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Coloring Fuzzy Graphs and Traffic Light Problem

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Abstract
Given a graph$G = (V, E)$, a coloring function C assigns an integer value $C(i)$ to each node $i \epsilon V$ in such a way that the extremes of any edge $\{i, j\} \epsilon E$ cannot share the same color, i.e., $C(i) \neq C(j)$. The classical concept of the (crisp) chromatic number of a graph G is generalized to fuzzy concept $\tilde{G}$ in this paper. Main approach is based on the successive coloring functions C_α of the crisp graphs $G_\alpha = (V; E_\alpha)$, the α −cuts of $\tilde{G}$; the traffic lights problem is analyzed following this approach.

Keywords: Fuzzy graph, Coloring, Traffic Light Problem.

1. Introduction

Graph coloring is one of the most studied problems of combinatorial optimization. Many problems of practical interest can be modeled as coloring problems. The general form of this application involves forming a graph with nodes representing items of interest. The basic graph coloring problem is to group items in as few groups as possible, subject to the constraint that no incompatible items end up in the same group. Formally, given a graph $G = (V, E)$, a coloring function is a mapping $C : V \rightarrow N$ identifying $C(i)$ as the color of node $i \epsilon V$, in such a way that two adjacent nodes cannot share the same color, i.e., $C(i) \neq C(j)$ if $\{i, j\} \epsilon E$. These nodes i and j will be denoted as incompatible and, in this context, graph G

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will be denoted the incompatibility graph. A k −coloring C^k is a coloring function with no more than k different colors $C^k: V \to \{1, \dots, k\}$. A graph is k −colored if it admits a k −coloring. The minimum value k such that G is k-colored is the chromatic number of G and it is denoted as $\chi(G)$. The graph coloring problem (for short, the coloring problem) consists of determining the chromatic number of a graph and an associated coloring function. This problem is known to be NP-hard.

An important area of application of the coloring problem is management science. Classical applications include wiring of printed circuits , loading problems , resource allocation , frequency assignment problem , a wide variety of scheduling problems and computer register allocation. In these problems, the objective is to minimize the number of colors assigned to the nodes, where the nodes represent certain items and adjacent nodes must have different colors. In some circumstances, however, this scheme seems to be very restrictive in the sense that the problems which can be modeled as coloring problems cannot be very complex.

This paper is focused on the conceptual framework of the applications of the fuzzy-set theory on those problems that can be stated as coloring problems. Fuzzy-set theory, introduced by Zadeh [2], is a mathematical tool to handle uncertainties like vagueness, ambiguity, and imprecision in linguistic variables. The first definition of fuzzy graph was proposed by Kaufmann, from the fuzzy relations introduced by Zadeh[2]. Although Rosenfeld[3] introduced another elaborated definition, including fuzzy nodes and fuzzy edges.

In this paper we deal with graphs with crisp nodes and fuzzy edges. We will use the classical definition of fuzzy set A defined on a non empty set X as the family $A = \{(x;\ \mu_A(x)) | x \in X\}$, where $\mu_A: X \to I$ is the membership function and $\mu_A(x)$ reflects the ambiguity of the assertion x belongs to A. A fuzzy number is a fuzzy set defined on X ⊆ R. In classical fuzzy-set theory the set I is usually defined as the interval [0; 1], in such a way that $\mu_A(x) = 0$ indicates that x does not belong to A , $\mu_A(x) = 1$ indicates that x strictly belongs to A, and any intermediate value represents the degree in which x could belong to A. However, the set I could be a discrete set of the form $I = \{0, 1, \dots, k\}$, where $\mu_A(x) \le \mu_A(\acute{x})$ indicates that the degree of membership of x to A is lower than the degree of membership of $\acute{x}$. In general, the set I can be any ordered set, not necessarily numerical; for instance, $I = \{null;\ low;\ medium;\ high;\ total\}$.

Let $\widetilde{G} = (V, \widetilde{E})$ be a fuzzy graph, where V is the node set, the fuzzy edge set $\tilde{E}$ is characterized by the matrix $\mu = (\mu_{ij})_{i,j \epsilon V}: \mu_{ij} = \mu_{\tilde{E}}(\{i,j\})\ \forall i, j \epsilon V$ such that $i \neq j$ and $\mu_{\tilde{E}}: V \times V \to I$ is the membership function.

Each element $\mu_{ij} \in I$ represents the intensity level of the edge $\{i, j\}$ for any $i;\ j \in V$ with $i \neq j$. In this sense, a fuzzy graph can also be denoted as $\tilde{G} = (V, \mu)$. The set I is linearly ordered in such a way that the expression $\mu_{ij} \prec \mu_{\acute{i}\acute{j}}$ stands for "the intensity level of edge $\{i, j\}$ is lower than the intensity level of edge $\{\acute{i}, \acute{j}\}$".

The fuzzy graph $\tilde{G}$ can be considered as a generalization of the incompatibility graph G, since, taking $I = \{0;\ 1\}$, $\tilde{G}$ becomes a crisp graph if matrix μ is defined as:

$$\mu_{ij} = \begin{cases} 1 & if\ \{i,j\} \epsilon E \\ 0 & otherwise \end{cases} \quad \forall i, j \epsilon V$$

In this paper Main approach to the coloring problem of fuzzy graphs be introduces in section 2 which is the natural extension of the coloring problem to fuzzy graphs; the traffic lights problem will be used to motivate and illustrate this problem in Section 2.1, and the extended coloring function and chromatic number of a fuzzy graph will be defined in Section 2.2.

2. The fuzzy coloring problem

In order to introduce the concepts of the coloring function of a fuzzy graph and its associated chromatic number, a scheduling problem is presented.

2.1. The traffic lights problem

The traffic lights problem consists of controlling a traffic lights system in such a way that certain level of security will be attained. This problem has been studied as an intersection graph in [4]. Other authors, see [1],[5],[6] have modeled it as an assignment set problem. The traffic lights problem can also be modeled as a graph coloring problem. The following example illustrates this approach.

Example 2.1. The traffic Now at the corner of two streets is depicted in Fig1. Certain lanes are compatible with one another, such as AD and CB, while others are incompatible, such as AB and CD. In order to avoid collisions, we wish to install a traffic light system to control the Now of vehicles. This problem can be modeled by means of an incompatibility graph $G = (V, E)$ whose nodes are given by the lanes, and a pair of lanes defines an edge if they are incompatible, i.e., they can cause a collision. In this case,
$V = \{AB;\ AD;\ CB;\ CD;\ DB\}$;
$E = \{\{AB;\ CD\};\ \{AD;\ DB\};\ \{CD;\ DB\}\}$:

The incompatibility graph G is depicted in Fig2. Any k-coloring C^k of the graph G identifies a control policy of the lights system. The entire cycle of the lights system is divided into k time periods or slots (with any time-length). For any slot $c \in \{1, \dots, k\}$ circulation movements i such that $C^k\ (i) = c$ are the only ones allowed. Therefore, the chromatic number $\chi(G)$ gives the minimum number of time periods required to control the system. The chromatic number of G is $\chi(G) = 2$ and a 2-coloring is:
$C^2(AB) = 1;\ C^2(AD) = 2;\ C^2(CB) = 1;$
$C^2(CD) = 2;\ C^2(DB) = 1:$

Obviously, the control policy of the lights depends on the incompatibility of the lanes. The concept of incompatibility could be fuzzy and it could be graduated. This graduation, which does not need to be numerical, is associated to the desired security level for the traffic Now at the corner. The maximum security level is attained when all lanes are considered incompatible and the graph is complete; in this case, the chromatic number is the number of lanes and the control policy of the lights assure that only one movement is allowed in any slot of the cycle. On the other hand, the minimum security level is attained when the incompatibility edge set is empty; in this case, the chromatic number is 1 and all movements are allowed at any instant.

In Example 2.1, for instance, lanes CD and DB are more incompatible than lanes AB and DB. Let I = {n, l, m, h, t}, where n, l, m, h and t denote the incompatibility degrees $null, low, medium, high\ and\ total$, respectively. The problem stated in Example 2.1 could be modeled by means of the fuzzy graph $\tilde{G} = (V, \mu)$, where:

$$V = \{AB;\ AD;\ CB;\ CD;\ DB\}; \qquad \mu = \begin{pmatrix} - & n & l & h & l \\ n & - & n & m & h \\ l & n & - & n & n \\ h & m & n & - & h \\ l & h & n & h & - \end{pmatrix}$$

This fuzzy graph is depicted in Fig3. One way to consider these fuzzy incompatibilities is to define a coloring function for fuzzy graphs. This concept is introduced in the next subsection.

2.2. The coloring function of a fuzzy graph

Given a fuzzy graph $\tilde{G} = (V, \mu)$, a natural approach in order to obtain some knowledge about it, is to analyze the sequence of the so called α-cuts. A fuzzy set A defined on X can be characterized from its α-cuts family $A_\alpha = \{x \in X | \mu_A(x) \geq \alpha\} \quad \alpha \in I$.

This family of sets is monotone, i.e., it vertices $A_\alpha \subseteq A_\beta \quad \forall \alpha, \beta \in I$ such that $\alpha \leq \beta$. On the other hand, given a finite monotone family $\{A_{\alpha^p} | p \in \{1, \ldots, m\}\}$, a fuzzy set A can be defined from the membership function $\mu_A(x) = sup\{\alpha^p | x \in A_{\alpha^p}\} \quad \forall x \in X$. Let $\{G_\alpha = (V; E_\alpha) | \alpha \in I\}$ be the family of α-cuts sets of $\tilde{G}$, where the α-cut of a fuzzy graph is the crisp graph $G_\alpha = (V; E_\alpha)$ with $E_\alpha = \{\{i,j\} \big| i,j \in V,\ \mu_{ij} \geq \alpha\}$.

Hence, any (crisp) k-coloring C_α^k can be defined on G_α. The k-coloring function of $\tilde{G}$, is defined through this sequence. For each $\alpha \in I$, let χ_α denote the chromatic number of G_α The chromatic number of $\tilde{G}$ is defined through a monotone family of sets.

Definition 2.1. Given a fuzzy graph $\tilde{G} = (V; \mu)$, its chromatic number is the fuzzy number $\chi(\tilde{G}) = \{(x; \nu(x)) | x \in X\}$; where $X = \{1, \ldots, |V|\}$, $\nu(x) = sup\{\alpha \in I | x \in A_\alpha\} \quad \forall x \in X \quad and \quad A_\alpha = \{1, \ldots, \chi_\alpha\} \quad \forall \alpha \in I$.

The chromatic number of a fuzzy graph is a normalized fuzzy number whose modal value is associated with the empty edge-set graph. Its meaning depends on the sense of index α, and it can be interpreted in the following way: for lower values of α there are many incompatible links between nodes and, consequently, more colors are needed in order to consider these incompatibilities; on the other hand, for higher values of α there are fewer incompatible links between nodes and less colors are needed. The chromatic number sums up all this information in order to manage the fuzzy problem.

The fuzzy coloring problem consists of determining the chromatic number of a fuzzy graph and an associated coloring function. In this approach, for any level α, the minimum number of colors needed to color the crisp graph G_α will be computed. In this way, the fuzzy chromatic number will be defined as a fuzzy number through its α-cuts.

In Example 2.1, five crisp graphs $G_\alpha = (V; E_\alpha)$ are obtained by considering the values $\alpha \in$ I . For each $\alpha \in$ I , Table 1 contains the edge set E_α , the chromatic number χ_α and a χ_α-coloring C_α

It can be shown that the chromatic number of $\tilde{G}$ is $\chi(\tilde{G}) = \{(1,t),(2,h),(3,m),(4,n),(5,n)\}$. The interpretation of $\chi(\tilde{G})$ is the following: lower values of α are associated to lower driver aptitude levels and, consequently, the traffic lights must be controlled conservatively and the chromatic number is high; on the other hand, for higher values of α the driver aptitude levels increase and the chromatic number is lower, allowing a less conservative control of the traffic lights and a more fluid traffic flow.

In order to solve the fuzzy coloring problem, any algorithm which computes the chromatic number of every (crisp) graph G_α can be used. For fuzzy graphs of small to medium size an exact algorithm can be used. Nevertheless, due to the NP-hardness of the coloring problem, for fuzzy graphs of medium to large size some heuristics may be needed.

3. Tables and Figures

Table 1. , C_α for Example 2.1:

α	E_α	χ_α	$C_\alpha(AB)$	$C_\alpha(AD)$	$C_\alpha(CB)$	$C_\alpha(CD)$	$C_\alpha(DB)$
n	{ [illegible]	5	1	2	3	4	5
l	{ [illegible]	3	1	1	2	2	3
m	{ [illegible]	3	1	3	1	2	1
h	{ [illegible]	2	1	2	1	2	1
t	$\emptyset$	1	1	1	1	1	1

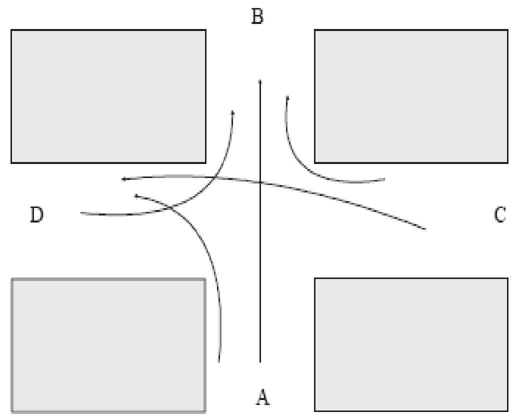

Figure 1. Traffic Flow for Example 2.1

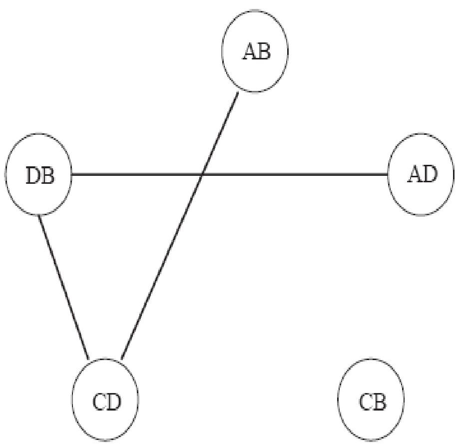

Figure 2 . Incompatibility Graph G for Example 2.1

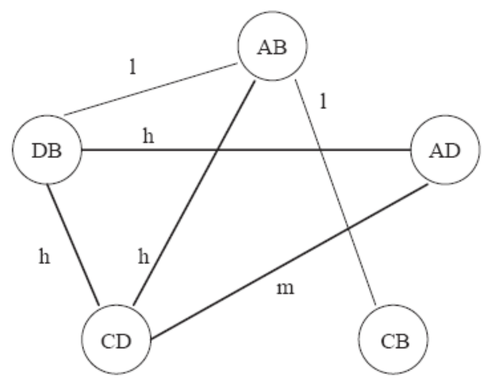

Figure 2 . Fuzzy Graph $\tilde{G}$ for Example 2.1

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