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Quantum L^p-spaces and inequalities of Hardy's type

Alaa E. Hamza[a], Maryam A. Alghamdi[b], Suha A. Alasmi[b,*]

[a]*Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt.*

[b]*Department of Mathematics, College of Science, University of Jeddah, Jeddah, 21589, Saudi Arabia.*

Abstract

In this paper, we revisit the L^p-spaces, $p \geqslant 1$, associated with a general quantum difference operator and prove some convergence theorems in the quantum setting. Furthermore, two inequalities of Hardy's type are established. Finally, many illustrative examples concerning with q-difference operator, Hahn difference operator and power quantum difference operator are given.

Keywords: Quantum difference operator, quantum calculus, Hahn difference operator, Jackson q-difference operator.

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1. Introduction

Quantum difference operators allows us to deal with nondifferentiable functions in the usual sense. They have an essential role due to their applications in several mathematical areas such as orthogonal polynomials, basic hypergeometric function, combinatorics, the calculus of variations and the theory of relativity. New results in quantum calculus can be found in [8] and the references cited therein. In this paper, we consider the quantum difference operator D_β which is defined by

$$D_\beta f(t) = \frac{f(\beta(t)) - f(t)}{\beta(t) - t}$$

for every t with $\beta(t) \neq t$ and $D_\beta f(t) = f'(t)$ when $\beta(t) = t$ provided that $f'(t)$ exists in the usual sense. Here, β is a continuous function on an interval I for which $\beta(t) \in I$ for any $t \in I$, and f is an arbitrary function from I to a Banach space $\mathbb{X}$. If $\beta(t) = qt, q \in (0,1)$, then $D_\beta = D_q$, the Jackson q-difference operator and if $\beta(t) = qt + \omega, q \in (0,1), \omega > 0$, then $D_\beta = D_{q,w}$, the Hahn difference operator. Theory of quantum difference equations helps us to avoid proving results twice, once for Jackson q-difference equations and once for Hahn difference equations (see [8]). For related results and applications to quantum difference operators, see [7]. We denote by

$$\beta^k(t) := \underbrace{\beta \circ \beta \circ \beta \circ \cdots \circ \beta}_{k \text{ times}}(t),$$

*Corresponding author

Email addresses: hamzaaeg2003@yahoo.com (Alaa E. Hamza), maalghamdi@uj.edu.sa (Maryam A. Alghamdi), suhaalharbi@outlook.com (Suha A. Alasmi)

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$k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all natural numbers. For convenience $\beta^0(t) = t$ for all $t \in I$.

It is well known that a continuous function $\beta : [a, b] \to [a, b]$ has at least a fixed point. This is due to Brouwer, see [15].

Throughout the paper, we assume β is a continuous function on I that has a unique fixed point $s_0 \in I$ and satisfies the following inequality:

$$(t - s_0)(\beta(t) - t) \leqslant 0 \text{ for all } t \in I.$$

Moreover, $\mathbb{X}$ denotes a Banach space endowed with a norm $\|\|$. Also, the β-interval is defined to be

$$[a, b]_\beta = \{\beta^k(a); k \in \mathbb{N}_0\} \cup \{\beta^k(b); k \in \mathbb{N}_0\} \cup \{s_0\}.$$

Finally, $[a, b]^*_\beta$ is defined by

$$[a, b]^*_\beta = [a, b]_\beta \backslash \{s_0\}.$$

For $d \in [a, b]_\beta$, the following facts are commonly known to be true.

(1) For $d > s_0$, we have $\beta^k(d)$ is decreasing to s_0 as $k \to \infty$.
(2) For $d < s_0$, we have $\beta^k(d)$ is increasing to s_0 as $k \to \infty$.

Accordingly, it is convenient to set $\beta^\infty(t) = s_0, t \in [a, b]_\beta$. For more details about quantum difference calclus, we refer the reader to [8]. We only mention some fundamental definitions and theorems that will be useful in our investigations.

Theorem 1.1. *Assume that* $f : I \longrightarrow \mathbb{X}$ *and* $g : I \longrightarrow \mathbb{R}$ *are* β*-differentiable functions at* $t \in I$. *Then*

(i) *the product* $fg : I \longrightarrow \mathbb{X}$ *is* β*-differentiable at* t *and*

$$D_\beta(fg)(t) = (D_\beta f(t))g(t) + f(\beta(t))D_\beta g(t) = (D_\beta f(t))g(\beta(t)) + f(t)D_\beta g(t);$$

(ii) f/g *is* β*-differentiable at* t *and*

$$D_\beta(f/g)(t) = \frac{\left(D_\beta f(t)\right) g(t) - f(t)D_\beta g(t)}{g(t)g(\beta(t))}, \quad g(t)g(\beta(t)) \neq 0.$$

Definition 1.2. Let $f : I \longrightarrow \mathbb{X}$ and $a, b \in I$. The β-integral of f from a to b is defined by

$$\int_a^b f(t)d_\beta t = \int_{s_0}^b f(t)d_\beta t - \int_{s_0}^a f(t)d_\beta t,$$

where

$$\int_{s_0}^h f(t)d_\beta t = \sum_{k=0}^\infty (\beta^k(h) - \beta^{k+1}(h))f(\beta^k(h)), \quad h \in I,$$

provided that the series converges at $h = a$ and $h = b$. f is called β-integrable on I if the series converges at a, b for all $a, b \in I$. Clearly, if f is continuous at $s_0 \in I$, then f is β-integrable on I.

Theorem 1.3. *Let* f *be continuous at* s_0. *Define the function*

$$F(t) = \int_{s_0}^t f(s)d_\beta s, \quad t \in I.$$

Then F *is continuous at* $s_0, D_\beta F(t)$ *exists for all* $t \in I$, *and* $D_\beta F(t) = f(t)$.

Corollary 1.4. *If* $f : I \longrightarrow \mathbb{X}$ *is continuous at* s_0, *then*

$$\int_t^{\beta(t)} f(\tau)d_\beta\tau = (\beta(t) - t)f(t), \quad t \in I.$$

Theorem 1.5. *If $f : I \to \mathbb{X}$ is β- differentiable on I, then*

$$\int_a^b D_\beta f(t) d_\beta t = f(b) - f(a) \quad \textit{for all } a, b \in I.$$

Theorem 1.6. *Assume f, g are β-differentiable functions on I and $D_\beta f, D_\beta g$ both continuous at s_0. Then*

$$\int_a^b f(t) D_\beta g(t) d_\beta t = f(b)g(b) - f(a)g(a) - \int_a^b (D_\beta f(t)) g(\beta(t)) d_\beta t, \quad a, b \in I.$$

Here, at least one of the functions of f and g is a real-valued function.

In 1970, Leindler, in [13], proved his inequality, which is stated as follows: if $\lambda(n), h(n) \geqslant 0, n \in \mathbb{N}_0$ and $p > 1$, then

$$\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{k=1}^{n} h(k) \right)^p \leqslant p^p \sum_{n=1}^{\infty} \lambda^{1-p}(n) \left(\sum_{k=n}^{\infty} \lambda(k) \right)^p h^p(n). \tag{1.1}$$

Also, in 1928, Copson [5] demonstrated that

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{k=1}^{n} a(k)\lambda(k) \right)^p \leqslant \left(\frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n), \tag{1.2}$$

where $\Lambda(n) = \sum_{k=1}^{n} \lambda(k)$, $\lambda(n)$, $a(n)$ are positive sequences and $p \geqslant c > 1$. The corresponding inequalities in the continuous case are

$$\int_0^\infty \left(\int_x^\infty g(y)\, dy \right)^p dx \leqslant p^p \int_0^\infty (xg(x))^p\, dx \tag{1.3}$$

and

$$\int_0^\infty \frac{\lambda(t)}{\Lambda^c(t)} \Phi^p(t) dt \leqslant \left(\frac{p}{c-1} \right)^p \int_0^\infty \frac{\lambda(t)}{\Lambda^{c-p}(t)} g^p(t) dt, \tag{1.4}$$

where $\Lambda(t) = \int_0^t \lambda(s)\, ds$ and $\Phi(t) = \int_0^t \lambda(s) g(s)\, ds$. Here, λ and g are continuous nonnegative functions. See [10] and [6].

Our paper is organized as follows. Section 2 is devoted to the quantum L^p- spaces, $p \geqslant 1$, which were considered in [4]. In this section, we follow a different approach in the proofs. In Section 3, we answer the following question: assuming f_n is a sequence of β-integrable functions on $[s_0, b]_\beta$, that converges to an integrable function f, under what conditions, do the β-integrals of f_n converge to the β-integral of f? Moreover, two inequalities of Hardy type are proved in quantum setting, which correspond to the classical inequalities (1.3) and (1.4). Finally, in Section 5, many illustrative examples are given.

2. Quantum L^p-spaces

Let $p \geqslant 1$ be a fixed real number, $a, b \in \mathbb{R}$ and $a < b$. Assume that $s_0 \in [a, b)$ is the fixed point of β. We denote by $L_p([a, b]_\beta, \mathbb{X})$ the family of all functions $f : [a, b]_\beta \to \mathbb{X}$ such that

$$\int_a^b \|f(t)\|^p d_\beta t < \infty.$$

We define the equivalence relation $\sim$ on the family $L_p[a,b]_\beta$ by

$$f \sim g \Leftrightarrow f = g \text{ on } [a,b]^*_\beta.$$

We denote by $L^p([a,b]_\beta, \mathbb{X}) = \{[f] : f \in L_p([a,b]_\beta, \mathbb{X})\}$, the family of all equivalence classes induced by $\sim$. For $f \in L_p([a,b]_\beta, \mathbb{X})$, we define the functional $\| [f] \|_p$ by

$$\| [f] \|_p = \left(\int_a^b \|f(t)\|^p d_\beta t \right)^{\frac{1}{p}}.$$

This definition is well defined, because $f \sim g$, implies $\int_a^b \|f(t)\|^p d_\beta t = \int_a^b \|g(t)\|^p d_\beta t$, and consequently, $\| [f] \|_p = \| [g] \|_p$. Note that using the inequality

$$(a+b)^p \leqslant 2^p (a^p + b^p), \;\; a, b \geqslant 0,$$

we obtain $f + g \in L_p([a,b]_\beta, \mathbb{X})$ for any $f, g \in L_p([a,b]_\beta, \mathbb{X})$. This is true since

$$\int_a^b \|f(t) + g(t)\|^p d_\beta t \leqslant 2^p \left(\int_a^b \|f(t)\|^p d_\beta t + \int_a^b \|g(t)\|^p d_\beta t \right).$$

We define the two operations

(1) $[f] + [g] = [f + g]$;
(2) $\alpha.[f] = [\alpha f], \alpha \in \mathbb{R}$.

One can see that $L^p([a,b]_\beta, \mathbb{X})$ is a linear space under the two operations. As known, the zero element is $[0]$ and the inverse of $[f]$ is $[-f]$. As usual, we write f instead of $[f]$. By making use of Young's inequality, [14], namely,

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b > 0,$$

where $p > 1$ and $q = \frac{p}{p-1}$, one can show Hölder inequality which is stated as follows.

Lemma 2.1 ([4, 9])**.** *If* $f \in L^p([a,b]_\beta, \mathbb{R})$ *and* $g \in L^q([a,b]_\beta, \mathbb{X})$, *where* $p > 1, q = \frac{p}{p-1}$, *then* $fg \in (L^1[a,b]_\beta, \mathbb{X})$ *and*

$$\|fg\|_1 \leqslant \|f\|_p \, \|g\|_q,$$

that is,

$$\int_a^b \|f(t)g(t)\| d_\beta t \leqslant \left(\int_a^b |f(t)|^p d_\beta t \right)^{\frac{1}{p}} \left(\int_a^b \|g(t)\|^q d_\beta t \right)^{\frac{1}{q}}.$$

In the following Lemma, we state Minkowski's inequality. We refer the reader to [4, 9].

Lemma 2.2. *Let* $p \geqslant 1$ *and* $f, g \in L^p([a,b]_\beta, \mathbb{X})$. *Then* $f + g \in L^p([a,b]_\beta, \mathbb{X})$ *and*

$$\|f + g\|_p \leqslant \|f\|_p + \|g\|_p,$$

that is,

$$\left(\int_a^b \|f(t) + g(t)\|^p d_\beta t \right)^{\frac{1}{p}} \leqslant \left(\int_a^b \|f(t)\|^p d_\beta t \right)^{\frac{1}{p}} + \left(\int_a^b \|g(t)\|^p d_\beta t \right)^{\frac{1}{p}}.$$

Another proof of the following Theorem can be found in [4].

Theorem 2.3. $L^p([a,b]_\beta,\mathbb{X})$ *is a normed space.*

Proof. Let $f \in L^p([a,b]_\beta,\mathbb{X})$. We check that $\|f\|^p \geqslant 0$. Indeed, we have

$$\|f\|_p^p = \int_a^b \|f(t)\|^p d_\beta t = \sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b))\ \|f\|^p(\beta^k(b)) - \sum_{k=0}^{\infty} (\beta^k(a) - \beta^{k+1}(a))\ \|f\|^p(\beta^k(a)) \geqslant 0.$$

Assume now $f \in L^p([a,b]_\beta,\mathbb{X})$ such that $\|f\|_p = 0$. Then $\int_a^b \|f(t)\|^p d_\beta t = 0$. Thus

$$\sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b))\ \|f\|^p(\beta^k(b)) = 0 \quad \text{and} \quad \sum_{k=0}^{\infty} (\beta^k(a) - \beta^{k+1}(a))\ \|f\|^p(\beta^k(a)) = 0,$$

which in turn implies that $f = 0$ on $[a,b]^*_\beta$, i.e., $[f] = [0]$. Finally, for $f \in L^p([a,b]_\beta,\mathbb{X})$ and $\alpha \in \mathbb{R}$, we can see easily $\|\alpha f\|_p = |\alpha|\ \|f\|_p$. By the previous discussions and Lemma 2.2, we get the required result. □

In the following Theorem we prove the completeness of L^p-spaces. Another proof is found also in [4].

Theorem 2.4. $L^p([a,b]_\beta,\mathbb{X})$ *is a Banach space.*

Proof. Assume that $\{f_n\}$ is a Cauchy sequence in $L^p[a,b]_\beta$, i.e.,

$$\int_a^b \|f_n(t) - f_m(t)\|^p d_\beta t \to 0 \text{ as } n,m \to \infty.$$

Setting $g_{n,m}(t) = \|f_n(t) - f_m(t)\|^p$, it follows that

$$\left(\sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b))\ g_{n,m}(\beta^k(b)) - \sum_{k=0}^{\infty} (\beta^k(a) - \beta^{k+1}(a))\, g_{n,m}(\beta^k(a)) \right) \longrightarrow 0 \text{ as } n,m \to \infty,$$

which in turn implies that,

$$\sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b))\ g_{n,m}(\beta^k(b)) \to 0 \text{ as } n,m \to \infty,$$

and

$$\sum_{k=0}^{\infty} (\beta^k(a) - \beta^{k+1}(a))\, g_{n,m}(\beta^k(a)) \to 0 \text{ as } n,m \to \infty.$$

So,

$$\|f_n(y) - f_m(y)\| \to 0 \text{ as } n,m \to \infty,\ y \in [a,b]^*_\beta.$$

Then $f_n(y)$ is a Cauchy sequence in $\mathbb{R}, y \in [a,b]^*_\beta$. Hence, $f_n(y)$ is pointwise convergent to some function $f(y)$ defined on $[a,b]^*_\beta$. Let f^* be any extension of f to $[a,b]_\beta$. Fix $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$, such that

$$\sum_{k=0}^{\infty} \left((\beta^k(b) - \beta^{k+1}(b))\ g_{n,m}(\beta^k(b)) - (\beta^k(a) - \beta^{k+1}(a))\ g_{n,m}(\beta^k(a)) \right) < \epsilon^p, \quad n,m \geqslant n_0.$$

This implies

$$\sum_{k=0}^{l} \left[(\beta^k(b) - \beta^{k+1}(b))\ \|f_n - f_m\|^p(\beta^k(b)) - (\beta^k(a) - \beta^{k+1}(a))\ \|f_n - f_m\|^p(\beta^k(a)) \right] < \epsilon^p, \quad n,m \geqslant n_0,$$

for each l. Taking $m \to \infty$, we get

$$\sum_{k=0}^{l}\left[(\beta^k(b)-\beta^{k+1}(b))\,\|f_n-f\|^p(\beta^k(b))-(\beta^k(a)-\beta^{k+1}(a))\,\|f_n-f\|^p(\beta^k(a))\right]<\epsilon^p,\;\; n\geqslant n_0,$$

for each l. Taking $l \to \infty$, we conclude that

$$\sum_{k=0}^{\infty}\left[(\beta^k(b)-\beta^{k+1}(b))\,\|f_n-f\|^p(\beta^k(b))-(\beta^k(b)-\beta^{k+1}(b))\,\|f_n-f\|^p(\beta^k(a))\right]<\epsilon^p,\; n\geqslant n_0.$$

We deduce that

$$\int_a^b \|f_n(t)-f^*(t)\|^p d_\beta t<\epsilon^p,\;\; n\geqslant n_0. \tag{2.1}$$

This implies that $f_n - f^* \in (L^p[a,b]_\beta,\mathbb{X})$, $n \geqslant n_0$, and consequently $f^* \in L^p[a,b]_\beta$. Also from (2.1), we get $\|f_n - f^*\|_p < \epsilon, n \geqslant n_0$. □

Theorem 2.5. *If* $h : [s_0,b]_\beta \to \mathbb{R}$ *is continuous at* s_0 *and* $f \in L^p([s_0,b]_\beta,\mathbb{X})$, *for some* $p \geqslant 1$, *then* $hf \in L^p([s_0,b]_\beta,\mathbb{X})$.

Proof. There is $M > 0$ such that $|h(t)|^p \leqslant M, t \in [s_0,b]_\beta$. We have

$$\begin{aligned}\int_{s_0}^{b}\|hf(t)\|^p\,d_\beta t&=\sum_{k=0}^{\infty}(\beta^k(b)-\beta^{k+1}(b))\,\|h(\beta^k(b))f(\beta^k(b))\|^p\\&\leqslant M\sum_{k=0}^{\infty}(\beta^k(b)-\beta^{k+1}(b))\,\|f(\beta^k(b))\|^p<\infty.\end{aligned}$$

□

3. Convergence theorems

Our objective in this section is to establish some convergence theorems.

Theorem 3.1. *Let* $f_n, f \in L^1([s_0,b]_\beta,\mathbb{X})$. *Assume* $\lim_{n\to\infty} f_n(\beta^k(b)) = f(\beta^k(b))$ *uniformly with respect to* $k \in \mathbb{N}_0$. *Then*

$$\lim_{n\to\infty}\int_{s_0}^{\beta^k(b)} f_n(s)\,d_\beta s=\int_{s_0}^{\beta^k(b)} f(s)\,d_\beta s$$

uniformly with respect to $k \in \mathbb{N}_0$.

Proof. Let $\epsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}_0$, the inequality

$$\|f_n(\beta^k(b))-f(\beta^k(b))\|<\frac{\epsilon}{b-s_0},$$

holds whenever $n \geqslant n_0$. It follows that

$$\begin{aligned}\left\|\int_{s_0}^{\beta^k(b)} f_n(s)\,d_\beta s-\int_{s_0}^{\beta^k(b)} f(s)\,d_\beta s\right\|&\leqslant\int_{s_0}^{\beta^k(b)}\|f_n(s)-f(s)\|\,d_\beta s\\&=\sum_{j=0}^{\infty}(\beta^{j+k}(b)-\beta^{j+k+1}(b))\|f_n(\beta^{j+k}(b))-f(\beta^{j+k}(b))\|\\&\leqslant\epsilon, n\geqslant n_0,\; k\in\mathbb{N}_0.\end{aligned}$$

□

Definition 3.2. We say that a sequence $f_n : [s_0, b]_\beta \to \mathbb{X}$ is equicontinuous at s_0 if for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\|f_n(\beta^k(b)) - f_n(s_0)\| < \epsilon, k \geqslant k_0, n \in \mathbb{N}.$$

Theorem 3.3. *Let f_n be an equicontinuous sequence at s_0 and f be any function on $[s_0, b]_\beta$ such that $\lim_{n\to\infty} f_n(\beta^k(b)) = f(\beta^k(b))$ uniformly with respect to $k \in \mathbb{N}_0 \cup \{\infty\}$. Then f is continuous at s_0.*

Proof. Let $\epsilon > 0$. There exist $k_0, n_0 \in \mathbb{N}$ such that

$$\|f_{n_0}(\beta^k(b)) - f(\beta^k(b))\| < \frac{\epsilon}{3}, \ k \in \mathbb{N}_0 \cup \{\infty\} \quad \text{and} \quad \|f_{n_0}(\beta^k(b)) - f_{n_0}(s_0)\| < \frac{\epsilon}{3}, \ k \geqslant k_0.$$

This implies

$$\|f(\beta^k(b)) - f(s_0)\| \leqslant \|f(\beta^k(b)) - f_{n_0}(\beta^k(b))\| + \|f_{n_0}(\beta^k(b)) - f_{n_0}(s_0)\| + \|f_{n_0}(s_0) - f(s_0)\| < \epsilon, \ k \geqslant k_0.$$

□

Theorem 3.4. *Assume $f_n : [s_0, b]_\beta \to \mathbb{R}$ is a monotone sequence of continuous functions at s_0. If f_n is convergent to a continuous function f at s_0, then*

$$\lim_{n\to\infty} f_n(\beta^k(b)) = f(\beta^k(b))$$

uniformly with respect to k.

Proof. Assume that $f_n(\beta^k(b)) \geqslant f_{n+1}(\beta^k(b)), n \in \mathbb{N}, k \in \mathbb{N}_0 \cup \{\infty\}$. Without loss of generality, we suppose that $f = 0$. For the sake of a contradiction, assume there exists $\epsilon_0 > 0$ and an increasing sequence $l_n \in \mathbb{N}$ such that $l_n \geqslant n$ and $f_n(\beta^{l_n}(b)) \geqslant \epsilon_0$. So, $f_m(\beta^{l_n}(b)) \geqslant \epsilon_0, m \leqslant n, n \in \mathbb{N}$. Fix m and take n to tend to ∞, we get $f_m(s_0) > \epsilon_0, m \in \mathbb{N}$. Again taking $m \to \infty$, we deduce $f(s_0) \geqslant \epsilon_0$ which is a contradiction. □

We combine Theorems 3.1 and 3.4, to conclude the monotone convergent theorem.

Theorem 3.5. *Assume $f_n : [s_0, b]_\beta \to \mathbb{R}$ is a monotone sequence of continuous functions at s_0. If f_n is convergent to a continuous function f at s_0, then*

$$\lim_{n\to\infty} \int_{s_0}^{\beta^k(b)} f_n(s)\, d_\beta s = \int_{s_0}^{\beta^k(b)} f(s)\, d_\beta s$$

uniformly with respect to $k \in \mathbb{N}_0$.

4. Quantum inequalities of Hardy type

This section includes two quantum inequalities of Hardy type.

Theorem 4.1. *Let $p > 1$ and $g \in L^p([s_0, b]_\beta, (0, \infty))$. Then*

$$\int_{s_0}^{b} \left(\int_{\beta(x)}^{b} g(y)\, d_\beta y \right)^p d_\beta x \leqslant p^p \int_{s_0}^{b} (x - s_0)^p (g(x))^p\, d_\beta x.$$

Proof. Set $F(x) = \left(\int_{\beta(x)}^{b} g(y)\, d_\beta y \right)^p$ and $I = \int_{s_0}^{b} F(x)\, d_\beta x$. Then, for $x = \beta^i(b) \in [s_0, b]_\beta, i \in \mathbb{N}_0$,

$$\begin{aligned} F(\beta^i(b)) &= \left(\sum_{k=0}^{\infty} (\beta^k(b) - \beta^{k+1}(b)) g(\beta^k(b)) - \sum_{k=0}^{\infty} (\beta^{k+i+1}(b) - \beta^{k+i+2}(b)) g(\beta^{k+i+1}(b)) \right)^p \\ &= \left(\sum_{k=0}^{i} (\beta^k(b) - \beta^{k+1}(b)) g(\beta^k(b)) \right)^p. \end{aligned}$$

This implies that

$$\begin{aligned}
I &= \sum_{n=0}^{\infty} (\beta^n(b) - \beta^{n+1}(b))F(\beta^n(b)) \\
&= \sum_{n=0}^{\infty} (\beta^n(b) - \beta^{n+1}(b)) \left(\sum_{k=0}^{n} (\beta^k(b) - \beta^{k+1}(b))g(\beta^k(b)) \right)^p \\
&= \sum_{n=1}^{\infty} (\beta^{n-1}(b) - \beta^n(b)) \left(\sum_{k=0}^{n-1} (\beta^k(b) - \beta^{k+1}(b))g(\beta^k(b)) \right)^p \\
&= \sum_{n=1}^{\infty} (\beta^{n-1}(b) - \beta^n(b)) \left(\sum_{k=1}^{n} (\beta^{k-1}(b) - \beta^k(b))g(\beta^{k-1}(b)) \right)^p .
\end{aligned}$$

Apply inequality (1.1) with $\lambda(n) = (\beta^{n-1}(b) - \beta^n(b))$ and $h(k) = (\beta^{k-1}(b) - \beta^k(b))g(\beta^{k-1}(b))$ to conclude that

$$\begin{aligned}
I &\leqslant p^p \sum_{n=1}^{\infty} \left(\beta^{n-1}(b) - \beta^n(b)\right)^{1-p} \left(\sum_{k=n}^{\infty} (\beta^{k-1}(b) - \beta^k(b)) \right)^p \left((\beta^{n-1}(b) - \beta^n(b))g(\beta^{n-1}(b))\right)^p \\
&= p^p \sum_{n=1}^{\infty} (\beta^{n-1}(b) - \beta^n(b)) \left(\sum_{k=n}^{\infty} (\beta^{k-1}(b) - \beta^k(b)) \right)^p \left(g(\beta^{n-1}(b))\right)^p \\
&= p^p \sum_{n=1}^{\infty} (\beta^{n-1}(b) - \beta^n(b))(\beta^{n-1}(b) - s_0)^p \left(g(\beta^{n-1}(b))\right)^p \\
&= p^p \int_{s_0}^{b} (x - s_0)^p g^p(x)\, d_\beta x.
\end{aligned}$$

□

Theorem 4.2. *Let* $p > 1$ *and* $g \in L^p([s_0, b]_\beta, (0, \infty))$. *Then*

$$\int_{s_0}^{b} \left(\frac{1}{b - \beta(x)} \int_{\beta(x)}^{b} g(y)\, d_\beta y \right)^p d_\beta x \leqslant \left(\frac{p}{p-1} \right)^p \int_{s_0}^{b} (g(x))^p\, d_\beta x.$$

Proof. Take $H(x) = \left(\frac{1}{b - \beta(x)} \int_{\beta(x)}^{b} g(y)\, d_\beta y \right)^p$ and $J = \int_{s_0}^{b} H(x)\, d_\beta x$. Let $x \in [s_0, b]_\beta$, that is $x = \beta^i(b)$ for some $i \in \mathbb{N}_0$. We have as before,

$$H(\beta^i(b)) = \frac{1}{(b - \beta^{i+1}(b))^p} \left(\sum_{k=0}^{i} (\beta^k(b) - \beta^{k+1}(b))g(\beta^k(b)) \right)^p$$

and

$$\begin{aligned}
J &= \sum_{n=0}^{\infty} (\beta^n(b) - \beta^{n+1}(b))H(\beta^n(b)) \\
&= \sum_{n=0}^{\infty} (\beta^n(b) - \beta^{n+1}(b)) \frac{1}{(b - \beta^{n+1})^p} \left(\sum_{k=0}^{n} (\beta^k(b) - \beta^{k+1}(b))g(\beta^k(b)) \right)^p \\
&= \sum_{n=1}^{\infty} (\beta^{n-1}(b) - \beta^n(b)) \frac{1}{(b - \beta^n(b))^p} \left(\sum_{k=0}^{n-1} (\beta^k(b) - \beta^{k+1}(b))g(\beta^k(b)) \right)^p \\
&= \sum_{n=1}^{\infty} (\beta^{n-1}(b) - \beta^n(b)) \frac{1}{(b - \beta^n(b))^p} \left(\sum_{k=1}^{n} (\beta^{k-1}(b) - \beta^k(b))g(\beta^{k-1}(b)) \right)^p .
\end{aligned}$$

Apply (1.2), with $\lambda(k) = \beta^{k-1}(b) - \beta^{k}(b)$, $a(k) = g(\beta^{k-1}(b))$, and $c = p$ to conclude that

$$J \leqslant \left(\frac{p}{p-1}\right)^{p} \sum_{n=1}^{\infty} (\beta^{n-1}(b) - \beta^{n}(b)) g^{p}(\beta^{n-1}(b)),$$

from which the desired inequality follows. □

5. Illustrative examples

Example 5.1. Let $q \in (0,1)$, $p > 1$, $b > 0$ and $g \in L^{p}([0,b]_{\beta},(0,\infty))$ with respect to the quantum q-difference operator, $\beta(t) = qt$. Here, 0 is the unique fixed point of β. Applying Theorem 4.1, the following inequality holds

$$\int_{0}^{b}\left(\int_{qx}^{b} g(y)\, d_{q}y\right)^{p} d_{q}x \leqslant p^{p}\int_{0}^{b} x^{p}(g(x))^{p}\, d_{q}x. \tag{5.1}$$

See [2, 11, 12]. It is well known that

$$\beta^{k}(t) = q^{k}t, k = 0,1,\dots$$

and

$$\beta^{k-1}(t) - \beta^{k}(t) = q^{k-1}(1-q)t, k = 1,\dots.$$

Inequality (5.1) yields the following inequality

$$\sum_{n=1}^{\infty} q^{n-1}\left(\sum_{k=1}^{n} q^{k-1} g(q^{k-1}b)\right)^{p} \leqslant \left(\frac{p}{1-q}\right)^{p} \sum_{n=1}^{\infty} q^{(n-1)(p+1)} g^{p}(q^{n-1}b). \tag{5.2}$$

If we substitute $g(x) = \dfrac{\sin x}{x}$ in (5.2), it yields the inequality

$$\sum_{n=1}^{\infty} q^{n-1}\left(\sum_{k=1}^{n} q^{k-1}\frac{\sin(q^{k-1}b)}{(q^{k-1}b)}\right)^{p} \leqslant \left(\frac{p}{1-q}\right)^{p} \sum_{n=1}^{\infty} q^{(n-1)(p+1)}\frac{\sin^{p}(q^{n-1}b)}{(q^{n-1}b)^{p}}.$$

We conclude that

$$\sum_{n=1}^{\infty} q^{n-1}\left(\sum_{k=1}^{n} \sin(q^{k-1}b)\right)^{p} \leqslant \left(\frac{p}{1-q}\right)^{p} \sum_{n=1}^{\infty} q^{n-1}\sin^{p}(q^{n-1}b).$$

Example 5.2. Let $q \in (0,1)$, $p > 1$, $\omega \geqslant 0$, and $b > s_0 = \dfrac{\omega}{1-q}$ and $g \in L^{p}([s_0,b]_{\beta},(0,\infty))$ with respect to the quantum Hahn difference operator, $\beta(t) = qt+\omega$. Here the fixed point of β is s_0. Then, applying Theorem 4.1, the following inequality holds

$$\int_{s_0}^{b}\left(\int_{qx+\omega}^{b} g(y)\, d_{q,\omega}y\right)^{p} d_{q,\omega}x \leqslant p^{p}\int_{s_0}^{b} (x-s_0)^{p}(g(x))^{p}\, d_{q,\omega}x. \tag{5.3}$$

It is readily seen that, $\beta^{k}(t) = q^{k}t + \omega[k]_q$, where $[k]_q = \dfrac{1-q^{k}}{1-q}$, $k = 0,1,\dots$. One can see that

$$\beta^{k}(t) = q^{k}(t-s_0)+s_0 \quad \text{and} \quad \beta^{k-1}(t) - \beta^{k}(t) = q^{k-1}(t-s_0)(1-q).$$

See [1, 3]. Inequality (5.3) yields

$$\begin{aligned}\sum_{n=1}^{\infty}(q^{n-1}(b-s_0)(1-q))\Big(\sum_{k=1}^{n}(q^{k-1}(b-s_0)(1-q))g(q^{k-1}(b-s_0)+s_0)\Big)^p\\ \leqslant p^p\sum_{n=1}^{\infty}(q^{n-1}(b-s_0)(1-q))(q^{n-1}(b-s_0))^p g^p(q^{n-1}(b-s_0)+s_0),\end{aligned}$$

which in turn implies that,

$$\sum_{n=1}^{\infty}q^{n-1}\Big(\sum_{k=1}^{n}q^{k-1}g(q^{k-1}(b-s_0)+s_0)\Big)^p\leqslant\left(\frac{p}{1-q}\right)^p\sum_{n=1}^{\infty}q^{(n-1)(p+1)}g^p(q^{n-1}(b-s_0)+s_0). \tag{5.4}$$

If we substitute $g(x)=\sin x$ in (5.4), it yields the inequality

$$\sum_{n=1}^{\infty}q^{n-1}\Big(\sum_{k=1}^{n}q^{k-1}\sin(q^{k-1}(b-s_0)+s_0)\Big)^p\leqslant\left(\frac{p}{1-q}\right)^p\sum_{n=1}^{\infty}q^{(n-1)(p+1)}\sin^p(q^{n-1}(b-s_0)+s_0).$$

Example 5.3. Let $q\in(0,1)$, $p>1$, $a\in 2\mathbb{N}+1$, $0<b<q^{\frac{1}{1-a}}$, and $g\in L^p([0,b]_\beta,(0,\infty))$, where $\beta(t)=qt^a$ is defined on $I=[0,b]$. See [1]. Then, applying Theorem 4.1, the following inequality holds

$$\int_0^b\left(\int_{qx^a}^b g(y)\,d_{a,q}y\right)^p d_{a,q}x\leqslant p^p\int_0^b x^p(g(x))^p\,d_{a,q}x. \tag{5.5}$$

It is well known that

$$\beta^k(t)=q^{[k]_a}t^{a^k}, k=0,1,\ldots,$$

and

$$\beta^{k-1}(t)-\beta^k(t)=q^{[k-1]_a}t^{a^{k-1}}(1-q^{a^{k-1}}t^{a^{k-1}(a-1)}), k=1,\ldots.$$

See [1, 2]. Then inequality (5.5) becomes

$$\begin{aligned}\sum_{n=1}^{\infty}q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})\left(\sum_{k=1}^{n}q^{[k-1]_a}b^{a^{k-1}}(1-q^{a^{k-1}}b^{a^{k-1}(a-1)})g(q^{[k-1]_a}b^{a^{k-1}})\right)^p\\ \leqslant p^p\sum_{n=1}^{\infty}q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})(q^{[n-1]_a}b^{a^{n-1}})^p g^p(q^{[n-1]_a}b^{a^{n-1}}).\end{aligned}$$

If we substitute $g(x)=\sin x$ in (5.5), it yields the inequality

$$\begin{aligned}\sum_{n=1}^{\infty}q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})\left(\sum_{k=1}^{n}q^{[k-1]_a}b^{a^{k-1}}(1-q^{a^{k-1}}b^{a^{k-1}(a-1)})\sin(q^{[k-1]_a}b^{a^{k-1}})\right)^p\\ \leqslant p^p\sum_{n=1}^{\infty}q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})(q^{[n-1]_a}b^{a^{n-1}})^p\sin^p(q^{[n-1]_a}b^{a^{n-1}}).\end{aligned}$$

Example 5.4. Let $q\in(0,1)$, $p>1$, $b>0$, and $g\in L^p([s_0,b]_\beta,(0,\infty))$ with respect to the quantum q-difference operator, $\beta(t)=qt$. Then, applying Theorem 4.2, the following inequality holds

$$\int_{s_0}^b\left(\frac{1}{b-qx}\int_{qx}^b g(y)\,d_qy\right)^p d_qx\leqslant\left(\frac{p}{p-1}\right)^{p-1}\int_{s_0}^b(g(x))^p\,d_qx. \tag{5.6}$$

Inequality (5.6) yields the following inequality

$$\sum_{n=1}^{\infty}\frac{q^{n-1}(1-q)b}{(b-q^{n}b)^{p}}\left(\sum_{k=1}^{n}q^{k-1}(1-q)b\ g(q^{k-1}b)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}q^{n-1}(1-q)b\ g^{p}(q^{n-1}b).$$

This implies

$$\sum_{n=1}^{\infty}\frac{q^{n-1}}{(1-q^{n})^{p}}\left(\sum_{k=1}^{n}q^{k-1}(1-q)g(q^{k-1}b)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}q^{n-1}g^{p}(q^{n-1}b). \tag{5.7}$$

If we substitute $g(x)=\frac{\sin x}{x}$ in (5.7), it yields the inequality

$$\sum_{n=1}^{\infty}\frac{q^{n-1}}{(1-q^{n})^{p}}\left(\sum_{k=1}^{n}q^{k-1}(1-q)\frac{\sin(q^{k-1}b)}{q^{k-1}b}\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}q^{n-1}\frac{\sin^{p}(q^{n-1}b).}{(q^{n-1}b)^{p}}$$

Thus, we obtain

$$\sum_{n=1}^{\infty}\frac{q^{n-1}}{(1-q^{n})^{p}}\left(\sum_{k=1}^{n}(1-q)\sin(q^{k-1}b)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}\frac{\sin^{p}(q^{n-1}b)}{q^{(n-1)(p-1)}},$$

from which we deduce that

$$\sum_{n=1}^{\infty}q^{n-1}\left(\frac{1-q}{1-q^{n}}\right)^{p}\left(\sum_{k=1}^{n}\sin(q^{k-1}b)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}\frac{\sin^{p}(q^{n-1}b)}{q^{(n-1)(p-1)}}.$$

It follows that

$$\sum_{n=1}^{\infty}q^{n-1}\frac{1}{[n]_q^{p}}\left(\sum_{k=1}^{n}\sin(q^{k-1}b)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}\frac{\sin^{p}(q^{n-1}b)}{q^{(n-1)(p-1)}}.$$

Example 5.5. Let $q\in(0,1)$, $p>1$, $\omega\geqslant 0$, and $b>s_0=\frac{\omega}{1-q}$ and $g\in L^{p}([s_0,b]_{\beta},(0,\infty))$ with respect to the quantum Hahn difference operator, $\beta(t)=qt+\omega$. Here the fixed point of β is s_0. Then, applying Theorem 4.2, the following inequality holds

$$\int_{s_0}^{b}\left(\frac{1}{b-qx-\omega}\int_{qx+\omega}^{b}g(y)\,d_{q,\omega}y\right)^{p}d_{q,\omega}x\leqslant\left(\frac{p}{p-1}\right)^{p-1}\int_{s_0}^{b}(g(x))^{p}\,d_{q,\omega}x. \tag{5.8}$$

Inequality (5.8), yields

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{q^{n-1}(b-s_0)(1-q)}{(b-q^{n}(b-s_0)-s_0)^{p}}\left(\sum_{k=1}^{n}q^{k-1}(b-s_0)(1-q)g(q^{k-1}(b-s_0)+s_0)\right)^{p}\\&\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}(q^{n-1}(b-s_0)(1-q))g^{p}(q^{n-1}(b-s_0)+s_0),\end{aligned}$$

which in turn implies that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{q^{n-1}(b-s_0)(1-q)}{(b-s_0)^{p}(1-q^{n})^{p}}\left(\sum_{k=1}^{n}q^{k-1}(b-s_0)(1-q)g(q^{k-1}(b-s_0)+s_0)\right)^{p}\\&\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}(q^{n-1}(b-s_0)(1-q))g^{p}(q^{n-1}(b-s_0)+s_0).\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} q^{n-1}\frac{1}{[n]_q^p}\left(\sum_{k=1}^{n} q^{k-1}g(q^{k-1}(b-s_0)+s_0)\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} q^{n-1}g^p(q^{n-1}(b-s_0)+s_0). \tag{5.9}$$

If we substitute $g(x) = \sin x$ in (5.9), it yields the inequality

$$\sum_{n=1}^{\infty} q^{n-1}\frac{1}{[n]_q^p}\left(\sum_{k=1}^{n} q^{k-1}\sin(q^{k-1}(b-s_0)+s_0)\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} q^{n-1}\sin^p(q^{n-1}(b-s_0)+s_0).$$

Example 5.6. Let $q \in (0,1)$, $p > 1$, $a \in 2\mathbb{N}+1$, $0 < b < q^{\frac{1}{1-a}}$, and $g \in L^p([0,b]_\beta,(0,\infty))$, when $\beta(t) = qt^a$ is defined on $I = [0,b]$. Then, applying Theorem 4.2, the following inequality holds

$$\int_0^b \left(\frac{1}{b-qx^a}\int_{qx^a}^b g(y)\, d_{a,q}y\right)^p d_{a,q}x \leqslant \left(\frac{p}{p-1}\right)^p \int_0^b (g(x))^p\, d_{a,q}x. \tag{5.10}$$

Then inequality (5.10) becomes

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})}{(b-q^{[n]_a}b^{a^n})^p}\left(\sum_{k=1}^{n} q^{[k-1]_a}b^{a^{k-1}}(1-q^{a^{k-1}}b^{a^{k-1}(a-1)})g(q^{[k-1]_a}b^{a^{k-1}})\right)^p\\ &\leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})g^p(q^{[n-1]_a}b^{a^{n-1}}).\end{aligned}$$

If we substitute $g(x) = \dfrac{\sin x}{x}$, it yields the inequality

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})}{(b-q^{[n]_a}b^{a^n})^p}\left(\sum_{k=1}^{n} q^{[k-1]_a}b^{a^{k-1}}(1-q^{a^{k-1}}b^{a^{k-1}(a-1)})\frac{\sin q^{[k-1]_a}b^{a^{k-1}}}{q^{[k-1]_a}b^{a^{k-1}}}\right)^p\\ &\leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})\frac{\sin^p q^{[n-1]_a}b^{a^{n-1}}}{(q^{[n-1]_a}b^{a^{n-1}})^p}.\end{aligned}$$

From which, we deduce

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{q^{[n-1]_a}b^{a^{n-1}}(1-q^{a^{n-1}}b^{a^{n-1}(a-1)})}{(b-q^{[n]_a}b^{a^n})^p}\left(\sum_{k=1}^{n} (1-q^{a^{k-1}}b^{a^{k-1}(a-1)})\sin q^{[k-1]_a}b^{a^{k-1}}\right)^p\\ &\leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} (1-q^{a^{n-1}}b^{a^{n-1}(a-1)})\frac{\sin^p q^{[n-1]_a}b^{a^{n-1}}}{(q^{[n-1]_a}b^{a^{n-1}})^{p-1}}.\end{aligned}$$

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