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# Generalized Hyers-Ulam stability of a bi-quadratic mapping in non-Archimedean spaces

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## Abstract

The main aim of this paper is to establish the generalized Hyers-Ulam stability of a bi-quadratic mappings in non-Archimedean spaces. That is, we prove the generalized Hyers-Ulam stability of a bi-quadratic functional equation of the form

$$\begin{aligned} &f(a_1(x_1+x_2), b_1(y_1+y_2)) + f(a_2(x_1+x_2), b_2(y_1-y_2)) + f(a_3(x_1-x_2), b_3(y_1+y_2)) \\ &\quad + f(a_4(x_1-x_2), b_4(y_1-y_2)) = C_{11}f(x_1,y_1) + C_{12}f(x_1,y_2) + C_{21}f(x_2,y_1) + C_{22}f(x_2,y_2) \end{aligned}$$

in non-Archimedean Banach spaces using Hyers direct method.

**Keywords:** Generalized Hyers-Ulam stability, functional equation in four variable, non-Archimedean Banach spaces.

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## 1. Introduction

We say that a functional equation is stable, if for every approximate solution, there exists an exact solution near to it. The concept of stability for a functional equation arises when one replaces a functional equation by an inequality which acts as a perturbation of the equation. The first stability problem of functional equation was raised by Ulam [44] in 1940. A simulating and famous talk presented by Ulam [44] in 1940, motivated the study of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems. Among those was the following question concerning the stability of homomorphisms.

**Theorem 1.1** ([44])**.** *Let $G_1$ be a group and let $G_2$ be a group endowed with a metric $\rho$. Given $\epsilon > 0$, does there exists a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies*

$$\rho(f(xy), f(x)\ f(y)) < \ \delta,$$

*for all $x, y \in G$, then we can find a homomorphism $h : G_1 \to G_2$ exists with*

$$\rho(f(x), h(x)) < \epsilon$$

*for all $x \in G_1$?.*

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Since then, this question has attracted the attention of many researchers. If the answer is affirmative, we say that the functional equation for homomorphisms is stable. In 1941, Hyers [15] was the first Mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam, the problem for the case of approximately additive mappings, when $G_1$ and $G_2$ are assumed to be Banach spaces. The result of Hyers is stated in the following celebrated Theorem.

**Theorem 1.2** ([15])**.** *Assume that* $G_1$ *and* $G_2$ *are Banach spaces. If a function* $f : G_1 \to G_2$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon \tag{1.1}$$

*for some* $\epsilon > 0$ *and for all* $x,\ y \in G_1$*, then the limit*

$$A(x) = \lim_{n\to\infty} 2^{-n}\ f(2^n\ x)$$

*exists for each* $x \in G_1$ *and* $A : G_1 \to\ G_2$ *is the unique additive function such that*

$$\|f(x) - A(x)\|\ \leqslant \epsilon \tag{1.2}$$

*for all* $x \in G_1$*. Moreover, if* $f(tx)$ *is continuous in* $t$ *for each fixed* $x \in G_1$*, then* $A$ *is linear.*

Taking the above fact into account, the additive functional equation

$$f(x+y) = f(x) + f(y)$$

is said to have **Hyers-Ulam stability** on $(G_1, G_2)$. In the above Theorem, an additive function $A$ satisfying the inequality (1.2) is constructed directly from the given function $f$ and it is the most powerful tool to study the stability of several functional equations. In course of time, the Theorem formulated by Hyers was generalized by Aoki [4] and Bourgin [5] for additive mappings.

There is no reason for the Cauchy difference $f(x+y) - f(x) - f(y)$ to be bounded as in the expression of (1.1). Towards this point, in the year 1978, Rassias [35] tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the Hyers-Ulam-stability for the Additive Cauchy Equation. This terminology is justified because the Theorem of Rassias has strongly influenced mathematicians studying stability problems of functional equation. In fact, Rassias proved the following Theorem.

**Theorem 1.3** ([35])**.** *Let* $X$ *and* $Y$ *be Banach spaces. Let* $\theta \in (0,\ \infty)$ *and let* $p \in [0,\ 1)$*. If a function* $f : X\ \to\ Y$ *satisfies*

$$\|f(x+y) - f(x) - f(y)\|\ \leqslant \theta\ \ (\|x\|^p + \|y\|^p)$$

*for all* $x,\ y \in X$*, then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x)\|\ \leqslant\ \frac{2\theta}{2-2^p}\ \|x\|^p$$

*for all* $x \in X$*. Moreover, if* $f(tx)$ *is continuous in* $t$ *for each fixed* $x \in X$*, then* $A$ *is linear.*

The findings of Rassias have exercised a delectable influence on the development of what is addressed as the generalized Hyers-Ualm-Rassias stability of functional equations. In 1982, Rassias [36] gave a further generalization of the result of Hyers and proved a theorem using weaker conditions controlled by a product of different powers of norms. His Theorem is presented as follows.

**Theorem 1.4** ([36])**.** *Let* $f : X\ \to\ Y$ *be a mapping from a normed vector space* $X$ *into a Banach space* $Y$ *subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon\ \ \|x\|^p\ \ \|y\|^p \tag{1.3}$$

*for all* $x, y \in X$, *where* $\epsilon$ *and* $p$ *are constants with* $\epsilon > 0$ *and* $0 \leqslant p < \frac{1}{2}$. *Then the limit*

$$A(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

*exists for all* $x \in X$ *and* $A : X \to Y$*is the unique additive mapping which satisfies*

$$\|f(x) - A(x)\| \leqslant \frac{\epsilon}{2 - 2^{2p}} \|x\|^{2p} \tag{1.4}$$

*for all* $x \in X$. *If* $p < 0$, *then the inequality* (1.3) *holds for* $x, y \neq 0$ *and* (1.4) *for* $x \neq 0$. *If* $p > 0$, *then the inequality* (1.3) *holds for all* $x, y \in X$ *and the limit*

$$A(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

*exists for all* $x \in X$. *If in addition* $f : X \to Y$ *is a mapping such that the transformation* $t \to f(tx)$ *is continuous in* $t \in R$ *for each fixed* $x \in X$, *then* $A$ *is* $R$*-linear mapping.*

In 1991, Gajda [9] answered the question for $p > 1$, which was raised by Rassias [35]. This new concept is known as the Hyers-Ulam-Rassias stability of functional equations. The terminology, Hyers-Ulam-Rassias stability, is originated from these historical backgrounds. The terminology can also be applied to the case of other functional equations. In 1994, a further generalization of Rassias theorem was obtained by Gavruta [10] (see also [11]).

The stability concept introduced by Rassias [35] is significantly influenced by a number of Mathematicians to investigate the stability problem for various functional equations and there are many interesting results concerning the Ulam stability problems in ([1–3, 7, 8, 16, 17, 19–21, 23, 24, 26–31, 34, 37–41, 43, 45]).

Jun and Kim [18] introduced the cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x), \tag{1.5}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.5). The function $f(x) = x^3$ satisfies the functional equation (1.5), which is called a cubic functional equation. Every solution of the cubic functional equation is said to be a cubic function. In [18] Jun and Kim proved that a function $f$ between real vector spaces $X$ and $Y$ is a solution of (1.5) if and only if there exits a unique function $C : X \times X \times X \to Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, and $C$ is symmetric for each fixed one variable and is additive for fixed two variables.

Moslehian and Rassias [25] proved the generalized Hyers-Ulam stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean spaces. Park and Bae [33] introduced the following quartic functional equation

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] - 24f(y) - 6f(x) \tag{1.6}$$

and obtained the stability results of (1.6). It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.6), which is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic function (see also [13, 14]). Eshaghi Gordji and Bavand Savadkouhi [14] proved the Hyers-Ulam-Rassias stability of cubic and quartic functional equations in non-Archimedean normed space and they have established the generalized Hyers-Ulam-Rassias stability for the mixed type cubic and quartic functional equation

$$f(x + 2y) + f(x - 2y) - 4[f(x + y) + f(x - y)] = 3\, f(2y) - 24\, f(y) - 6\, f(x)$$

in non-Archimedean normed spaces by Hyers direct method. In the same year, by fixed point approach, Park [32] established the generalized Hyers-Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$$f(x + 2y) + f(x - 2y) = 4[f(x + y) + f(x - y)] - 6\, f(x) + f(2y) + f(-2y) - 4\, f(y) - 4\, f(-y)$$

in non-Archimedean Banach spaces.

In 2014, Lee et al. [23] proved the generalized Hyers-Ulam stability of the mixed type additive-quadratic functional equation in Banach spaces. In the same year, Shen and Lan [40] proved the general solution of a new quadratic functional equation of the form

$$f\left(x-\frac{y+z}{2}\right)+f\left(x+\frac{y-z}{2}\right)+f(x+z)=3f(x)+\frac{1}{2}f(y)+\frac{3}{2}f(z). \tag{1.7}$$

Also studied the Ulam stability of this functional equation (1.7) in a real normed space and a non-Archimedean space. In 2015, Kim and Shin [22] proved the generalized Hyers-Ulam stability of the cubic functional equation

$$f(3x+y)+f(3x-y)=f(x+2y)+2f(x-y)+6f(2x)+3f(x)-6f(y)$$

in non-Archimedean normed spaces.

In 2018, Lee et al. [24] established the uniqueness theorems concerning the functional inequalities involving with an $n$-dimensional cubic-quadratic-additive equation of the form

$$\sum_{i=1}^{m} c_i\, f\left(a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_n\right)=0$$

by using the Hyers direct method. In the next year, Park and Rassias [34] solved the additive functional equations

$$f(x+y+z)-f(x+y)-f(z)=s\left[f(x+y-z)+f(x-y+z)-2f(x)\right] \tag{1.8}$$

and

$$f(x+y-z)+f(x-y+z)-2f(x)=s\left[f(x+y+z)-f(x+y)-f(z)\right], \tag{1.9}$$

where $s$ is a fixed nonzero complex number. Furthermore, they investigated the Hyers-Ulam stability of the additive functional equations (1.8) and (1.9) in complex Banach spaces. This is applied to prove the partial multipliers in Banach *-algebras, unital C*-algebras, Lie C*-algebras, JC*-algebras and C*-ternary algebras, associated with the additive functional equations (1.8) and (1.9).

Recently, Gharib et al. [12] investigated the asymptotic stability behaviour of the Pexider-Cauchy functional equation in non-Archimedean spaces. They also showed that, under some conditions, if $\|f(x+y)-g(x)-h(y)\| \leqslant \epsilon$, then $f, g$ and $h$ can be approximated by additive mapping in non-Archimedean normed spaces and dealt with a functional inequality and its asymptotic behaviour.

Very recently, Tamilvanan et al. [42] introduced a new type of generalized mixed-type quadratic-additive functional equation and obtained its general solution and investigated the Ulam stability of the mixed type of quadratic-additive functional equation in non-Archimedean fuzzy $\phi$-2-normed space and non-Archimedean Banach space using the direct and fixed point approaches by taking into even and odd mapping.

Recently, in 2021, Cieplinski [6] established the generalized Hyers-Ulam stability of a functional equation in four variables of the form

$$\begin{aligned} &f(a_1(x_1+x_2), b_1(y_1+y_2))+f(a_2(x_1+x_2), b_2(y_1-y_2))+f(a_3(x_1-x_2), b_3(y_1+y_2)) \\ &\quad+f(a_4(x_1-x_2), b_4(y_1-y_2))=C_{11}f(x_1,y_1)+C_{12}f(x_1,y_2)+C_{21}f(x_2,y_1)+C_{22}f(x_2,y_2) \end{aligned} \tag{1.10}$$

by fixed point method, then stability of some known equations will be also derived.

In this paper, we will establish the generalized Hyers-Ulam stability for a bi-quadratic functional equation of four variable (1.10) in non-Archimedean Banach spaces.

## 2. Preliminaries

In this section, we will provide some basic notations, Definitions, and Theorem, which will be very useful to prove our main results.

**Definition 2.1.** Let $A$ be a set. A function $\rho : A \times A \to [0, \infty)$ is called a generalized metric on $A$ if $\rho$ satisfies the following conditions:

(i). $\rho(x, y) = 0$ if and only if $x = y$;
(ii). $\rho(x, y) = \rho(y, x)$ for all $x, y \in A$;
(iii). $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$ for all $x, y, z \in A$.

By a non-Archimedean field we mean a field $\mathbb{K}$ equipped with a function (valuation) $|.|$ from $\mathbb{K}$ into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r| \,.\, |s|$, and

$$|r + s| \leqslant \max\{|r|, |s|\}$$

for all $r, s \in \mathbb{K}$. Clearly $|1| = |-1| = 1$ and $|n| \leqslant 1$ for all $n \in \mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $||$ taking everything except for 0 into 1 and $|0| = 0$.

**Definition 2.2.** The function $|.| : \mathbb{K} \to \mathbb{R}$ is called a non-Archimedean valuation or absolute value over the field $\mathbb{K}$ if it satisfies following conditions: for any $r, s \in \mathbb{K}$

(1) $|r| \geqslant 0$;
(2) $|r| = 0$ if and only if $r = 0$;
(3) $|rs| = |r| \,.\, |s|$;
(4) $|r + s| \leqslant \max\{|r|, |s|\}$;
(5) there exists a member $r_0 \in \mathbb{K}$ such that $|r_0| \neq 0, 1$.

A field $\mathbb{K}$ with a non-Archimedean valuation is called a non-Archimedean field.

**Corollary 2.3.** $|1| = |-1| = 1$ *and so, for any* $r \in \mathbb{K}$, *we have* $|-r| = |r|$. *Also, if* $|r| < |s|$ *for any* $r, s \in \mathbb{K}$, *then* $|r + s| = |s|$

In a non-Archimedean field, the triangle inequality is satisfied and so a metric is defined. But an interesting inequality changes the usual Archimedean sense of the absolute value. For any $n \in \mathbb{N}$, we have $|n.\, 1| \leqslant \mathbb{R}$. Thus, for any $r \in \mathbb{K}$, $n \in \mathbb{N}$ and nonzero divisor $k \in \mathbb{Z}$ of $n$, the following inequalities hold:

$$|nr| \leqslant |kr| \leqslant |r| \leqslant \left|\frac{r}{k}\right| \leqslant \left|\frac{r}{n}\right|.$$

**Definition 2.4.** Let $A$ be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $\|.\| : A \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r| \;\|x\|$ for all $r \in \mathbb{K}$, $x \in A$;
(iii) the strong triangle inequality (ultra-metric); namely, $\|x + y\| \leqslant \max\{\|x\|, \|y\|\}$ for all $x, y \in A$.

Then $(A, \|.\|)$ is called a non-Archimedean normed space.

Since $0 = \|0\| = \|x - x\| \leqslant \max\{\|x\|, \|-x\|\} = \|x\|$ for all $x \in A$, we have $\|x\| \geqslant 0$. Any vector space $A$ with a non-Archimedean norm $\|.\| : A \to \mathbb{R}$ is called a non- Archimedean space. If the metric $\rho(x, y) = \|x - y\|$ is induced by a non-Archimedean norm $\|.\| : A \to \mathbb{R}$ on a vector space $A$ which is complete, then $(A, \|.\|)$ is called a complete non-Archimedean space.

**Definition 2.5.** Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $A$. Then the sequence $\{x_n\}$ is called Cauchy, if for a given $\epsilon > 0$, there is a positive integer $N$ such that $\|x_n - x_m\| \leqslant \epsilon$ for all $m, n \geqslant N$.

**Definition 2.6.** Let $\{x_n\}$ be a sequence in a non-Archimedean normed space A. Then the sequence $\{x_n\}$ is called convergent, if for a given $\epsilon > 0$, there are a positive integer N and an $x \in A$ such that $\|x_n - x\| \leqslant \epsilon$ for all $n \geqslant N$. Then we call $x \in A$ a limit of the sequence $\{x_n\}$, and denote it by

$$\lim_{n\to\infty} x_n = x.$$

**Definition 2.7.** If every Cauchy sequence in A converges, then the non-Archimedean normed space A is called a non-Archimedean Banach space.

Due to the fact that $\|x_n - x_m\| \leqslant \max\left\{\|x_{j+1} - x_j\| \ : \ m \leqslant j \leqslant n-1\right\}$, $(n > m)$ a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

Since any non-Archimedean norm satisfies the triangle inequality, any non-Archimedean norm is a continuous function from its domain to real numbers.

Throughout this paper, let us consider A be a non-Archimedean normed vector space over a field $\mathbb{F}$ and B is a non-Archimedean Banach space over a field $\mathbb{K}$. Let, moreover, $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4 \in \mathbb{F}$ and $C_{11}, C_{12}, C_{21}, C_{22} \in \mathbb{K}$ be given scalars. Suppose for a function $f : A \times A \times A \times A \to B$, we define a mapping $D_f : A \to B$ by

$$\begin{aligned} D_f(x_1,x_2,y_1,y_2) = {} & f(a_1(x_1+x_2), b_1(y_1+y_2)) + f(a_2(x_1+x_2), b_2(y_1-y_2)) \\ & + f(a_3(x_1-x_2), b_3(y_1+y_2)) + f(a_4(x_1-x_2), b_4(y_1-y_2)) \\ & - C_{11}f(x_1,y_1) - C_{12}f(x_1,y_2) - C_{21}f(x_2,y_1) - C_{22}f(x_2,y_2) \end{aligned} \tag{2.1}$$

for each $x_1, x_2, y_1, y_2 \in A$. Denote by $\mathbb{N}$, as usual, the set of all positive integers, put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $C := C_{11} + C_{12} + C_{21} + C_{22}$. Assume, moreover, that $C \neq 0$.

## 3. Generalized Hyers-Ulam stability of the bi-quadratic mappings (1.10): an odd case

In this section, first we will prove the generalized Hyers-Ulam stability of the bi-quadratic functional equation $D_f(x, y) = 0$ in non-Archimedean Banach spaces.

**Theorem 3.1.** *Let* $\phi : A \times A \times A \times A \to [0, \infty)$ *such that*

$$\lim_{n\to\infty} \frac{\phi(2^n x_1, 2^n x_2, 2^n y_1, 2^n y_2)}{|16|^n} = 0 \tag{3.1}$$

*for all* $x_1, x_2, y_1, y_2 \in X$. *Suppose that* $f : A \times A \times A \times A \to B$ *is a mapping satisfying the equality*

$$\|D_f(x_1,x_2,y_1,y_2)\| \leqslant \phi(x_1,x_2,y_1,y_2) \tag{3.2}$$

*and let* $f(x_1, 0) = 0$ *and* $f(0, y_1) = 0$ *for all* $x_1, x_2, y_1, y_2 \in A$. *Then there exists a unique mapping* $T : A \times A \times A \times A \to B$ *such that*

$$\|f(x_1,y_1) - T(x_1,y_1)\| \leqslant \frac{1}{|16|} \lim_{n\to\infty} \max\left\{ \frac{\phi(2^j x_1, 2^j x_1, 2^j y_1, 2^j y_1)}{|16|^j} : 0 \leqslant j < n \right\} \tag{3.3}$$

*for all* $x_1, y_1 \in A$.

*Proof.* Let $a_1 = a_2 = \cdots = b_4 = 1$ and $C_{11} = \cdots = C_{22} = 4$ in (2.1) we get

$$\begin{aligned} D_f(x_1,x_2,y_1,y_2) = {} & f(x_1+x_2, y_1+y_2) + f(x_1+x_2, y_1-y_2) + f(x_1-x_2, y_1+y_2) \\ & + f(x_1-x_2, y_1-y_2) - 4[f(x_1,y_1) + f(x_1,y_2) + f(x_2,y_1) + f(x_2,y_2)]. \end{aligned} \tag{3.4}$$

Replacing $x_2$ by $x_1$ and $y_2$ by $y_1$ in (3.4), we have

$$\|f(2x_1,2y_1)-16f(x_1,y_1)\| \leqslant \phi(x_1,x_1,y_1,y_1)$$

for all $x_1,y_1 \in A$. Thus we obtain

$$\left\|f(x_1,y_1)-\frac{1}{16}f(2x_1,2y_1)\right\| \leqslant \frac{1}{|16|}\phi(x_1,x_1,y_1,y_1)$$

for all $x_1,y_1 \in A$. Replacing $x_1$ by $2^n x_1$ and $y_1$ by $2^n y_1$, we get

$$\left\|\frac{f(2^n x_1,2^n y_1)}{16^n}-\frac{f(2^{n+1}x_1,2^{n+1}y_1)}{16^{n+1}}\right\| \leqslant \frac{1}{|16|^{n+1}}\phi(2^n x_1,2^n x_1,2^n y_1,2^n y_1) \tag{3.5}$$

for all $x_1,y_1 \in A$. It follows from the inequality (3.5) and (3.1) that the sequence $\left\{\frac{f(2^n x_1,2^n y_1)}{16^n}\right\}$ is a Cauchy sequence. We conclude that the sequence $\left\{\frac{f(2^n x_1,2^n y_1)}{16^n}\right\}$ is convergent. Now define a mapping

$$T(x)=\lim_{n\to\infty}\frac{f(2^n x_1,2^n y_1)}{16^n}. \tag{3.6}$$

So, we have

$$\begin{aligned}\left\|f(x_1,y_1)-\frac{f(2^n x_1,2^n y_1)}{16^n}\right\| &= \left\|\sum_{j=0}^{n-1}\left(\frac{f(2^j x_1,2^j y_1)}{16^j}-\frac{f(2^{j+1}x_1,2^{j+1}y_1)}{16^{j+1}}\right)\right\|\\ &\leqslant \max\left\{\left\|\frac{f(2^j x_1,2^j y_1)}{16^j}-\frac{f(2^{j+1},2^{j+1}y_1)}{16^{j+1}}\right\| : 0\leqslant j<n\right\}\\ &\leqslant \frac{1}{|16|}\max\left\{\frac{\phi(2^j x_1,2^j x_1,2^j y_1,2^j y_1)}{|16|^j} : 0\leqslant j<n\right\},\end{aligned} \tag{3.7}$$

taking $n\to\infty$ in (3.7) and applying (3.6), we can see that the inequality (3.3) holds. By (3.2) we have

$$\begin{aligned}&\left\|\frac{1}{16^n}f(2^n(x_1+x_2),2^n(y_1+y_2))+\frac{1}{16^n}f(2^n(x_1+x_2),2^n(y_1-y_2))\right.\\ &\quad+\frac{1}{16^n}f(2^n(x_1-x_2),2^n(y_1+y_2))+\frac{1}{16^n}f(2^n(x_1-x_2),2^n(y_1-y_2))\\ &\quad\left.-\frac{4}{16^n}f(2^n x_1,2^n y_1)-\frac{4}{16^n}f(2^n x_1,2^n y_2)-\frac{4}{16^n}f(2^n x_2,2^n y_1)-\frac{4}{16^n}f(2^n x_2,2^n y_2)\right\|\\ &\leqslant \lim_{n\to\infty}\frac{\phi(2^n x_1,2^n x_2,2^n y_1,2^n y_2)}{|16|^n}=0.\end{aligned}$$

Hence the mapping T satisfies (2.1). Now let $T' : A\times A\times A\times A \to B$ be another mapping satisfying (3.3), then we have

$$\begin{aligned}\|T(x_1,y_1)-T'(x_1,y_1)\| &= \lim_{k\to\infty}\frac{1}{|16|^k}\|T(2^k x_1,2^k y_1)-T'(2^k x_1,2^k y_1)\|\\ &\leqslant \lim_{k\to\infty}\frac{1}{|16|^k}\max\left\{\|T(2^k x_1,2^k y_1)-f(2^k x_1,2^k y_1)\|,\|f(2^k x_1,2^k y_1)-T'(2^k x_1,2^k y_1)\|\right\}\\ &\leqslant \frac{1}{|16|}\lim_{k\to\infty}\lim_{n\to\infty}\max\left\{\frac{\phi(2^j x_1,2^j x_1,2^j y_1,2^j y_1)}{|16|^j} : k\leqslant j<n+k\right\}\\ &\leqslant \frac{1}{|16|}\lim_{k\to\infty}\sup\left\{\frac{\phi(2^j x_1,2^j x_1,2^j y_1,2^j y_1)}{|16|^j} : k\leqslant j<\infty\right\}=0.\end{aligned}$$

Hence $T=T'$. This completes the proof. □

**Corollary 3.2.** *Let $\theta$ and $r$ be a positive real numbers with $r < 4$. Let $f : A \times A \times A \times A \to B$ be a bi-quadratic mapping satisfying*

$$\|D_f(x_1, x_2, y_1, y_2)\| \leqslant \theta\left(\|x_1\|^r + \|x_2\|^r + \|y_1\|^r + \|y_2\|^r\right)$$

*and let $f(x_1, 0) = 0$ and $f(0, y_1) = 0$ for all $x_1, x_2, y_1, y_2 \in A$. Then there exists a unique bi-quadratic mapping $T : A \times A \times A \times A \to B$ such that*

$$\|f(x_1, y_1) - T(x_1, y_1)\| \leqslant \max\left\{\frac{2^{rj}}{16^{j+1}} : 0 \leqslant j < n\right\} \frac{2\theta}{16 - 2^r}\left(\|x_1\|^r + \|y_1\|^r\right)$$

*for all $x_1, y_1 \in A$.*

**Theorem 3.3.** *Let $\phi : A \times A \times A \times A \to [0, \infty)$ such that*

$$\lim_{n\to\infty} |16|^n \phi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{y_1}{2^n}, \frac{y_2}{2^n}\right) = 0 \tag{3.8}$$

*for all $x_1, x_2, y_1, y_2 \in A$. Suppose that $f : A \times A \times A \times A \to B$ is a mapping satisfying the equality*

$$\|D_f(x_1, x_2, y_1, y_2)\| \leqslant \phi(x_1, x_2, y_1, y_2) \tag{3.9}$$

*and let $f(x_1, 0) = 0$ and $f(0, y_1) = 0$ for all $x_1, x_2, y_1, y_2 \in A$. Then there exists a unique mapping $T : A \times A \times A \times A \to B$ such that*

$$\|f(x_1, y_1) - T(x_1, y_1)\| \leqslant \lim_{n\to\infty} \max\left\{|16|^j \phi\left(\frac{x_1}{2^{j+1}}, \frac{x_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}\right) : 0 \leqslant j < n\right\} \tag{3.10}$$

*for all $x_1, y_1 \in A$.*

*Proof.* Using the same technique as in Theorem 3.1, we also can easily prove this theorem. Then by the analog of Theorem 3.1, we have

$$\|f(2x_1, 2y_1) - 16f(x_1, y_1)\| \leqslant \phi(x_1, x_1, y_1, y_1)$$

for all $x_1, y_1 \in A$. Thus we obtain

$$\left\|16^n f\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}\right) - 16^{n+1} f\left(\frac{x_1}{2^{n+1}}, \frac{y_1}{2^{n+1}}\right)\right\| \leqslant |16|^n \phi\left(\frac{x_1}{2^{n+1}}, \frac{x_1}{2^{n+1}}, \frac{y_1}{2^{n+1}}, \frac{y_1}{2^{n+1}}\right) \tag{3.11}$$

for all $x_1, y_1 \in A$. It follows from (3.11) and (3.8) that the sequence $\left\{16^n f\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}\right)\right\}$ is Cauchy. We conclude that the sequence $\left\{16^n f\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}\right)\right\}$ is convergent. Now define a mapping

$$T(x) = \lim_{n\to\infty} 16^n f\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}\right). \tag{3.12}$$

Then we have

$$\begin{aligned}
\left\|f(x_1, y_1) - 16^n f\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}\right)\right\| &= \left\|\sum_{j=0}^{n-1}\left(16^j f\left(\frac{x_1}{2^j}, \frac{y_1}{2^j}\right) - 16^{j+1} f\left(\frac{x_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}\right)\right)\right\| \\
&\leqslant \max\left\{\left\|16^j f\left(\frac{x_1}{2^j}, \frac{y_1}{2^j}\right) - 16^{j+1} f\left(\frac{x_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}\right)\right\| : 0 \leqslant j < n\right\} \qquad (3.13) \\
&\leqslant \max\left\{|16|^j \phi\left(\frac{x_1}{2^{j+1}}, \frac{x_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}\right) : 0 \leqslant j < n\right\},
\end{aligned}$$

taking $n \to \infty$ in (3.13) and applying (3.12) we can see that the inequality (3.10) holds. By (3.9), we reach that

$$\begin{aligned}&\Bigg\|16^n f\left(\frac{x_1+x_2}{2^n}, \frac{y_1+y_2}{2^n}\right) + 16^n f\left(\frac{x_1+x_2}{2^n}, \frac{y_1-y_2}{2^n}\right)\\ &+16^n f\left(\frac{x_1-x_2}{2^n}, \frac{y_1+y_2}{2^n}\right) + 16^n f\left(\frac{x_1-x_2}{2^n}, \frac{y_1-y_2}{2^n}\right)\\ &-4\times 16^n f\left(\frac{x_1}{2^n}, \frac{y_1}{2^n}\right) - 4\times 16^n f\left(\frac{x_1}{2^n}, \frac{y_2}{2^n}\right) - 4\times 16^n f\left(\frac{x_2}{2^n}, \frac{y_1}{2^n}\right)\\ &-4\times 16^n f\left(\frac{x_2}{2^n}, \frac{y_2}{2^n}\right)\Bigg\| \leqslant \lim_{n\to\infty} |16|^n \phi\left(\frac{x_1}{2^n}, \frac{x_2}{2^n}, \frac{y_1}{2^n}, \frac{y_2}{2^n}\right) = 0.\end{aligned}$$

Hence the mapping $T$ satisfies (2.1). Now let $T' : A \times A \times A \times A \to B$ be another mapping satisfying (3.10), then we have

$$\begin{aligned}\|T(x_1,y_1) - T'(x_1,y_1)\| &= \lim_{k\to\infty} |16|^k \left\|T\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right) - T'\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right)\right\|\\ &\leqslant \lim_{k\to\infty} |16|^k \max\left\{\left\|T\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right) - f\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right)\right\|, \left\|f\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right) - T'\left(\frac{x_1}{2^k}, \frac{y_1}{2^k}\right)\right\|\right\}\\ &\leqslant \lim_{k\to\infty}\lim_{n\to\infty} \max\left\{|16|^j \phi\left(\frac{x_1}{2^{j+1}}, \frac{x_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}\right) : k \leqslant j < n+k\right\}\\ &\leqslant \lim_{k\to\infty} \sup\left\{|16|^j \phi\left(\frac{x_1}{2^{j+1}}, \frac{x_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}, \frac{y_1}{2^{j+1}}\right) : k \leqslant j < \infty\right\} = 0.\end{aligned}$$

Hence $T = T'$. Thus we reached the desired result. □

**Corollary 3.4.** *Let $\theta$ and $r$ be a positive real numbers with $r > 4$. Let $f : A \times A \times A \times A \to B$ be a bi-quadratic mapping satisfying*

$$\|D_f(x_1,x_2,y_1,y_2)\| \leqslant \theta\, (\|x_1\|^r + \|x_2\|^r + \|y_1\|^r + \|y_2\|^r)$$

*and let $f(x_1, 0) = 0$ and $f(0, y_1) = 0$ for all $x_1, x_2, y_1, y_2 \in A$. Then there is a unique bi-quadratic mapping $T : A \times A \times A \times A \to B$ such that*

$$\|f(x_1,y_1) - T(x_1,y_1)\| \leqslant \max\left\{\frac{16^j}{2^{r(j+1)}} : 0 \leqslant j < n\right\} \frac{2\theta}{2^r - 16} (\|x_1\|^r + \|y_1\|^r)$$

*for all $x_1, y_1 \in A$.*

## 4. Conclusion

In this paper, we established the sufficient criteria for the generalized Hyers-Ulam stability of a bi-quadratic mappings in non-Archimedean Banach spaces.

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