Choose $V = (X \setminus c_w(O))$. Since $c_w(O)$ is w-closed for every $g\pi(w)$-open set O of X, then V is a w-open set such that $x \in O$, $S \subset V$ and $O \cap V = \emptyset$. Consequently, (X, w) is a w-regular space. □

Theorem 4.5. *Let (X, w) be a $\mathbb{WSS}$. Consider the following statements.*

(i) *(X, w) is w-normal.*
(ii) *For any pair of disjoint w-closed sets N and B of X, there exist disjoint w-open set O and $g\pi(w)$-open set V of X such that $N \subset O$ and $B \subset V$.*
(iii) *For each w-closed set N and w-open set G with $N \subset G$, there exists a w-open set O such that $N \subset O \subset c_w(O) \subset G$.*

Then the implications (i)$\Longrightarrow$(ii)$\Longrightarrow$(iii) *hold. If $c_w(O)$ is w-closed for every w-open set O of X, then the statements are equivalent.*

Proof.

(i)$\Longrightarrow$(ii) Obvious, from Corollary 3.2.

(ii)$\Longrightarrow$(iii) Let N and G be w-closed and w-open sets, respectively, such that $N \subset G$. Suppose $B = (X \setminus G)$, then B is w-closed with $N \cap B = \emptyset$. By using (ii), there exist disjoint w-open set O and $g\pi(w)$-open set V of X such that $N \subset O$ and $B \subset V$. Hence $(X \setminus V) \subset G$. Since $(X \setminus V)$ is a $g\pi(w)$-closed set, then $c_w i_w(X \setminus V) \subset G$ and so $c_w(O) \subset c_w i_w(X \setminus V) \subset G$. Consequently, $N \subset O \subset c_w(O) \subset G$.

(iii)$\Longrightarrow$(i) Suppose $c_w(O)$ is w-closed for every w-open set O of X. Let N, B be disjoint w-closed sets. Take $G = (X \setminus B)$, then G is a w-open set with $N \subset G$. From (iii), there exists a w-open set O such that $N \subset O \subset c_w(O) \subset G$. Then there exist disjoint w-open sets O, $X \setminus c_w(O)$ such that $N \subset O$ and $B \subset X \setminus c_w(O)$. Therefore (X, w) is w-normal space. □

5. Types of functions between weak structure spaces

In the current section, we are going to assign $(w, \pi(\acute{w}))$(resp. $(w, \acute{w})\pi$ and $g(w, \acute{w})\pi$, almost $(w, \acute{w})\pi$) continuous functions between two weak structures spaces (X, w), $(\acute{X}, \acute{w})$ and consider some of their attributes.

Definition 5.1. Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces. Then a function $f\colon (X, w) \longrightarrow (\acute{X}, \acute{w})$ is called:

(i) $(w, \pi(\acute{w}))$-continuous if for each $x \in X$ and each $\pi(\acute{w})$-open set N containing $f(x)$ there exists w-open set O containing x such that $f(O) \subseteq N$;

(ii) $(w, \acute{w})\pi$-continuous if for each $x \in X$ and each $\acute{w}$-open set N containing $f(x)$ there exists a $\pi(w)$-open set O containing x such that $f(O) \subseteq N$;

(iii) $g(w, \acute{w})\pi$-continuous if for each $x \in X$ and each $\acute{w}$-open set N containing $f(x)$ there exists a $g\pi(w)$-open set O containing x such that $f(O) \subseteq N$.

Remark 5.1. From the Definition 5.1, we have the following implications but the reverse relations may not be true in general

$$(w, \pi(\acute{w}))\text{-continuous} \Longrightarrow (w, \acute{w})\text{-continuous} \Longrightarrow (w, \acute{w})\pi\text{-continuous} \Longrightarrow g(w, \acute{w})\pi\text{-continuous}.$$

Example 5.1.

(i) Let $X = \{a, b, c, d\}$, $w = \{\emptyset, \{a\}, \{b, c\}\}$. Define f: $(X, w) \longrightarrow (X, w)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. One may notice that f is a $(w, \acute{w})$-continuous function but it is not $(w, \pi(\acute{w}))$-continuous.

(ii) Consider $X = \{a, b, c, d\}$, $w = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}\}$ and $\acute{w} = \{\emptyset, \{a, b, c\}\}$. Define f: $(X, w) \longrightarrow (X, \acute{w})$ as follows: $f(a) = b$, $f(b) = f(d) = d$, $f(c) = c$. One may notice that f is a $(w, \acute{w})\pi$-continuous function but it is not $(w, \acute{w})$-continuous.

(iii) Consider $X = \{a, b, c, d\}$, $w = \{\emptyset, \{a, c\}, \{a, c, d\}, \{b, c, d\}\}$ and $\acute{w} = \{\emptyset, \{a, b, c\}, \{d\}\}$. Define f: $(X, w) \longrightarrow (X, \acute{w})$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$. One may notice that f is a $g(w, \acute{w})\pi$-continuous function but it is not $(w, \acute{w})\pi$-continuous.

Theorem 5.1. *Let w and $\acute{w}$ be weak structures on X and $\acute{X}$, respectively. If $f : (X, w) \longrightarrow (\acute{X}, \acute{w})$, then the following statements are equivalent.*

(i) *f is $(w, \acute{w})\pi$-continuous.*

(ii) $f^{-1}(N) = i_{\pi(w)}(f^{-1}(N))$ *for each $\acute{w}$-open set N in $\acute{X}$.*

(iii) $f^{-1}(T) = c_{\pi(w)}(f^{-1}(T))$ *for each $\acute{w}$-closed set T in $\acute{X}$.*

Proof.

(i)$\Longrightarrow$(ii) Let N be a $\acute{w}$-open set in $\acute{X}$ and $x \in f^{-1}(N)$, then $f(x) \in N$. Since f is $(w, \acute{w})\pi$-continuous, hence there exists a $\pi(w)$-open set O containing x such that $f(O) \subseteq N$. Thus $O \subseteq f^{-1}(N)$ and so $x \in i_{\pi(w)}(f^{-1}(N))$. Consequently, $f^{-1}(N) = i_{\pi(w)}(f^{-1}(N))$.

(ii)$\Longrightarrow$(ii) Let N be a $\acute{w}$-open set containing $f(x)$, then $x \in f^{-1}(N)$. Since $f^{-1}(N) = i_{\pi(w)}(f^{-1}(N))$, so $x \in i_{\pi(w)}(f^{-1}(N))$. Therefore there exists a $\pi(w)$-open set O such that $x \in O \subseteq f^{-1}(N)$. This implies that f is $(w, \acute{w})\pi$-continuous.

(i)$\Longrightarrow$(iii) Obvious. □

Theorem 5.2. *Let w and $\acute{w}$ be weak structures on X and $\acute{X}$, respectively. If $f : (X, w) \longrightarrow (\acute{X}, \acute{w})$, then the following statements are equivalent*

(i) $f^{-1}(N) = i_{\pi(w)}(f^{-1}(N))$ *for each $\acute{w}$-open set N in $\acute{X}$.*

(ii) $f^{-1}(i_{\acute{w}}(N)) \subseteq i_{\pi(w)}(f^{-1}(N))$ *for each subset N in $\acute{X}$.*

(iii) $c_{\pi(w)}(f^{-1}(N)) \subseteq f^{-1}(c_{\acute{w}}(N))$ *for each subset N in $\acute{X}$.*

(iv) $f(c_{\pi(w)}(O)) \subseteq c_{\acute{w}}(f(O))$ *for each subset O in X.*

Proof.

(ii)$\Longrightarrow$(i), (ii)$\Longleftrightarrow$(iii), and (iv)$\Longrightarrow$(iii) Obvious.

(i)$\Longrightarrow$(ii) Let $N \subseteq \acute{X}$. If $x \in f^{-1}(i_{\acute{w}}(N))$, then $f(x) \in i_{\acute{w}}(N)$. Hence there exists a $\acute{w}$-open set B such that $f(x) \in B \subseteq N$ and so $x \in f^{-1}(B) \subseteq f^{-1}(N)$. From (i), $f^{-1}(B) = i_{\pi(w)}(f^{-1}(B))$. This implies that there exists a $\pi(w)$-open set O such that $x \in O \subseteq f^{-1}(B) \subseteq f^{-1}(N)$. Consequently, $x \in i_{\pi(w)}(f^{-1}(N))$. This shows that $f^{-1}(i_{\acute{w}}(N)) \subseteq i_{\pi(w)}(f^{-1}(N))$.

(iii)$\Longrightarrow$(iv) Let $O \subseteq X$, then $c_{\pi(w)}(f^{-1}(f(O))) \subseteq f^{-1}(c_{\acute{w}}(f(O)))$. Hence $c_{\pi(w)}(O) \subseteq f^{-1}(c_{\acute{w}}(f(O)))$ and so $f(c_{\pi(w)}(O)) \subseteq c_{\acute{w}}(f(O))$. □

Theorem 5.3. *Let* $f\colon (X, w) \longrightarrow (\acute{X}, \acute{w})$ *be a* $(w, \acute{w})\pi$*-continuous, where* (X, w), $(\acute{X}, \acute{w})$ *be weak structure spaces. If* N *is a* $\acute{w}$*-open set and* $x \in [c_{\pi(w)}(f^{-1}(N)) \setminus f^{-1}(N)]$, *then* $f(x) \in c_{\acute{w}}(N)$.

Proof. Let $x \in X$ and N be a $\acute{w}$-open set such that $x \in [c_{\pi(w)}(f^{-1}(N)) \setminus f^{-1}(N)]$. Suppose $f(x) \notin c_{\acute{w}}(N)$. Then there exists a $\acute{w}$-open set B containing $f(x)$ such that $B \cap N = \emptyset$. Since f is a $(w, \acute{w})\pi$-continuous function, then there exists a $\pi(w)$-open set O containing x such that $f(O) \subseteq B$. Hence $f(O) \cap N = \emptyset$ and so $O \cap f^{-1}(N) = \emptyset$. Since $x \in c_{\pi(w)}(f^{-1}(N))$, then $G \cap f^{-1}(N) \neq \emptyset$, for every $\pi(w)$-open set G containing x. So $O \cap i_w(f^{-1}(N)) \neq \emptyset$. This is a contradiction. Therefore $f(x) \in c_{\acute{w}}(N)$. □

The proof of next theorems are similar to that of Theorems 5.1 and 5.2.

Theorem 5.4. *Let* w *and* $\acute{w}$ *be weak structures on* X *and* $\acute{X}$, *respectively. If* $f\colon (X, w) \longrightarrow (\acute{X}, \acute{w})$, *then the following statements are equivalent.*

(i) f *is* $g(w, \acute{w})\pi$*-continuous.*
(ii) $f^{-1}(N) = i_{g\pi(w)}(f^{-1}(N))$ *for each* $\acute{w}$*-open set* N *in* $\acute{X}$.
(iii) $f^{-1}(T) = c_{g\pi(w)}(f^{-1}(T))$ *for each* $\acute{w}$*-closed set* T *in* $\acute{X}$.

Theorem 5.5. *Let* w *and* $\acute{w}$ *be weak structures on* X *and* $\acute{X}$, *respectively. If* $f\colon(X, w) \longrightarrow (\acute{X}, \acute{w})$, *then the following statements are equivalent.*

(i) $f^{-1}(N) = i_{g\pi(w)}(f^{-1}(N))$ *for each* $\acute{w}$*-open set* N *in* $\acute{X}$.
(ii) $f^{-1}(i_{\acute{w}}(N)) \subseteq i_{g\pi(w)}(f^{-1}(N))$ *for each subset* N *in* $\acute{X}$.
(iii) $c_{g\pi(w)}(f^{-1}(N)) \subseteq f^{-1}(c_{\acute{w}}(N))$ *for each subset* N *in* $\acute{X}$.
(iv) $f(c_{g\pi(w)}(O)) \subseteq c_{\acute{w}}(f(O))$ *for each subset* O *in* X.

Theorem 5.6. *Let* w *and* $\acute{w}$ *be weak structures on* X *and* $\acute{X}$, *respectively. If* $f\colon (X, w) \longrightarrow (\acute{X}, \acute{w})$, *then the following statements are equivalent.*

(i) f *is* $(w, \pi(\acute{w}))$*-continuous.*
(ii) $f^{-1}(i_{\pi(\acute{w})}(N)) \subseteq i_w(f^{-1}(N))$ *for each subset* N *in* $\acute{X}$.
(iii) $c_w(f^{-1}(N)) \subseteq f^{-1}(c_{\pi(\acute{w})}(N))$ *for each subset* N *in* $\acute{X}$.
(iv) $f(c_w(O)) \subseteq c_{\pi(\acute{w})}(f(O))$ *for each subset* O *in* X.

Definition 5.2. Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces. Then a function $f\colon (X, w) \longrightarrow (\acute{X}, \acute{w})$ is called almost $(w, \acute{w})\pi$-continuous if for each point $x \in X$ and each $\pi(\acute{w})$-open set N containing $f(x)$, there exists a w-open set O containing x such that $f(O) \subseteq i_{\acute{w}}c_{\acute{w}}(N)$.

Theorem 5.7. *Let* w *and* $\acute{w}$ *be weak structures on* X *and* $\acute{X}$, *respectively. If* $f : (X, w) \longrightarrow (\acute{X}, \acute{w})$, *then the following statements are equivalent.*

(i) f *is almost* $(w, \acute{w})\pi$*-continuous.*
(ii) $f^{-1}(N) \subseteq i_w(f^{-1}(i_{\acute{w}}c_{\acute{w}}(N)))$ *for each* $\acute{w}$*-open set* N.
(iii) $c_w(f^{-1}(c_{\acute{w}}i_{\acute{w}}(T))) \subseteq f^{-1}(T)$ *for each* $\acute{w}$*-closed set* T.
(iv) $f^{-1}(N) \subseteq i_w(f^{-1}(i_{\acute{w}}c_{\acute{w}}(N)))$ *for each* $\pi(\acute{w})$*-open set* N.
(v) $c_w(f^{-1}(c_{\acute{w}}i_{\acute{w}}(T))) \subseteq f^{-1}(T)$ *for each* $\pi(\acute{w})$*-closed set* T.
(vi) f *is almost* w*-continuous.*

Proof.

(ii)$\Longrightarrow$(iii) and (iv)$\Longleftrightarrow$(v) Obvious.

(ii)$\Longrightarrow$(vi) Let $x \in X$ and N be a $\acute{w}$-open set containing $f(x)$. From (ii), we have $x \in i_w(f^{-1}(i_{\acute{w}}c_{\acute{w}}(N)))$. Hence there exists a w-open set O such that $x \in O \subseteq f^{-1}(i_{\acute{w}}c_{\acute{w}}(N))$ and so $f(O) \subseteq i_{\acute{w}}c_{\acute{w}}(N)$.

(vi)$\Longrightarrow$(ii) Let N be a $\acute{w}$-open set containing $f(x)$. Thus there exists a w-open set O containing x such that $f(O) \subseteq i_{\acute{w}}c_{\acute{w}}(N)$. Hence $O = i_w(O)$ and so $x \in i_w(f^{-1}(i_{\acute{w}}c_{\acute{w}}(N)))$.

(i)$\Longrightarrow$(iv) Let N be a $\pi(\acute{w})$-open set and $x \in f^{-1}(N)$, then $f(x) \in N$. In view of (vi), there exists a w-open set O containing x such that $f(O) \subseteq i_{\acute{w}}c_{\acute{w}}(N)$ and so $x \in O \subseteq f^{-1}(i_{\acute{w}}c_{\acute{w}}(N))$. Therefore, $x \in i_w(f^{-1}(i_{\acute{w}}c_{\acute{w}}(N)))$.

(iv)$\Longrightarrow$(ii) Obvious, since every $\acute{w}$-open set is $\pi(\acute{w})$-open.

(vi)$\Longrightarrow$(i) Let $x \in X$ and N be a $\pi(\acute{w})$-open set containing $f(x)$, then $f(x) \in i_{\acute{w}}c_{\acute{w}}(N)$. Hence there exists a $\acute{w}$-open set G such that $f(x) \in G \subseteq c_{\acute{w}}(N)$. Since f is an almost $(w, \acute{w})$-continuous function, so there exists a w-open set O containing x such that $f(O) \subseteq i_{\acute{w}}c_{\acute{w}}(G)$. This implies that $f(O) \subseteq i_{\acute{w}}c_{\acute{w}}(N)$. □

Remark 5.2. From the Definition 5.2, we have the following implication but the reverse relation may not be true in general.

$$(w, \acute{w})\text{-continuous} \Longrightarrow \text{almost } (w, \acute{w})\pi\text{-continuous}.$$

Example 5.2. Consider $X = \{a, b, c, d\}$, $w = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}\}$, and $\acute{w} = \{\emptyset, \{a, c\}, \{b, c\}\}$. Define $f : (X, w) \longrightarrow (\acute{X}, \acute{w})$ as follows: $f(a) = b$, $f(b) = f(d) = d$, $f(c) = c$. One may notice that f is an almost $(w, \acute{w})\pi$-continuous function but it is not $(w, \acute{w})$-continuous.

6. Strongly closed weak graph of weak structures

In topological spaces, Husain and Long in [22, 26] offered the functions with closed graph and strongly closed graph, respectively and displayed their properties. In the next part, functions with closed w-graph and strongly closed w-graph by utilizing w-open sets and the w-closure operator are acquired with some of their principle advantages.

Definition 6.1. Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces. A subset A of $X \times \acute{X}$ is $(w, \acute{w})$-closed if for each $(x, \check{x}) \in [(X \times \acute{X}) \setminus A]$, there exist w-open set O containing x and $\acute{w}$-open set V containing $\check{x}$ such that $(O \times V) \cap A = \emptyset$.

Definition 6.2. Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces and $f \colon (X, w) \longrightarrow (\acute{X}, \acute{w})$ be any function. Then the set $\mathcal{G}(f) = \{(x, f(x)) \mid x \in X\}$ of the product space $X \times \acute{X}$ is called the graph of f.

Definition 6.3. Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces. A function $f{:}(X, w) \longrightarrow (\acute{X}, \acute{w})$ is said to have the

(i) closed weak graph if its graph $\mathcal{G}(f)$ is $(w, \acute{w})$-closed subset in the product space $X \times \acute{X}$;
(ii) strongly closed weak graph if $x \in X$, $\check{x} \in \acute{X}$, $f(x) \neq \check{x}$ implies there exist w-open set O containing x and $\acute{w}$-open set V containing $\check{x}$ such that $(O \times c_{\acute{w}}(V)) \cap \mathcal{G}(f) = \emptyset$.

Now, we turn to grouping several facts about the functions with strongly closed weak graph, closed weak graph, and their relations to other functions.

Remark 6.1. Each function with a strongly closed weak graph also has closed weak graph.

Lemma 6.1. *Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces. A function $f \colon (X, w) \longrightarrow (\acute{X}, \acute{w})$ has a closed weak graph iff for each $x \in X$, $\check{x} \in \acute{X}$ such that $f(x) \neq \check{x}$ there exist w-open set O containing x and $\acute{w}$-open set V containing $\check{x}$ such that $f(O) \cap V = \emptyset$.*

Proof.

Necessity: Suppose $x \in X$, $\breve{x} \in \acute{X}$ such that $f(x) \neq \breve{x}$. Let f has a closed weak graph, then $\mathcal{G}(f) = \{(x, f(x)) \mid x \in X\}$ is $(w, \acute{w})$-closed, i.e., there exist w-open set O containing x and $\acute{w}$-open set V containing $\breve{x}$ such that $(O \times V) \cap \mathcal{G}(f) = \emptyset$. Hence $f(O) \cap V = \emptyset$.

Sufficiency: Let $(x, \breve{x}) \in [(X \times \acute{X}) \setminus \mathcal{G}(f)]$, then there exist w-open set O containing x and $\acute{w}$-open set V containing $\breve{x}$ such that $f(O) \cap V = \emptyset$. Hence $f(x) \neq \breve{x}$ and so $(O \times V) \cap \mathcal{G}(f) = \emptyset$. Consequently, f has a closed weak graph. □

Lemma 6.2. *Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces. A function $f: (X, w) \longrightarrow (\acute{X}, \acute{w})$ has a strongly closed weak graph, if for each $x \in X$, $\breve{x} \in \acute{X}$ such that $f(x) \neq \breve{x}$ there exist a w-open set O containing x and a $\acute{w}$-open set V containing $\breve{x}$ such that $f(O) \cap c_{\acute{w}}(V) = \emptyset$.*

Proof. It is similar to that of Lemma 6.1. □

Theorem 6.1. *Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces. If $f : (X, w) \longrightarrow (\acute{X}, \acute{w})$ is a function with a strongly closed weak graph, then for each $x \in X$, $f(x) = \cap\{c_{\acute{w}}(f(O)) : O$ is a w-open set containing $x\}$.*

Proof. Let f be a function with a strongly closed weak graph and the result is false. Suppose that there exists a point $\breve{x} \in \acute{X}$ such that $\breve{x} \neq f(x)$ and $\breve{x} \in \cap\{c_{\acute{w}}(f(O)) : O$ is a w-open set containing $x\}$. Hence, $V \cap f(O) \neq \emptyset$, for every w-open set V containing $\breve{x}$. It follows that $c_{\acute{w}}(V) \cap f(O) \neq \emptyset$. This is a contradiction. Consequently, the theorem holds. □

Theorem 6.2. *Let (X, w) be $\mathbb{WSS}$ and $(\acute{X}, \acute{w})$ be $\acute{w}\acute{T}_2$ space. If $f : (X, w) \longrightarrow (\acute{X}, \acute{w})$ is a $(w, \acute{w})$-continuous function, then f has a strongly closed weak graph.*

Proof. Let $(x, \breve{x}) \notin \mathcal{G}(f)$. Then $f(x) \neq \breve{x}$. Since $(\acute{X}, \acute{w})$ is a $\acute{w}\acute{T}_2$ space, there exist $\acute{w}$-open sets N and V such that $f(x) \in N$, $\breve{x} \in V$ and $c_{\acute{w}}(N) \cap c_{\acute{w}}(V) = \emptyset$. Since f is a $(w, \acute{w})$-continuous function, there exists w-open set O such that $f(O) \subseteq N \subseteq c_{\acute{w}}(N)$. This implies that $f(O) \cap c_{\acute{w}}(V) = \emptyset$. Consequently, f has a strongly closed weak graph. □

Theorem 6.3. *Let (X, w) be $\mathbb{WSS}$ and $(\acute{X}, \acute{w})$ be a $\acute{w}T_2$ space. If $f: (X, w) \longrightarrow (\acute{X}, \acute{w})$ is a $(w, \acute{w})$-continuous function, then f has a closed weak graph.*

Proof. Obvious in view of Remark 6.1 and Theorem 6.2. □

Theorem 6.4. *Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces and $f: (X, w) \longrightarrow (\acute{X}, \acute{w})$ be an injection $(w, \acute{w})$-continuous function. If f has a closed weak graph, then (X, w) is a wT_2 space.*

Proof. Let $x, \breve{x} \in X$ with $x \neq \breve{x}$. Since f is an injection function, then $f(x) \neq f(\breve{x})$. Therefore $(x, f(\breve{x})) \in [(X \times \acute{X}) \setminus \mathcal{G}(f)]$. Since f has a closed weak graph, there exist a w-open set O containing x and a $\acute{w}$-open set V containing $f(\breve{x})$ such that $(O \times V) \cap \mathcal{G}(f) = \emptyset$. Thus, $f(O) \cap V = \emptyset$. In view of f is a $(w, \acute{w})$-continuous function, there exists a w-open set H containing $\breve{x}$ such that $f(H) \subseteq V$. This implies that $f(O \cap H) \subseteq f(O) \cap f(H) = \emptyset$. Consequently, $O \cap H = \emptyset$ and so (X, w) is a wT_2 space. □

Theorem 6.5. *If $f: (X, w) \longrightarrow (\acute{X}, \acute{w})$ is an injection function and f has a strongly closed weak graph, then (X, w) is awT_2 space.*

Proof. Let $x_1, x_2 \in X$ and $x_1 \neq x_2$. Since f is an injection function, $f(x_1) \neq f(x_2)$. Hence $(x_1, f(x_2)) \notin \mathcal{G}(f)$. Since $\mathcal{G}(f)$ is a strongly closed weak graph, there exist a w-open set O containing x_1 and a $\acute{w}$-open set V containing $f(x_2)$ such that $f(O) \cap c_{\acute{w}}(V) = \emptyset$. Therefore, $x_2 \notin O$. Thus, there exists a w-open set U containing x_2 such that $x_1 \notin U$. Hence, (X, w) is wT_2 space. □

Theorem 6.6. *Let (X, w), $(\acute{X}, \acute{w})$ be weak structure spaces and $f:(X, w) \longrightarrow (\acute{X}, \acute{w})$ be a surjective function. If f has a strongly closed weak graph, then $(\acute{X}, \acute{w})$ is $\acute{w}T_2$ space.*

Proof. Let $z_1, z_2 \in \acute{X}$ with $z_1 \neq z_2$. Then there exists $x \in X$ such that $f(x) = z_1$. Therefore, $(x, z_2) \in [(X \times \acute{X}) \setminus \mathcal{G}(f)]$. Since f has a strongly closed weak graph, there exist a w-open set O containing x and $\acute{w}$-open set V containing z_2 such that $f(O) \cap c_{\acute{w}}(V) = \emptyset$. This implies that $z_1 \notin c_{\acute{w}}(V)$. Thus there exists a $\acute{w}$-open set U containing z_1 such that $U \cap V = \emptyset$. Consequently, $(\acute{X}, \acute{w})$ is a $\acute{w}T_2$ space. □

Theorem 6.7. *A weak structure space* (X, w) *is* $w\grave{T}_2$ *iff the identity function* id_w*:* $X \longrightarrow X$ *has a strongly closed weak graph* $\mathcal{G}(id_w)$*.*

Proof. According to Theorems 6.2 and 6.4, the required proof will be fulfilled. □

As an immediate consequence of Theorems 6.5 and 6.6, the next corollary is satisfied.

Corollary 6.1. *If* $f\colon (X, w) \longrightarrow (\acute{X}, \acute{w})$ *is a bijective function with a strongly closed weak graph, then both* (X, w) *and* $(\acute{X}, \acute{w})$ *are* $w\grave{T}_2$ *and* $\acute{X}\grave{T}_2$ *spaces, respectively.*

7. Conclusions

In this article, we have introduced the ideas of generalized π-weak closed and π-weak open sets as a generalization of gw-closed sets in weak structures. Also, we have studied some of their behaviors. Further, we have investigated w-regular and w-normal spaces and some kinds of weak continuous functions such as $(w, \pi(\acute{w}))$-continuous ($(w, \acute{w})\pi$-continuous, $g(w, \acute{w})\pi$-continuous, and almost $(w, \acute{w})\pi$-continuous) functions between weak structure spaces. Finally, we have discussed a notion of $(w, \acute{w})$-closed in the product space and the functions with closed w-graph and strongly closed w-graph with some of their principle advantages.

In the upcoming works, we will examine the functionally separation axioms [29] in frame of weak structures. Also, we will generalize the concepts defined herein using somewhere dense sets given in [4, 15, 16] to weak structures.

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