J. Math. Computer Sci., 31 (2023), 262–273

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Spline collocation methods for solving some types of nonlinear parabolic partial differential equations

Bewar A. Mahmood[a,*], Shko A. Tahir[b], Karwan H. F. Jwamer[b]

[a]*Department of Mathematics, College of Science, University of Duhok, Duhok, Iraq.*

[b]*Department of Mathematics, College of Science, University of Sulaiumai, Sulaiumai, Iraq.*

Abstract

In this work, some types of nonlinear parabolic partial differential equations have been studied by means of the collocation method with cubic B-splines, without transformation or linearization. Here, the convergence analysis of the current scheme is also theoretically investigated. A few numerical examples are given to illustrate the viability and effectiveness of the proposed technique. The error norms l_2 and l_∞ are used to assess the accuracy of the current method. In this respect, the proposed method, keeping the real features of such problems, is able to save the behavior of nonlinear terms without facing any conventional drawbacks. Furthermore, it is mathematically shown and numerically seen that there is a good agreement between the approximation and the exact solutions. The current approach reduces the cost of calculation as well as the need for storage space at various parameters.

Keywords: Cubic B-spline, collocation method, numerical solution, SSP-RK54 scheme, nonlinear partial differential equations.

2020 MSC: 35Q35, 80M25, 33F05, 34K28.

1. Introduction

Many problems in physics and engineering are often described via nonlinear partial differential equations. These equations appear in many scientific fields, such as optics, plasma physics, fluid mechanics, condensed matter physics, and the heat diffusion equation [15, 16, 19, 26, 29, 39]. Unfortunately, solving these equations explicitly in analytical form is only possible for special simple cases. As a result, numerical methods play an important role in solving these problems. Many numerical techniques have been introduced in the literature to obtain an approximate solution to these equations. These methods include variational iteration method [34], variational iteration algorithm-I with an auxiliary parameter [3], the differential transform method [11], A fourth-order finite difference scheme [12], the discrete Adomian decomposition method [7], the residual power series[42], the finite difference method [23], non-polynomial cubic spline method [4], extended cubic B-spline approximation [6], finite-difference MacCormack method [17], C^1 Cubic quasi-interpolation splines [10], differential transform method and Padé approximant [43], double Laplace transform and double Laplace decomposition methods [31]. Many mathematicians have

*Corresponding author
Email address: `bewar.ahmed@uod.ac` (Bewar A. Mahmood)

doi: 10.22436/jmcs.031.03.03

Received: 2023-01-16 *Revised:* 2023-03-29 *Accepted:* 2023-04-05

solved such problems to date, for more details, see[1, 5, 8, 9, 13, 24, 25, 27, 35]. Few authors have studied the spline method to solve partial differential equations, for instance, in [2, 21, 22, 30, 32, 36, 37, 40]. In this article, the cubic B-spline technique is used to solve a parabolic partial differential equation as follows:

$$\frac{\partial w}{\partial t}+\alpha w^{\delta}\frac{\partial w}{\partial x}-\mu\frac{\partial^2 w}{\partial x^2}=\beta w\left(1-w^{\delta}\right)\left(w^{\delta}-\gamma\right), L_1\leqslant x\leqslant L_2, t\geqslant 0, \tag{1.1}$$

with initial condition

$$w(x,t_0)=w_0(x),$$

and the Dirichlet boundary conditions are given by

$$\begin{cases} w\left(L_1,t\right)=g_1(t),\\ w\left(L_2,t\right)=g_2(t),\end{cases} \tag{1.2}$$

where $w(x,t)$ the unknown function of the space x and the time t with $\alpha,\beta,\delta,$ and γ are parameters that $\beta\geqslant 0,\delta>0,0<\gamma<1$. By rewriting equation (1.1), we can determine the linear and nonlinear parts as follows:

$$\begin{aligned}&\mathcal{L}(w_{xx},w,x,t)=\mu w_{xx}(x,t)-\beta\gamma w(x,t),\\ &\mathcal{N}(w_{xx},w,x,t)=-\alpha w^{s}(x,t)w_x(x,t)+\beta w(x,t)w^{s}(x,t)-\beta w(x,t)w^{2s}(x,t)+\beta\gamma w(x,t)w^{s}(x,t).\end{aligned}$$

Some of the terms in equation (1.1) are nonlinear term. We studied it without using transformation or linearization, and also the proposed method was directly derived from Space X by using the natural spline conditions. By testing three problems and comparing them with exact solutions, the effectiveness of the method is demonstrated, showing that the cubic B-spline is appropriate and in good agreement with previous studies available in the literature. The remainder of the article is arranged as follows. In Section 2 the cubic B-spline method is presented. In Section 3, the convergence of the method is investigated. Section 4 contains numerical results to demonstrate the reliability and relevance of the technique and finally, the conclusions and discussions are reported in Section 5.

2. Description of the methods

To approximate $w(x,t)$ by cubic B-spline collocation method, let's consider equally dividing the domain into knots such that a mesh $a=x_0<x_1,\dots,x_{m-1}<x_m=b, x_i=a+ih$, and $h=x_{i+1}-x_i=\frac{b-a}{m}, i=0,1,2,\dots,m$ is the length of each interval. The exact solution $w(x,t)$ in the cubic B-spline collocation method is approximated by $W_m(x,t)$ in the following form:

$$W_m(x,t)=\sum_{i=-1}^{m+1}\alpha_i(t)\eta_i(x), \tag{2.1}$$

where $\alpha_j(t)$ are unknown time-dependent quantities to be determined from the boundary conditions and collocation from the differential equation. The cubic B-spline functions $\eta_i(x)$ at these knots are given as follows:

$$\eta_i(x)=\frac{1}{h^3}\begin{cases}\left(x-x_{i-2}\right)^3, & x\in\left[x_{i-2},x_{i-1}\right],\\ h^3+3h^2\left(x-x_{i-1}\right)+3h\left(x-x_{i-1}\right)^2-3\left(x-x_{i-1}\right)^3, & x\in\left[x_{i-1},x_i\right],\\ h^3+3h^2\left(x_{i+1}-x\right)+3h\left(x_{i+1}-x\right)^2-3\left(x_{i+1}-x\right)^3, & x\in\left[x_i,x_{i+1}\right],\\ \left(x_{i+2}-x\right)^3, & x\in\left[x_{i+1},x_{i+2}\right],\\ 0, & \text{otherwise},\end{cases} \tag{2.2}$$

where, $\{\eta_{-1},\eta_0,\eta_1,\dots,\eta_{m-1},\eta_m,\eta_{m+1}\}$ forms a basis over the interval $a\leqslant x\leqslant b$. The values of $\eta_j(x)$ and its derivative may be tabulated as in Table 1.

Table 1: coefficient of the cubic B-splines and its derivatives at nodes x_j.

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+1}
$\eta_j(x)$	0	1/6	4/6	1/6	0
$\eta_j'(x)$	0	$1/2h$	0	$-1/2h$	0
$\eta_j''(x)$	0	$1/h^2$	$-2/h^2$	$1/h^2$	0

Let w be a function with a $[a, b]$ range definition. The cubic spline W_m interpolating the function w at points $x_0, \dots, x_m$ is the unique function in $C^2[a, b]$ satisfying the following conditions

$$\begin{cases} W_m(x_i, t) = w(x_i, t) \ \text{for } i = 0, \dots, m, \\ W_m''(a, t) = W_m''(b, t). \end{cases} \tag{2.3}$$

The cubic spline W_m that interpolates and satisfies condition (2.3), we have

$$W_m(x_k) = \sum_{i=-1}^{m+1} \alpha_i(t)\eta_i(x_k) = w(x_k, t), 0 \leqslant k \leqslant m, \tag{2.4}$$

with

$$W_m''(a, t) = \frac{1}{h^2}\alpha_{-1}(t) - \frac{2}{h^2}\alpha_0(t) + \frac{1}{h^2}\alpha_1(t) \quad \text{and} \quad W_m''(b, t) = \frac{1}{h^2}\alpha_{m-1}(t) - \frac{2}{h^2}\alpha_m(t) + \frac{1}{h^2}\alpha_{m+1}(t).$$

Now, by using natural cubic splines at boundary conditions in the given interval, which require that the second derivative is zero, $W_m''(a, t) = W_m''(b, t) = 0$, we obtain:

$$\alpha_{-1}(t) = 2\alpha_0(t) - \alpha_1(t) \quad \text{and} \quad \alpha_{m+1}(t) = 2\alpha_m(t) - \alpha_{m-1}(t) \tag{2.5}$$

also from (2.4) and by interpolating conditions at boundary points $x_0 = a$ and $x_m = b$, we get

$$W_m(x_0, t) = \frac{1}{6}\Big(\alpha_{-1}(t) + 4\alpha_0(t) + \alpha_1(t)\Big) = w(x_0, t),$$
$$W_m(x_m, t) = \frac{1}{6}\Big(\alpha_{m-1}(t) + 4\alpha_m(t) + \alpha_{m+1}(t)\Big) = w(x_m, t).$$

In addition to relations (2.5), we get to

$$\alpha_0(t) = w(x_0, t) = g_1(t), \quad \alpha_m(t) = w(x_m, t) = g_2(t).$$

Using (2.1) and (2.2), the approximate values of $W_m(x)$ and their derivatives at the nodes are determined as follows:

$$W_t(x, t) = \sum_{i=-1}^{m+1} \alpha_i'(t)\eta_i(x), \quad W_x(x, t) = \sum_{i=-1}^{m+1} \alpha_i(t)\eta_i'(x), \quad W_{xx}(x, t) = \sum_{i=-1}^{m+1} \alpha_i(t)\eta_i''(x). \tag{2.6}$$

Here, $\alpha_i'(t)$ is the derivative of $\alpha_i(t)$ with respect to time t, the present scheme consists of substituting $W(x, t)$ and its derivatives in (1.1) by the expression of $W_m(x, t)$ and its derivatives given by (2.1) and (2.6), Then, by the equation at points x_i for $i = 0, 1, 2, \dots, m$, we get:

$$W_t(x_0, t) = \mu W_{xx}(x_0, t) + \beta\gamma W(x_0, t) + \mathcal{N}\big(\varphi(t), x_0, t\big), \tag{2.7}$$
$$W_t(x_m, t) = \mu W_{xx}(x_m, t) + \beta\gamma W(x_m, t) + \mathcal{N}\big(\varphi(t), x_m, t\big), \tag{2.8}$$

where the function $\mathcal{N}$ represents the nonlinear part. Using (2.5), (2.7), and (2.8), we get:

$$\begin{cases} \alpha_0'(t) = -\beta\gamma g_1(t) + \mathcal{N}\big(\varphi(t), x_0, t\big), \\ \alpha_m'(t) = -\beta\gamma g_2(t) + \mathcal{N}\big(\varphi(t), x_m, t\big), \end{cases} \tag{2.9}$$

where

$$\mathcal{N}(\varphi(t),x_0,t)=\beta g_1(t)\big(g_1(t)\big)^s\left(1-\big(g_1(t)^s+\gamma\right)-\frac{\alpha}{h}\big(g_1(t)\big)^s\big(g_1(t)-\alpha_1(t)\big),$$
$$\mathcal{N}(\varphi(t),x_m,t)=\beta g_2(t)\big(g_2(t)\big)^s\left(1-\big(g_2(t)^s+\gamma\right)-\frac{\alpha}{h}\big(g_2(t)\big)^s\big(\alpha_{m-1}(t)-g_2(t)\big).$$

Now, by (2.6) and (2.9), by evaluating these equations at points x_1 and x_{m-1}, we get:

$$\begin{aligned}\frac{4}{6}\alpha_1'(t)+\frac{1}{6}\alpha_2'(t)&=\left(\frac{-2\beta\gamma}{3}-\frac{2\mu}{h^2}\right)\alpha_1(t)+\left(\frac{-\beta\gamma}{6}+\frac{\mu}{h^2}\right)\alpha_2(t)+\left(\frac{\mu}{h^2}-\frac{-\beta\gamma}{6}\right)g_1(t)\\&\quad+\mathcal{N}(\varphi(t),x_1,t)-\frac{1}{6}\mathcal{N}(\varphi(t),x_0,t),\\\frac{1}{6}\alpha_{m-2}'(t)+\frac{4}{6}\alpha_{m-1}'(t)&=\left(\frac{-\beta\gamma}{6}+\frac{\mu}{h^2}\right)\alpha_{m-2}(t)+\left(\frac{-2\beta\gamma}{3}-\frac{2\mu}{h^2}\right)\alpha_{m-1}(t)+\left(\frac{\mu}{h^2}-\frac{-\beta\gamma}{6}\right)g_2(t)\\&\quad+\mathcal{N}(\varphi(t),x_{m-1},t)-\frac{1}{6}\mathcal{N}(\varphi(t),x_m,t),\end{aligned}\tag{2.10}$$

and at points $x_i, i=2,3,\dots,m-2$, one obtains

$$\begin{aligned}\frac{1}{6}\alpha_{i-i}'(t)+\frac{4}{6}\alpha_i'(t)+\frac{1}{6}\alpha_{i+1}'(t)&=\left(\frac{-\beta\gamma}{6}+\frac{\mu}{h^2}\right)\alpha_{i-1}(t)+\left(\frac{-2\beta\gamma}{3}-\frac{2\mu}{h^2}\right)\alpha_i(t)\\&\quad+\left(\frac{-\beta\gamma}{6}+\frac{\mu}{h^2}\right)\alpha_{i+1}+\mathcal{N}(\varphi(t),x_i,t),\end{aligned}\tag{2.11}$$

where

$$\begin{aligned}&\mathcal{N}(\varphi(t),x_1,t)\\&=\left(-\beta\Big(g_1(t)+\alpha_1(t)+\alpha_2(t)\Big)\Big(\frac{1}{6}g_1(t)+\frac{4}{6}\alpha_1(t)+\frac{1}{6}\alpha_2(t)\Big)^s\right)\\&\quad\times\left(\Big(\frac{1}{6}g_1(t)+\frac{4}{6}\alpha_1(t)+\frac{1}{6}\alpha_2(t)\Big)^s-\gamma\right)-\frac{\alpha}{2h}\Big(\frac{1}{6}g_1(t)+\frac{4}{6}\alpha_1(t)+\frac{1}{6}\alpha_2(t)\Big)^s\Big(g_1(t)-\alpha_2(t)\Big),\\&\mathcal{N}(\varphi(t),x_m,t)\\&=\left(-\beta\Big(\alpha_{m-2}(t)+4\alpha_{m-1}(t)+g_2(t)\Big)\Big(1-\Big(\frac{1}{6}\alpha_{m-2}(t)+\frac{4}{6}\alpha_{m-1}(t)+\frac{1}{6}g_1(t)\Big)^s\Big)\right)\\&\quad\times\left(\Big(\frac{1}{6}\alpha_{m-2}(t)+\frac{4}{6}\alpha_{m-1}(t)+\frac{1}{6}g_2(t)\Big)^s-\gamma\right)\\&\quad-\frac{\alpha}{2h}\Big(\frac{1}{6}\alpha_{m-2}(t)+\frac{4}{6}\alpha_{m-1}(t)+\frac{1}{6}g_2(t)\Big)^s\Big(\alpha_{m-2}(t)-g_2(t)\Big),\\&\mathcal{N}(\varphi(t),x_i,t)\\&=\left(-\beta\Big(\alpha_{i-1}(t)+4\alpha_i(t)+\alpha_{i+1}(t)\Big)\Big(1-\Big(\frac{1}{6}\alpha_{i-1}(t)+\frac{4}{6}\alpha_i(t)+\frac{1}{6}\alpha_{i+1}(t)\Big)^s\Big)\right)\\&\quad-\frac{\alpha}{2h}\Big(\frac{1}{6}\alpha_{i-1}(t)+\frac{4}{6}\alpha_i(t)+\frac{1}{6}\alpha_{i+1}(t)\Big)^s\Big(\alpha_{i-1}(t)-\alpha_{i+1}(t)\Big),\end{aligned}$$

Simplifying the equations (2.10)-(2.11) leads to the following system of ordinary differential equations:

$$\begin{cases}A\varphi'(t)=D\varphi(t)+\phi(\varphi(t)),\\A\varphi\,(t_0)=\varphi_0,\end{cases}\tag{2.12}$$

where the matrices A and D are of size $(m-1)\times(m-1)$ and given by:

$$A=\frac{1}{6}\begin{pmatrix} 4 & 1 & 0 & \cdots & 0\\ 1 & 4 & 1 & & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & & 1 & 4 & 1\\ 0 & \cdots & 0 & 1 & 4 \end{pmatrix},\quad D=\begin{bmatrix} D_0 & D_1 & 0 & \cdots & 0\\ D_2 & D_0 & D_1 & & \vdots\\ 0 & - & \ddots & \ddots & 0\\ \vdots & & D_2 & D_0 & D_1\\ 0 & \cdots & 0 & D_2 & D_0 \end{bmatrix},$$

where $D_0=\frac{-2\beta\gamma}{3}-\frac{2\mu}{h^2}$, $D_1=\frac{-\beta\gamma}{6}+\frac{\mu}{h^2}$, and $D_2=\frac{-\beta\gamma}{6}+\frac{\mu}{h^2}$ and the vector valued function Φ is given by

$$\begin{aligned}\phi(\varphi(t)) &= [\phi_1(\varphi(t)),\phi_2(\varphi(t)),\ldots,\phi_{m-2}(\varphi(t)),\phi_{m-1}(\varphi(t))]^T,\\ \phi_1(\varphi(t)) &= \left(\frac{\mu}{h^2}-\frac{-\beta\gamma}{6}\right)g_1(t)+N\big(\varphi(t),x_1,t\big)-\frac{1}{6}N\big(\varphi(t),x_0,t\big),\\ \Phi_{m-1}(\phi(t)) &= \left(\frac{\mu}{h^2}-\frac{-\beta\gamma}{6}\right)g_2(t)+N\big(\varphi(t),x_{m-1},t\big)-\frac{1}{6}N\big(\varphi(t),x_m,t\big).\end{aligned}$$

Finally, we have to solve the resultant system (2.12) by using the well-known SSP-RK 54 algorithm [37], and as a result, the solution $w(x,t)$ is obtained at the necessary time level after the initial vector φ_0 determined for a given time. The initial vector φ_0 can be computed using the following initial and boundary conditions at $t=0$:

$$\begin{cases} W_m(x_0,t_0)=w_0(x_0), & \text{for } i=0,\\ W_m(x_i,t_0)=w_0(x_i), & \text{for } i=1,\ldots,m-1,\\ W_m(x_m,t_0)=w_0(x_m), & \text{for } i=m. \end{cases}$$

This gives a $(m-1)\times(m-1)$ system of equations of the following form:

$$A\varphi(t_0)=\varphi_0, \tag{2.13}$$

where φ_0 is the vector given by

$$\varphi_0=\left[w_0(x_1)-\frac{1}{6}w_0(x_0),w_0(x_2),\ldots,w_0(x_m-2),w_0(x_m-1)-\frac{1}{6}w_0(x_m)\right]^T.$$

Here, we use the Thomas algorithm to find a solution to the system (2.13).

3. Convergence of the method

To study the convergence conditions of the present scheme by presenting an error analysis of the current scheme and proving its convergence, let's start with a few lemmas and theorems that give the properties of B-spline functions and interpolation polynomials of spline type, which are very important for the proof of convergence of the numerical method.

Theorem 3.1. *Let $W_m\in S_{k,\Delta}$ be a unique splice interpolating the solution $w(x,t)$ of initial -boundary value problem (1.1)-(1.2), then*

$$\|D^i\left(w(x,t)-W_m(x,t)\right\|_\infty\leqslant m_i h^{4-i}, i=0,1,2,$$

where, $w(.,.)\in C^2[0,1]$, m_i is constant, and $D^i=\frac{\partial}{\partial x_i}$.

Proof. See [22]. ◻

Lemma 3.2. *The B-spline set* $\{B_{-1}, B_0, B_{m+1}\}$ *defined in Eq.* (2.2), *satisfies following inequality:*

$$\sum_{n=-1}^{m+1} |B_m(x)| \leqslant 10 \quad a \leqslant x \leqslant b.$$

Proof. See [22]. □

Theorem 3.3. *Let* $W_t(x,t)$ *be the collocation approximate from the space* $S_{3,\Delta}$ *to the solution* $w(x,t)$ *of the problem* (1.1)-(1.2) *exists, then*

$$\|w(x,t) - W_t(x,t)\|_\infty \leqslant Ch^2, \tag{3.1}$$

where C *is content and* h *is the piecewise uniform spacing and sufficiently small.*

Proof. To prove this theorem, we use the approximation of w in $S_{k,\Delta}$ defined as $W_M(x,t)$ and write

$$\|w(x,t) - W_t(x,t)\|_\infty \leqslant \|w(x,t) - W_M(x,t)\|_\infty + \|W_M(x,t) - W_t(x,t)\|_\infty \leqslant 0,$$

by Theorem 3.1 for $t \geqslant 0$ we have the following bound as bellow:

$$\|W(x,t) - W_M(x,t)\|_\infty \leqslant m_0 h^4,$$

corresponding to Lemma (3.2) the function $W_m(x,t)$ can be written in terms of B-spline basis as given in (2.3). Therefore with a constant c independent of h, we have $e \leqslant Ch^2$, where e is the error by using mathematical induction, we can obtain the estimates for $e_m^{k+1} \leqslant Ch^2$ at $m = -1, 0, \dots, N+1$. From (3.1) we obtain:

$$W_t(x,t) - W_m(x,t) = \sum_{m=-1}^{N+1} (\alpha_t(t) - \alpha_m(t)) \cdot B_m(x)$$

and thus

$$\|W_t(x,t) - W_m(x,t)\|_\infty \leqslant 10Ch^2,$$

where C is a constant independent of h. □

4. Numerical examples and results

In this part, we solve a few problems using the suggested method, and we'll demonstrate that it yields accurate approximations. The efficiency of our suggested scheme is evaluated by determining the l_2 error norm, maximum absolute error l_∞, and absolute errors by using the following formulae:

$$l_2 = \sqrt{h\left(\sum_{j=0}^{m} \left|w_j^{exact} - W_j^{num}\right|^2\right)}, \quad l_\infty = \max_j \left|w_j^{exact} - v_j^{num}\right|.$$

Example 4.1. Consider the following Burgers-Huxley equation:

$$\frac{\partial w}{\partial t} + w^2\frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} = \frac{2}{3}w^3\left(1 - w^2\right),$$

with the initial condition

$$w(x, t=0) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{3}x\right)\right]^{1/2},$$

and the Dirichlet boundary conditions

$$w\left(x = x_l, t\right) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{9}\left(3x_l + t\right)\right)\right]^{1/2}, \quad w\left(x = x_u, t\right) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{9}\left(3x_w + t\right)\right)\right]^{1/2},$$

with $x_l = -14, x_w = 6$. The analytical solution [28, 41] is

$$w_\alpha(x,t) = \left[\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{9}(3x + t)\right)\right]^{1/2}.$$

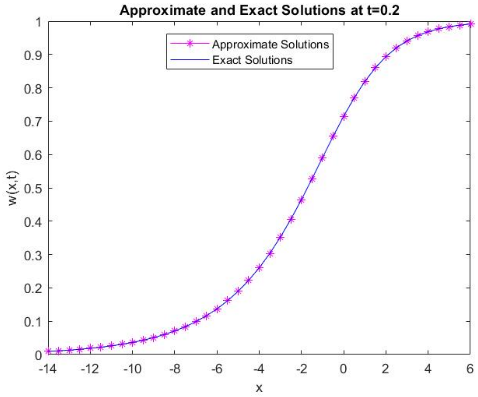

Figure 1: Approximate and exact solutions of Example 4.1, when $h = 0.05$ and $\Delta t = 0.001$.

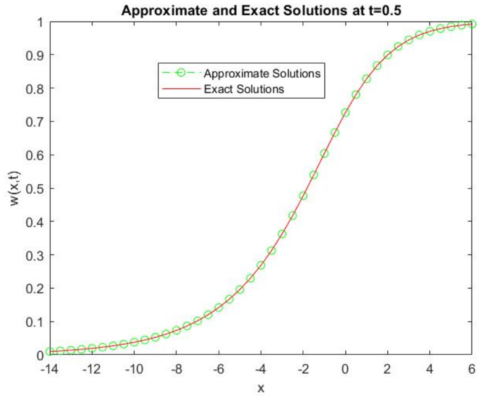

Figure 2: Approximate and exact solution of Example 4.1, when $h = 0.05$ and $\Delta t = 0.001$.

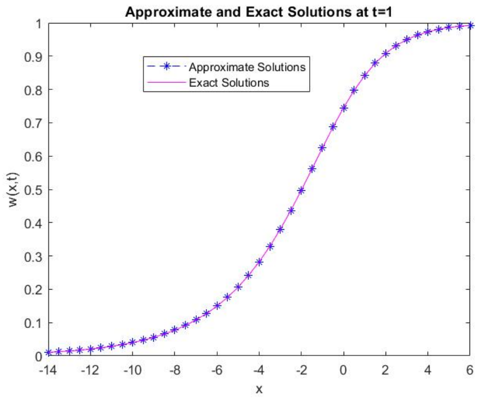

Figure 3: Approximate and exact solutions of Example 4.1, when $h = 0.05$ and $\Delta t = 0.001$.

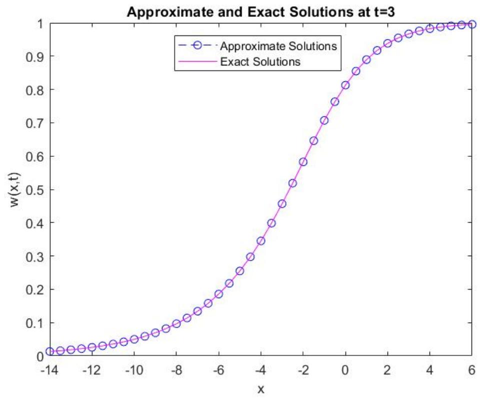

Figure 4: Approximate and exact solutions of Example 4.1, when $h = 0.05$ and $\Delta t = 0.001$.

Table 2: Shows the comparison of absolute errors for Example 4.1.

	$x = -12$		$x = -5$	
t	proposed method	[38]	proposed method	[38]
0	0	0	0	0
1	1.77E-07	2.10E-07	4.75E-06	3.20E-06
2	1.92E-07	4.80E-07	5.28E-06	5.07E-06
3	2.68E-07	1.90E-07	5.86E-06	7.50E-06
4	2.25E-07	8.50E-07	6.50E-06	1.28E-05
5	2.44E-07	1.24E-06	7.19E-06	1.59E-05
6	2.64E-07	1.32E-06	7.94E-06	2.02E-05
7	2.85E-07	7.30E-07	8.75E-06	2.09E-05
8	3.07E-07	1.28E-06	9.60E-06	3.08E-05
9	3.31E-07	4.80E-07	1.05E-05	3.42E-05
10	3.56E-07	1.08E-06	1.15E-05	3.77E-05

Table 2 presents various values of errors with different parameters. Clearly, we can see that the error results are in good agreement. Figures 1-4 show the shape of relative error between the numerical and exact solutions. Here, we observe from comparing the results in these figures that the numerical and theoretical results are in good agreement as given in Theorem 3.3.

Example 4.2. Consider the following Burgers-Huxley equation in the domain $[0, 1]$,

$$\frac{\partial w}{\partial t} + \alpha w \frac{\partial w}{\partial x} - \frac{\partial^2 w}{\partial x^2} = \beta w(1-w)(w-\gamma),$$

with the initial condition

$$w(x, 0) = \left\{\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\omega_1 x\right)\right\},$$

and the boundary conditions

$$w(0, t) = \left\{\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(-\omega_1\omega_2 x\right)\right\}, \quad w(1, t) = \left\{\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left(\omega_1\left(1-\omega_2 t\right)\right\}.$$

The exact solution is provided in [28, 41] by

$$w(x, t) = \left\{\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\{\omega_1\left(x-\omega_2 t\right)\}\right\},$$

that

$$\omega_1 = \frac{-\alpha + \sqrt{\alpha^2 + 8\beta}}{8}\gamma, \quad \omega_2 = \frac{\gamma\alpha}{2} - \frac{(2-\gamma)\left(-\alpha + \sqrt{\alpha^2 + 8\beta}\right.}{8},$$

$\alpha = 1, \beta = 1, \gamma = 0.001$.

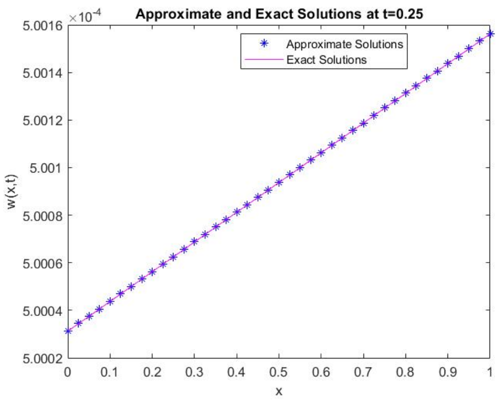

Figure 5: Approximate and exact solutions of Example 4.2, when $h = 0.05$ and $\Delta t = 0.001$.

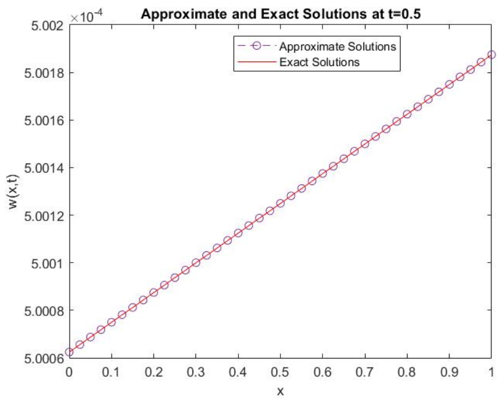

Figure 6: Approximate and exact solutions of Example 4.2, when $h = 0.05$ and $\Delta t = 0.001$.

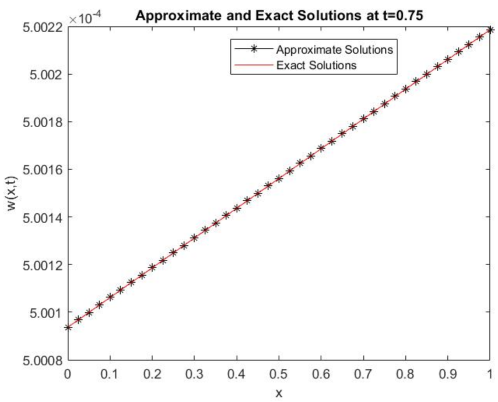

Figure 7: Approximate and exact solutions of Example 4.2, when $h = 0.05$ and $\Delta t = 0.001$.

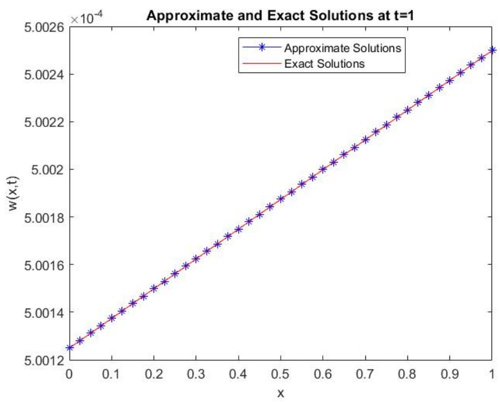

Figure 8: Approximate and exact solutions of Example 4.2, when $h = 0.05$ and $\Delta t = 0.001$.

Example 4.3. In this example, let us consider the Fitzhugh-Nagumo in the domain $[0, 1]$ as

$$\frac{\partial w}{\partial t} = \mu \frac{\partial^2 w}{\partial x^2} + f(w); \quad f(w) = w(1-w)(w-c),$$

with arbitrary constants c, μ. An exact solution is available in [20] as

$$w(x,t) = \frac{1}{1+\exp\left[\frac{-x}{\sqrt{2\mu}} - \left(\frac{1}{2}-c\right)t\right]}.$$

The analytic solution can be used to describe the initial condition

$$w(x, t=0) = g_1(x) = \frac{1}{1+\exp\left[\frac{-x}{\sqrt{2\mu}}\right]}$$

and boundary conditions

$$w(0,t) = \frac{1}{1+\exp\left[-\left(\frac{1}{2}-c\right)t\right]}, \quad w(1,t) = \frac{1}{1+\exp\left[\frac{1}{\sqrt{2\mu}} - \left(\frac{1}{2}-c\right)t\right]}.$$

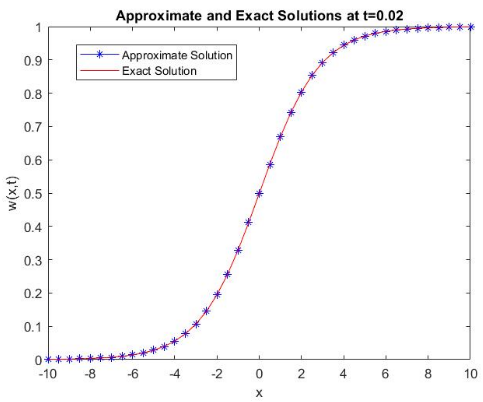

Figure 9: Approximate and exact solution of Example 4.3, when $h = 0.01$ and $\Delta t = 0.0001$.

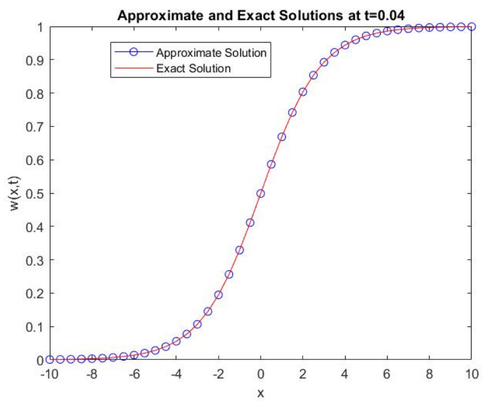

Figure 10: Approximate and exact solution of Example 4.3, when $h = 0.01$ and $\Delta t = 0.0001$.

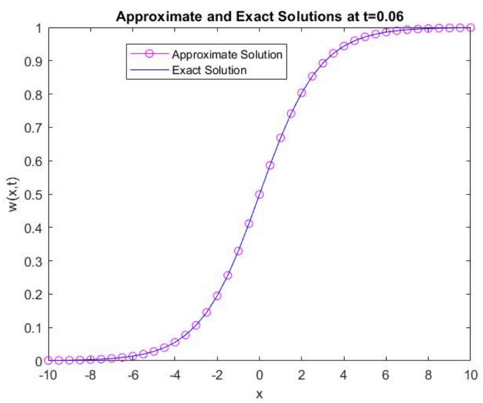

Figure 11: Approximate and exact solution of Example 4.3, when $h = 0.01$ and $\Delta t = 0.0001$.

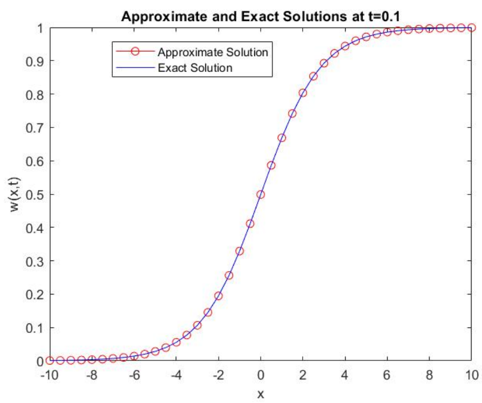

Figure 12: Approximate and exact solution of Example 4.3, when $h = 0.01$ and $\Delta t = 0.0001$.

Table 3: The absolute errors for various values of x and t for Example 4.2.

x	t	Proposed method	ADM [18]
0.1	0.05	1.87E-08	1.94E-07
	0.1	3.74E-08	3.87E-07
	1	3.75E-07	3.88E-06
0.5	0.05	1.87E-08	1.94E-07
	0.1	3.74E-08	3.87E-07
	1	3.75E-07	3.88E-06
0.9	0.05	1.87E-08	1.94E-07
	0.1	3.74E-08	3.87E-07
	1	3.75E-07	3.88E-06

Table 3 shows results using the present scheme at different grid sizes with various parameters. It is concluded from the errors are quite small. We have also seen from the corresponding Figures 5-8 the numerical and exact solutions are very accurate and efficient.

Table 4: Evaluation of error norms when $h = 0.01, \Delta t = 0.0001, \mu = 1$, and $c = 0.75$ for Example 4.3.

Errors	Proposed method		[20]	
	$t = 0.01$	$t = 1$	$t = 0.01$	$t = 1$
l_2	$2.4E-07$	$1.6E-07$	$1.0E-06$	$3.0E-06$
l_∞	$3.5E-07$	$2.7E-07$	$2.0E-06$	$3.0E-06$

Thanks to Theorem 3.3, we also note in this example that the corresponding Figures 9-12 convergence are presented by the proposed numerical scheme. They show that the numerical solutions are very good in similarity to the exact solution, clearly, the error tends to be zero.

5. Conclusions and discussions

In this research, the collocation method with cubic B-splines scheme has been investigated for solving the nonlinear parabolic partial differential equations without any transformation or linearization. Here, the convergence analysis of the present scheme is also studied theoretically. Several numerical examples are provided to demonstrate the viability and effectiveness of the proposed technique. The error norms l_2 and l_∞ are used to evaluate the method's accuracy. From the viewpoints of our results, the behavior of such problems, without any linearization, and thus by preserving the nonlinear features of nature, which could be understood by using the current method. Moreover, it can be seen that there is a good agreement between the computational results and the exact solutions. The current method reduces the computational cost and the need for storage space. One of the most outstanding aspects of the proposed approach is that the use of various parameters produces more effective and accurate numerical results. In further works, the proposed procedure can be effectively used for various nonlinear problems with the Neumann boundary condition.

Acknowledgment

The authors would like to thank UoD and UoS for their contributions to this work.

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