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A new type of the hybrid algebra between Abelian groups and UP (BCC)-algebras: UP (BCC)-modules

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Abstract

The goal of this study is to introduce the concept of a new type of the hybrid algebra between Abelian groups and UP (BCC)-algebras: UP (BCC)-modules. We introduce the concept of fuzzy UP (BCC)-submodules of UP (BCC)-modules and provide properties and find the necessary and sufficient conditions for this concept. We define fuzzy sets in UP (BCC)-modules of many forms, supplying their properties and their relation to fuzzy UP (BCC)-submodules. We also define and study the fuzzy UP (BCC)-submodule generated by a set of fuzzy sets in UP (BCC)-modules, as well as provide for their properties and their relation to fuzzy UP (BCC)-submodules. Finally, we apply the concept of fuzzy UP (BCC)-ideals of UP (BCC)-algebras while providing properties and find the results of the composition and the product between fuzzy UP (BCC)-ideals and fuzzy UP (BCC)-submodules.

Keywords: UP (BCC)-algebra, UP (BCC)-module, fuzzy UP (BCC)-ideal, fuzzy UP (BCC)-submodule.

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1. Introduction and preliminaries

In 2017, Iampan [9] established the UP-algebras branch of logical algebra, and it is well known that the KU-algebras class [24] is a proper subclass of the class of UP-algebras. It has been studied by a number of researchers. For instance, Somjanta et al. [32] studied fuzzy sets in UP-algebras, Kesorn et al. [17] studied intuitionistic fuzzy sets in UP-algebras, Tanamoon et al. [36] introduced the notion of Q-fuzzy sets in UP-algebras, and Senapati et al. [29, 30] applied cubic sets and interval-valued intuitionistic fuzzy structures in UP-algebras. Fuzzy UP-subalgebras (fuzzy UP-filters, fuzzy UP-ideals, fuzzy strong UP-ideals) with thresholds of UP-algebras were presented by Dokkhamdang et al. [7]. Ansari et al. [2] established a graph of equivalence classes of commutative UP-algebras and proposed the idea of graphs associated with commutative UP-algebras. Songsaeng and Iampan [33–35] investigated $\mathcal{N}$-fuzzy sets, fuzzy proper

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UP-filters, and neutrosophic sets in UP-algebras. The idea of BCC-algebras (see [18]) and the idea of UP-algebras (see [9]) are the same ideas, according to Jun et al. [16] in 2022. In order to show respect for Komori, who initially defined it in 1984, we shall refer to it as BCC rather than UP in this paper.

A module is an action of algebraic structures on groups. It has been studied in different kinds of algebraic structures. Examples of these are L-modules [20], BCK-modules [1], extended BCK-modules, and multiplication extended BCK-modules [5]. In 1975, Naegoita and Ralescu [22] introduced the concept of fuzzy submodules. In 1987, Pan [23] introduced the construction of fuzzy finitely generated modules. The product of a fuzzy ideal and a fuzzy submodule and the sum of two fuzzy submodules were defined by Kumar et al. [19] in 1995. Several researchers have studied the concept of fuzzy submodules in different fields. Zahed [38] studied L-fuzzy residual quotient modules and P-primary submodules. In 2011, Bakhshi [4] introduced fuzzy BCK-submodules and studied their properties. He also studies the fuzzy BCK-submodule generated by a set of fuzzy sets in BCK-modules. Moreover, he presented some operations of fuzzy BCK-submodules. As was already indicated, we think that studying fuzzy submodules is important and fascinating, which is how we came to investigate this topic.

In 1965, Zadeh [37] was the first to propose the idea of fuzzy sets. Many applications in the field of mathematics and other fields have been made possible by the fuzzy set theories created by Zadeh and others. Several studies have been carried out on the extensions of the concept of fuzzy sets and their application to a wide range of algebras, for example, in 2016, Somjanta et al. [32] introduced the notion of fuzzy sets in BCC-algebras. In 2019, Burandate et al. [6] applied the notion of fuzzy sets with respect to a triangular norm to BCC-algebras. Songsaeng and Iampan [34] introduced fuzzy proper BCC-filters of BCC-algebras.

The goal of this study is to introduce the concept of a new type of the hybrid algebra between Abelian groups and BCC-algebras: BCC-modules. In this article, after Section 1, where we have outlined the origins and inspiration, we have divided the content into the following six sections. In Section 2, we introduce the concept of BCC-modules and provide various properties for use in the following sections. In Section 3, we introduce the concept of fuzzy BCC-submodules (FBCCSMs) of BCC-modules and provide properties and find the necessary and sufficient conditions for this concept. In Section 4, we define fuzzy sets in BCC-modules of many forms, supplying their properties and their relation to FBCCSMs. In Section 5, we define and study the FBCCSM generated by a set of fuzzy sets in BCC-modules, as well as provide for their properties and their relation to FBCCSMs. Finally, in Section 6, we provide properties and use the notion of fuzzy BCC-ideals (FBCCIs) of BCC-algebras. Additionally, we discover the composition and product outcomes between FBCCSMs and FBCCIs.

Without the need of the expression (1.1), the idea of BCC-algebras (see [18]) may be described as follows.

Definition 1.1 ([8]). If the following axioms are true, an algebra of the form $\mathcal{U} = (\mathcal{U}; \cdot, 1)$ of type $(2, 0)$ is said to be a *BCC-algebra*, where $\mathcal{U}$ is a nonempty set, $\cdot$ is a binary operation on $\mathcal{U}$, and 1 is a fixed element of $\mathcal{U}$:

$$(\forall \upsilon, \iota, \sigma \in \mathcal{U})((\iota \cdot \sigma) \cdot ((\upsilon \cdot \iota) \cdot (\upsilon \cdot \sigma)) = 1), \tag{BCC-1}$$

$$(\forall \upsilon \in \mathcal{U})(1 \cdot \upsilon = \upsilon), \tag{BCC-2}$$

$$(\forall \upsilon \in \mathcal{U})(\upsilon \cdot 1 = 1), \tag{BCC-3}$$

$$(\forall \upsilon, \iota \in \mathcal{U})(\upsilon \cdot \iota = 1, \iota \cdot \upsilon = 1 \Rightarrow \upsilon = \iota). \tag{BCC-4}$$

A partial ordering $\leqslant$ is defined on a BCC-algebra $\mathcal{U} = (\mathcal{U}; \cdot, 1)$ by

$$(\forall \upsilon, \iota \in \mathcal{U})(\upsilon \leqslant \iota \Leftrightarrow \upsilon \cdot \iota = 1).$$

See [2, 3, 10, 11, 21, 25–27, 29, 30] for additional BCC-algebra research and examples.

In a BCC-algebra $\mathcal{U} = (\mathcal{U}; \cdot, 1)$, the following assertions are valid (see [9, 10]).

$$(\forall \upsilon \in \mathcal{U})(\upsilon \cdot \upsilon = 1), \tag{1.1}$$

$$
\begin{aligned}
&(\forall \upsilon, \iota, \sigma \in \mathcal{U})(\upsilon \cdot \iota = 1, \iota \cdot \sigma = 1 \Rightarrow \upsilon \cdot \sigma = 1),\\
&(\forall \upsilon, \iota, \sigma \in \mathcal{U})(\upsilon \cdot \iota = 1 \Rightarrow (\sigma \cdot \upsilon) \cdot (\sigma \cdot \iota) = 1),\\
&(\forall \upsilon, \iota, \sigma \in \mathcal{U})(\upsilon \cdot \iota = 1 \Rightarrow (\iota \cdot \sigma) \cdot (\upsilon \cdot \sigma) = 1),\\
&(\forall \upsilon, \iota, \sigma \in \mathcal{U})(\upsilon \circ (\iota \circ \upsilon) = 1, \text{ in particular, } (\iota \circ \sigma) \circ (\upsilon \circ (\iota \circ \sigma)) = 1),\\
&(\forall \upsilon, \iota \in \mathcal{U})((\iota \cdot \upsilon) \cdot \upsilon = 1 \Leftrightarrow \upsilon = \iota \cdot \upsilon),\\
&(\forall \upsilon, \iota \in \mathcal{U})(\upsilon \cdot (\iota \cdot \iota) = 1),\\
&(\forall a, \upsilon, \iota, \sigma \in \mathcal{U})((\upsilon \cdot (\iota \cdot \sigma)) \cdot (\upsilon \cdot ((a \cdot \iota) \cdot (a \cdot \sigma))) = 1),\\
&(\forall a, \upsilon, \iota, \sigma \in \mathcal{U})((((a \cdot \upsilon) \cdot (a \cdot \iota)) \cdot \sigma) \cdot ((\upsilon \cdot \iota) \cdot \sigma) = 1),\\
&(\forall \upsilon, \iota, \sigma \in \mathcal{U})(((\upsilon \cdot \iota) \cdot \sigma) \cdot (\iota \cdot \sigma) = 1),\\
&(\forall \upsilon, \iota, \sigma \in \mathcal{U})(\upsilon \cdot \iota = 1 \Rightarrow \upsilon \cdot (\sigma \cdot \iota) = 1),\\
&(\forall \upsilon, \iota, \sigma \in \mathcal{U})(((\upsilon \cdot \iota) \cdot \sigma) \cdot (\upsilon \cdot (\iota \cdot \sigma)) = 1),\\
&(\forall a, \upsilon, \iota, \sigma \in \mathcal{U})(((\upsilon \cdot \iota) \cdot \sigma) \cdot (\iota \cdot (a \cdot \sigma)) = 1).
\end{aligned}
$$

Example 1.2 ([27]). Suppose that $\mathcal{U}$ is a nonempty set and that X is in the $\mathcal{P}(\mathcal{U})$, where $\mathcal{P}(\mathcal{U})$ denotes the power set of $\mathcal{U}$. Let $\mathcal{P}_X(\mathcal{U}) = \{Y \in \mathcal{P}(\mathcal{U}) \mid X \subseteq Y\}$. Put $A \cdot B = B \cap (A' \cup X)$, $\forall A, B \in \mathcal{P}_X(\mathcal{U})$ to define the binary operation $\cdot$ on $\mathcal{P}_X(\mathcal{U})$, where A' means the complement of a subset A. Hence, $(\mathcal{P}_X(\mathcal{U}), \cdot, X)$ is a BCC-algebra. Let $\mathcal{P}^X(\mathcal{U}) = \{Y \in \mathcal{P}(\mathcal{U}) \mid Y \subseteq X\}$. Put $A * B = B \cup (A' \cap X)$, $\forall A, B \in \mathcal{P}^X(\mathcal{U})$ to define the binary operation $*$ on $\mathcal{P}^X(\mathcal{U})$. Hence, $(\mathcal{P}^X(\mathcal{U}), *, X)$ is a BCC-algebra.

Definition 1.3. A BCC-algebra $\mathcal{U} = (\mathcal{U}; \cdot, 1)$ is said to be

(i) *bounded* if there is an element $0 \in \mathcal{U}$ such that $0 \leqslant \upsilon$, $\forall \upsilon \in \mathcal{U}$, that is,

$$(\forall \upsilon \in \mathcal{U})(0 \cdot \upsilon = 1); \tag{Bounded}$$

(ii) *meet-commutative* [28] if

$$(\forall \upsilon, \iota \in \mathcal{U})(\upsilon \wedge \iota = \iota \wedge \upsilon), \tag{Meet-commutative}$$

where

$$(\forall \upsilon, \iota \in \mathcal{U})(\upsilon \wedge \iota = (\iota \cdot \upsilon) \cdot \upsilon). \tag{Meet}$$

2. Introducing BCC-modules

In this section, we introduce a system of the hybrid algebra between BCC-algebras and Abelian groups in a form similar to the well-known modules. This new algebraic system is called BCC-modules, which is defined as follows.

Definition 2.1. By a *left BCC-module* (briefly, *BCC-module*) over a BCC-algebra $\mathcal{U} = (\mathcal{U}; \cdot, 1)$, we mean an Abelian group $G = (G; +, 0)$ with an operation $\mathcal{U} \times G \to G$ with $(\upsilon, n) \mapsto \upsilon n$ that satisfies the following axioms:

$$
\begin{aligned}
&(\forall \upsilon, \iota \in \mathcal{U}, \forall n \in G)((\upsilon \wedge \iota) n = \upsilon(\iota n)), && \text{(BCCM-1)}\\
&(\forall \upsilon \in \mathcal{U}, \forall n, m \in G)(\upsilon(n + m) = \upsilon n + \upsilon m), && \text{(BCCM-2)}\\
&(\forall n \in G)(1n = 0). && \text{(BCCM-3)}
\end{aligned}
$$

The followings are examples that support the above definition.

Example 2.2. Let A be a nonempty set and $\mathcal{U} = \mathcal{P}(A)$. Then $(\mathcal{U}; +, \emptyset)$ is an Abelian group with $n + m = (n - m) \cup (m - n)$ for any $n, m \in \mathcal{U}$. By Example 1.2, we get $(\mathcal{U}; \cdot, \emptyset)$ is a BCC-algebra. Hence, $\mathcal{U}$ is a BCC-module over itself with $\upsilon n = \upsilon \cap n$, $\forall \upsilon, n \in \mathcal{U}$.

Proof.

(UPM-1) Let $\upsilon, \iota, \mathfrak{n} \in \mathcal{U}$. Then

$$\begin{aligned}(\upsilon \wedge \iota)\mathfrak{n} &= ((\iota \cdot \upsilon) \cdot \upsilon)\mathfrak{n}\\ &= ((\iota \cdot \upsilon) \cdot \upsilon) \cap \mathfrak{n}\\ &= ((\iota' \cap \upsilon)' \cap \upsilon) \cap \mathfrak{n}\\ &= ((\iota \cup \upsilon') \cap \upsilon) \cap \mathfrak{n}\\ &= (\iota \cup \upsilon') \cap \upsilon \cap \mathfrak{n}\\ &= (\iota \cup \upsilon') \cap (\upsilon \cap \mathfrak{n})\\ &= (\iota \cap (\upsilon \cap \mathfrak{n})) \cup (\upsilon' \cap (\upsilon \cap \mathfrak{n}))\\ &= ((\iota \cap \upsilon) \cap \mathfrak{n}) \cup ((\upsilon' \cap \upsilon) \cap \mathfrak{n})\\ &= ((\iota \cap \upsilon) \cap \mathfrak{n}) \cup (\emptyset \cap \mathfrak{n})\\ &= ((\upsilon \cap \iota) \cap \mathfrak{n}) \cup \emptyset\\ &= (\upsilon \cap \iota) \cap \mathfrak{n}\\ &= \upsilon \cap (\iota \cap \mathfrak{n})\\ &= \upsilon(\iota\mathfrak{n}).\end{aligned}$$

(UPM-2) Let $\upsilon, \mathfrak{n}, \mathfrak{m} \in \mathcal{U}$. Then

$$\begin{aligned}\upsilon(\mathfrak{n}+\mathfrak{m}) &= \upsilon((\mathfrak{n}-\mathfrak{m}) \cup (\mathfrak{m}-\mathfrak{n}))\\ &= \upsilon \cap ((\mathfrak{n} \cap \mathfrak{m}') \cup (\mathfrak{m} \cap \mathfrak{n}'))\\ &= (\upsilon \cap (\mathfrak{n} \cap \mathfrak{m}')) \cup (\upsilon \cap (\mathfrak{m} \cap \mathfrak{n}'))\\ &= ((\upsilon \cap \mathfrak{n}) \cap \mathfrak{m}') \cup ((\upsilon \cap \mathfrak{m}) \cap \mathfrak{n}')\\ &= ((\upsilon \cap \mathfrak{n}) \cap (\upsilon' \cup \mathfrak{m}')) \cup ((\upsilon \cap \mathfrak{m}) \cap (\upsilon' \cup \mathfrak{n}'))\\ &= ((\upsilon \cap \mathfrak{n}) \cap (\upsilon \cap \mathfrak{m})') \cup ((\upsilon \cap \mathfrak{m}) \cap (\upsilon \cap \mathfrak{n})')\\ &= ((\upsilon \cap \mathfrak{n}) - (\upsilon \cap \mathfrak{m})) \cup ((\upsilon \cap \mathfrak{m}) - (\upsilon \cap \mathfrak{n}))\\ &= (\upsilon\mathfrak{n} - \upsilon\mathfrak{m}) \cup (\upsilon\mathfrak{m} - \upsilon\mathfrak{n})\\ &= \upsilon\mathfrak{n} + \upsilon\mathfrak{m}.\end{aligned}$$

(UPM-3) Let $\mathfrak{n} \in \mathcal{U}$. Then

$$\emptyset\mathfrak{n} = \emptyset \cap \mathfrak{n} = \emptyset.$$

Hence, $\mathcal{U}$ is a BCC-module over itself. □

Example 2.3. Let A be a nonempty set and $\mathcal{U} = \mathcal{P}(A)$. Then $(\mathcal{U};+,A)$ is an Abelian group with $\mathfrak{n}+\mathfrak{m} = (\mathfrak{n} \cap \mathfrak{m}) \cup (\mathfrak{m} \cup \mathfrak{n})'$ for any $\mathfrak{n}, \mathfrak{m} \in \mathcal{U}$. By Example 1.2, we get $(\mathcal{U};*,A)$ is a BCC-algebra. Hence, $\mathcal{U}$ is a BCC-module over itself with $\upsilon\mathfrak{n} = \upsilon \cup \mathfrak{n}$, $\forall \upsilon, \mathfrak{n} \in \mathcal{U}$.

Proof. Let $\upsilon, \iota, \mathfrak{n}, \mathfrak{m} \in \mathcal{U}$. Then we have following.

(UPM-1) Let $\upsilon, \iota, \mathfrak{n} \in \mathcal{U}$. Then

$$\begin{aligned}(\upsilon \wedge \iota)\mathfrak{n} &= ((\iota \cdot \upsilon) \cdot \upsilon)\mathfrak{n}\\ &= ((\iota \cdot \upsilon) \cdot \upsilon) \cup \mathfrak{n}\\ &= ((\iota' \cup \upsilon)' \cup \upsilon) \cup \mathfrak{n}\\ &= ((\iota \cap \upsilon') \cup \upsilon) \cup \mathfrak{n}\\ &= ((\iota \cup \upsilon) \cap (\upsilon' \cup \upsilon)) \cup \mathfrak{n}\end{aligned}$$

$$
\begin{aligned}
&= ((\upsilon \cup \iota) \cap A) \cup n \\
&= (\upsilon \cup \iota) \cup n \\
&= \upsilon \cup (\iota \cup n) \\
&= \upsilon(\iota n).
\end{aligned}
$$

(UPM-2) Let $\upsilon, n, m \in \mathcal{U}$. Then

$$
\begin{aligned}
\upsilon(n+m) &= \upsilon \cup (n+m) \\
&= \upsilon \cup ((n \cap m) \cup (m \cup n)') \\
&= \upsilon \cup ((n \cap m) \cup (m' \cap n')) \\
&= (\upsilon \cup (n \cap m)) \cup (m' \cap n') \\
&= A \cap (\upsilon \cup (n \cap m)) \cup (m' \cap n')) \\
&= (\upsilon \cup (n \cap m)) \cup \upsilon') \cap (\upsilon \cup (n \cap m)) \cup (m' \cap n')) \\
&= (\upsilon \cup (n \cap m)) \cup (\upsilon' \cap (m' \cap n')) \\
&= (\upsilon \cup (n \cap m)) \cup (\upsilon \cup (m \cup n))' \\
&= ((\upsilon \cup n) \cap (\upsilon \cup m)) \cup ((\upsilon \cup m) \cup (\upsilon \cup n))' \\
&= (\upsilon n \cap \upsilon m) \cup (\upsilon m \cup \upsilon n)' \\
&= \upsilon n + \upsilon m.
\end{aligned}
$$

(UPM-3) Let $n \in \mathcal{U}$. Then

$$An = A \cup n = A.$$

Hence, $\mathcal{U}$ is a BCC-module over itself. □

Definition 2.4. A BCC-module G over $\mathcal{U}$ is said to be

(i) *unitary* (when $\mathcal{U}$ is bounded) if

$$(\forall n \in G)(0n = n); \tag{Unitary}$$

(ii) *separability* if

$$(\forall \upsilon \in \mathcal{U}, \forall n \in G)(\upsilon n = n); \tag{Separability}$$

(iii) *distributive* if

$$(\forall \upsilon, \iota \in \mathcal{U}, \forall n, m \in G)(\upsilon n + \iota m = (\upsilon \wedge \iota)(n+m)). \tag{Distributive}$$

For convenience, we define G as a BCC-module G over $\mathcal{U}$ until further described, where we shall let $\mathcal{U} = (\mathcal{U}; \cdot, 1)$ be a BCC-algebra and $G = (G; +, 0)$ an Abelian group.

Proposition 2.5. *Let* $\upsilon, \upsilon_i \in \mathcal{U}$ *and* $n, n_i \in G$, $\forall i \in \{1, 2, \ldots, k\}$. *Then*

(i) $(\forall \upsilon \in \mathcal{U}, \forall n \in G)((1 \wedge \upsilon)n = 0)$;

(ii) $(\forall \upsilon \in \mathcal{U})(\upsilon 0 = 0)$;

(iii) $(\forall \upsilon \in \mathcal{U}, \forall n \in G)((\upsilon \wedge 1)n = 0)$;

(iv) $(\forall \upsilon \in \mathcal{U}, \forall n \in G)((\upsilon \wedge \upsilon)n = \upsilon n)$;

(v) $(\forall \upsilon \in \mathcal{U}, \forall n \in G)(-\upsilon n = \upsilon(-n))$;

(vi) $(\forall \upsilon \in \mathcal{U}, \forall n, m \in G)(\upsilon(n - m) = \upsilon n - \upsilon m)$;

(vii) $-(\sum_{i=1}^{k} \upsilon_i n_i) = \sum_{i=1}^{k} \upsilon_i(-n_i)$.

Proof.

(i) Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned} (1 \wedge \upsilon)n &= 1(\upsilon n) && \text{(BCCM-1)} \\ &= 0. && \text{(BCCM-3)} \end{aligned}$$

(ii) Let $\upsilon \in \mathcal{U}$. Then

$$\begin{aligned} \upsilon 0 + \upsilon 0 &= \upsilon(0 + 0) && \text{(BCCM-1)} \\ &= \upsilon 0. \end{aligned}$$

Thus $\upsilon 0$ is an idempotent element in G, that is, $\upsilon 0 = 0$.

(iii) Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned} (\upsilon \wedge 1)n &= \upsilon(1n) && \text{(BCCM-1)} \\ &= \upsilon 0 && \text{(BCCM-3)} \\ &= 0. && \text{(ii)} \end{aligned}$$

(iv) Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned} (\upsilon \wedge \upsilon)n &= ((\upsilon \cdot \upsilon) \cdot \upsilon)n && \text{(Meet)} \\ &= (1 \cdot \upsilon)n && \text{(1.1)} \\ &= \upsilon n. && \text{(BCC-2)} \end{aligned}$$

(v) Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned} \upsilon n + \upsilon(-n) &= \upsilon(n + (-n)) && \text{(BCCM-2)} \\ &= \upsilon 0 \\ &= 0. && \text{(ii)} \end{aligned}$$

Thus $\upsilon(-n)$ is the inverse element of υn, that is, $-\upsilon n = \upsilon(-n)$.

(vi) Let $\upsilon \in \mathcal{U}$ and $n, m \in G$. Then

$$\begin{aligned} \upsilon(n - m) &= \upsilon(n + (-m)) \\ &= \upsilon n + \upsilon(-m) && \text{(BCCM-2)} \\ &= \upsilon n + (-\upsilon m) && \text{(v)} \\ &= \upsilon n - \upsilon m. \end{aligned}$$

(vii) Let $\upsilon_i \in \mathcal{U}$ and $n_i \in G$, $\forall i = 1, 2, 3, \ldots, k$. Then

$$\begin{aligned} \sum_{i=1}^{k} \upsilon_i n_i + \sum_{i=1}^{k} \upsilon_i(-n_i) &= (\upsilon_1 n_1 + \upsilon_2 n_2 + \cdots + \upsilon_k n_k) + (\upsilon_1(-n_1) + \upsilon_2(-n_2) + \cdots + \upsilon_k(-n_k)) \\ &= \upsilon_1 n_1 + \upsilon_2 n_2 + \cdots + \upsilon_k n_k + \upsilon_1(-n_1) + \upsilon_2(-n_2) + \cdots + \upsilon_k(-n_k) \\ &= (\upsilon_1 n_1 + \upsilon_1(-n_1)) + (\upsilon_2 n_2 + \upsilon_2(-n_2)) + \cdots + (\upsilon_k n_k + \upsilon_k(-n_k)) \end{aligned}$$

$$
\begin{aligned}
&= \upsilon_1(n_1 + (-n_1)) + \upsilon_2(n_2 + (-n_2)) + \cdots + \upsilon_k(n_k + (-n_k)) \\
&= \upsilon_1(0) + \upsilon_2(0) + \cdots + \upsilon_k(0) \\
&= 0 + 0 + \cdots + 0 \\
&= 0.
\end{aligned}
$$

Thus $\sum_{i=1}^{k} \upsilon_i(-n_i)$ is the inverse element of $\sum_{i=1}^{k} \upsilon_i n_i$, that is, $-(\sum_{i=1}^{k} \upsilon_i n_i) = \sum_{i=1}^{k} \upsilon_i(-n_i)$. □

Definition 2.6. A subgroup N of G is called a *BCC-submodule* of G if N is a BCC-module over $\mathfrak{U}$ under the same multiplication, which is defined on $\mathfrak{U}$ and G.

Theorem 2.7. *A nonempty subset A of G is a BCC-submodule if and only if $a - b, \upsilon a \in A$, $\forall \upsilon \in \mathfrak{U}$ and $a, b \in A$.*

Proof. Assume that A is a BCC-submodule of G. Let $\upsilon \in \mathfrak{U}$ and $a, b \in A$. Since $A = (A; +, 0)$ is a group, we have $a - b \in A$. Since A is a BCC-module over $\mathfrak{U}$, we have $\upsilon a \in A$.

Conversely, assume that $a - b, \upsilon a \in A$, $\forall \upsilon \in \mathfrak{U}$ and $a, b \in A$. Since $a - b \in A$, $\forall a, b \in A$, it follows from subgroup criterion that A is a subgroup of G. Since $\upsilon a \in A$, $\forall \upsilon \in \mathfrak{U}$ and $a \in A$, we have the map $(\upsilon, a) \mapsto \upsilon a \in A$ is well-defined. Since G satisfies (BCCM-1), (BCCM-2), and (BCCM-3), we have A satisfies those axioms. Hence, A is a BCC-module over $\mathfrak{U}$, that is, it is a BCC-submodule of G. □

3. Fuzzy sets in BCC-modules

In this section, we introduce the concept of FBCCSMs of BCC-modules and provides properties and finds the necessary and sufficient conditions for this concept.

A *fuzzy set* (FS) [37] in a nonempty set $\mathfrak{U}$ is defined to be a function $\mu : \mathfrak{U} \to [0, 1] \subseteq \mathbb{R}$. If a FS in $\mathfrak{U}$ is a constant function, we refer to it as being *constant*. We define $0_{\mathfrak{U}}$ and $1_{\mathfrak{U}}$ represent the constant FSs in $\mathfrak{U}$ that map every element of $\mathfrak{U}$ to 0 and every element of $\mathfrak{U}$ to 1, respectively.

Definition 3.1. A FS ϖ in G is called a *fuzzy BCC-submodule* (FBCCSM) of G if

$$(\forall n, m \in G)(\varpi(n + m) \geqslant \min\{\varpi(n), \varpi(m)\}), \tag{FBCCSM-1}$$
$$(\forall n \in G)(\varpi(-n) = \varpi(n)), \tag{FBCCSM-2}$$
$$(\forall \upsilon \in \mathfrak{U}, \forall n \in G)(\varpi(\upsilon n) \geqslant \varpi(n)). \tag{FBCCSM-3}$$

From now on, we define $F(G), FS(G)$, and $F(\mathfrak{U})$ as the set of all FSs and FBCCSMs of a BCC-module G over $\mathfrak{U}$, and the set of all FSs in $\mathfrak{U}$, respectively.

The following definition describes the binary relation $\leqslant$ on $F(G)$:

$$(\forall \varpi, \beta \in F(G))(\varpi \leqslant \beta \Leftrightarrow (\forall n \in G)(\varpi(n) \leqslant \beta(n))).$$

The definition of the binary relation $\leqslant$ on $F(\mathfrak{U})$ is the same as that of $F(G)$.

Example 3.2. Let $\mathfrak{U} = \{0, 1, 2, 3\}$ be a set that has the binary operations $\cdot$ and $+$, which are each described by the corresponding tables:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	1
3	0	0	0	0

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $\mathcal{U} = (\mathcal{U}; \cdot, 0)$ is a BCC-algebra and $\mathcal{U} = (\mathcal{U}; +, 0)$ is an Abelian group. Thus $\mathcal{U}$ is a BCC-module over itself with an operation defined by the following table:

	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Now, let $\wp_0, \wp_1 \in [0, 1]$ be such that $\wp_0 < \wp_1$. We define a FS ϖ on $\mathcal{U}$ as follows:

$$\varpi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \wp_1 & \wp_0 & \wp_0 & \wp_0 \end{pmatrix}.$$

Hence, ϖ is a FBCCSM of $\mathcal{U}$.

Theorem 3.3. *If $\varpi \in \mathsf{F}(G)$ satisfies (FBCCSM-3), then*

$$(\forall n \in G)(\varpi(0) \geqslant \varpi(n)). \tag{3.1}$$

Proof. Assume that ϖ satisfies (FBCCSM-3). By (BCCM-3), we have $\varpi(0) = \varpi(1n) \geqslant \varpi(n), \forall n \in G$. □

Theorem 3.4. *Let $\varpi \in \mathsf{F}(G)$. Then $\varpi \in \mathsf{FS}(G)$ if and only if it satisfies (FBCCSM-3) and*

$$(\forall n, m \in G)(\varpi(n - m) \geqslant \min\{\varpi(n), \varpi(m)\}). \tag{3.2}$$

Proof. Assume that $\varpi \in \mathsf{FS}(G)$. Then (FBCCSM-3) holds. Let $n, m \in G$. Then

$$\begin{aligned} \varpi(n - m) &= \varpi(n + (-m)) \\ &\geqslant \min\{\varpi(n), \varpi(-m)\} && \text{(FBCCSM-1)} \\ &= \min\{\varpi(n), \varpi(m)\}. && \text{(FBCCSM-2)} \end{aligned}$$

Conversely, assume that (FBCCSM-3) and (3.2) hold. Let $n \in G$. Then

$$\begin{aligned} \varpi(-n) &= \varpi(0 - n) && 0 - n = -n \\ &= \varpi(1n - n) && \text{(BCCM-3)} \\ &\geqslant \min\{\varpi(1n), \varpi(n)\} && (3.2) \\ &\geqslant \min\{\varpi(n), \varpi(n)\} && \text{(FBCCSM-3)} \\ &= \varpi(n) \end{aligned}$$

and

$$\begin{aligned} \varpi(n) &= \varpi(-(-n)) && -(-n) = n \\ &= \varpi(0 - (-n)) && 0 - n = -n \\ &= \varpi(1(-n) - (-n)) && \text{(BCCM-3)} \\ &\geqslant \min\{\varpi(1(-n)), \varpi(-n)\} && (3.2) \\ &\geqslant \min\{\varpi(-n), \varpi(-n)\} && \text{(FBCCSM-3)} \\ &= \varpi(-n). \end{aligned}$$

Thus $\varpi(-n) = \varpi(n)$, this means that (FBCCSM-2) holds. Let $n, m \in G$. Then

$$\begin{aligned} \varpi(n + m) &= \varpi(n - (-m)) && -(-n) = n \\ &\geqslant \min\{\varpi(n), \varpi(-m)\} && (3.2) \\ &= \min\{\varpi(n), \varpi(m)\}, && \text{(FBCCSM-2)} \end{aligned}$$

this means that (FBCCSM-1) holds. Hence, ϖ is a FBCCSM of G, that is, $\varpi \in \mathsf{FS}(G)$. □

Theorem 3.5. *Let* G *be unitary and* $\varpi \in \mathsf{F}(\mathsf{G})$. *Then* $\varpi \in \mathsf{FS}(\mathsf{G})$ *if and only if it satisfies* (3.1) *and*

$$(\forall \upsilon, \iota \in \mathcal{U}, \forall \mathfrak{n}, \mathfrak{m} \in \mathsf{G})(\varpi(\upsilon\mathfrak{n} - \iota\mathfrak{m}) \geqslant \min\{\varpi(\mathfrak{n}), \varpi(\mathfrak{m})\}). \tag{3.3}$$

Proof. Assume that $\varpi \in \mathsf{FS}(\mathsf{G})$. Let $\upsilon, \iota \in \mathcal{U}$ and $\mathfrak{n}, \mathfrak{m} \in \mathsf{G}$. Then

$$\begin{aligned} \varpi(0) &= \varpi(1\mathfrak{n}) && \text{(BCCM-3)} \\ &\geqslant \varpi(\mathfrak{n}) && \text{(FBCCSM-3)} \end{aligned}$$

and

$$\begin{aligned} \varpi(\upsilon\mathfrak{n} - \iota\mathfrak{m}) &\geqslant \min\{\varpi(\upsilon\mathfrak{n}), \varpi(\iota\mathfrak{m})\} && (3.2) \\ &\geqslant \min\{\varpi(\mathfrak{n}), \varpi(\mathfrak{m})\}. && \text{(FBCCSM-3)} \end{aligned}$$

Conversely, assume that (3.1) and (3.3) hold. Let $\upsilon, \iota \in \mathcal{U}$ and $\mathfrak{n}, \mathfrak{m} \in \mathsf{G}$. Then

$$\begin{aligned} \varpi(\upsilon\mathfrak{n}) &= \varpi(\upsilon\mathfrak{n} - 0) \\ &= \varpi(\upsilon\mathfrak{n} - 1(0)) && \text{(BCCM-3)} \\ &\geqslant \min\{\varpi(\mathfrak{n}), \varpi(0)\} && (3.3) \\ &= \varpi(\mathfrak{n}) && (3.1) \end{aligned}$$

and

$$\begin{aligned} \varpi(\mathfrak{n} - \mathfrak{m}) &= \varpi(0\mathfrak{n} - 0\mathfrak{m}) && \text{(Unitary)} \\ &\geqslant \min\{\varpi(\mathfrak{n}), \varpi(\mathfrak{m})\}, && (3.3) \end{aligned}$$

this means that (FBCCSM-3) and (3.2) hold. Hence, by Theorem 3.4, we have ϖ is a FBCCSM of G, that is, $\varpi \in \mathsf{FS}(\mathsf{G})$. □

Definition 3.6. Let $\varpi \in \mathsf{F}(\mathsf{G})$. For all $\wp \in [0,1]$, the set

$$\mathsf{U}(\varpi; \wp) = \{\mathfrak{n} \in \mathsf{G} \mid \varpi(\mathfrak{n}) \geqslant \wp\}$$

is called an *upper* $\wp$*-level subset* of ϖ.

Theorem 3.7. *Let* $\varpi \in \mathsf{F}(\mathsf{G})$. *Then* $\varpi \in \mathsf{FS}(\mathsf{G})$ *if and only if* $\forall \wp \in [0,1], \emptyset \neq \mathsf{U}(\varpi;\wp)$ *is a BCC-submodule of* G.

Proof. Assume that $\varpi \in \mathsf{FS}(\mathsf{G})$ and let $\wp \in [0,1]$ be such that $\mathsf{U}(\varpi;\wp)$ is nonempty. Let $\upsilon \in \mathcal{U}$ and $\mathfrak{n}, \mathfrak{m} \in \mathsf{U}(\varpi;\wp)$. Then $\varpi(\mathfrak{n}) \geqslant \wp$ and $\varpi(\mathfrak{m}) \geqslant \wp$. By (3.2) of Theorem 3.4, we have $\varpi(\mathfrak{n}-\mathfrak{m}) \geqslant \min\{\varpi(\mathfrak{n}), \varpi(\mathfrak{m})\} \geqslant \wp$. Thus $\mathfrak{n} - \mathfrak{m} \in \mathsf{U}(\varpi;\wp)$. By (FBCCSM-3), we have $\varpi(\upsilon\mathfrak{n}) \geqslant \varpi(\mathfrak{n})$. Thus $\upsilon\mathfrak{n} \in \mathsf{U}(\varpi;\wp)$. By Theorem 2.7, we have $\mathsf{U}(\varpi;\wp)$ is a BCC-submodule of G.

Conversely, assume that $\forall \wp \in [0,1], \emptyset \neq \mathsf{U}(\varpi;\wp)$ is a BCC-submodule of G. Let $\upsilon \in \mathcal{U}$ and $\mathfrak{n}, \mathfrak{m} \in \mathsf{G}$. Put $\wp = \min\{\varpi(\mathfrak{n}), \varpi(\mathfrak{m})\}$. Then $\varpi(\mathfrak{n}) \geqslant \wp$ and $\varpi(\mathfrak{m}) \geqslant \wp$, so $\mathfrak{n}, \mathfrak{m} \in \mathsf{U}(\varpi;\wp) \neq \emptyset$. By assumption, $\mathsf{U}(\varpi;\wp)$ is a BCC-submodule of G. Since $\mathsf{U}(\varpi;\wp)$ is a subgroup of G, we have $\mathfrak{n} - \mathfrak{m} \in \mathsf{U}(\varpi;\wp)$. Thus $\varpi(\mathfrak{n} - \mathfrak{m}) \geqslant \wp = \min\{\varpi(\mathfrak{n}), \varpi(\mathfrak{m})\}$, this means that (3.2) holds. Put $s = \varpi(\mathfrak{n})$. Then $\mathfrak{n} \in \mathsf{U}(\varpi;s) \neq \emptyset$. By assumption, $\mathsf{U}(\varpi;s)$ is a BCC-submodule of G. Since $\mathsf{U}(\varpi;s)$ is a BCC-submodule over $\mathcal{U}$, we have $\upsilon\mathfrak{n} \in \mathsf{U}(\varpi;s)$. Thus $\varpi(\upsilon\mathfrak{n}) \geqslant s = \varpi(\mathfrak{n})$, this means that (FBCCSM-3) holds. Hence, by Theorem 3.4, we have ϖ is a FBCCSM of G, that is, $\varpi \in \mathsf{FS}(\mathsf{G})$. □

4. Some properties of fuzzy sets in BCC-modules

In this section, we define FSs in BCC-modules of many forms, supplying their properties and their relation to FBCCSMs.

Definition 4.1. Let $k \in \mathbb{N}, \varpi, \varpi_i \in F(G)$, $\forall i \in \{1,2,\dots,k\}$, and $\upsilon \in \mathfrak{U}$. We define FSs $\sum_{i=1}^{k} \varpi_i, -\varpi$, and $\upsilon\varpi$ in G as follows:

$$\begin{aligned}
&(\forall n \in G)((\sum_{i=1}^{k} \varpi_i)(n) = \sup_{n=\sum_{i=1}^{k} a_i} \{\min_{i=1}^{k}\{\varpi_i(a_i)\}\}),\\
&(\forall n \in G)((-\varpi)(n) = \varpi(-n)),\\
&(\forall n \in G)((\upsilon\varpi)(n) = \sup_{n=\upsilon m} \{\varpi(m)\}).
\end{aligned}$$

If $\varpi_i = \varpi$, $\forall i \in \{1,2,\dots,k\}$, then $\sum_{i=1}^{k} \varpi_i$ is denoted by $k\varpi$.

Definition 4.2. For all $i \in \{1,2,\dots,k\}, \varpi_i \in F(G)$ is said to have the *same tip* if $\varpi_i(0) = \varpi_j(0)$, $\forall i,j \in \{1,2,\dots,k\}$.

Proposition 4.3. *Let* $\varpi_i, \varpi, \beta, \gamma \in F(G)$, $\forall i \in \{1,2,\dots,k\}$. *Then*

(i) $(1\varpi)(0) \geqslant \varpi(n)$, $\forall n \in G$;

(ii) *if* G *is unitary, then* $0\varpi = \varpi$;

(iii) *if* $\varpi \leqslant \beta$, *then* $\upsilon\varpi \leqslant \upsilon\beta$, $\forall \upsilon \in \mathfrak{U}$;

(iv) *if* G *is unitary and* $0\varpi \leqslant 0\beta$, *then* $\upsilon\varpi \leqslant \upsilon\beta$, $\forall \upsilon \in \mathfrak{U}$;

(v) $(\upsilon \wedge \iota)\varpi = \upsilon(\iota\varpi)$, $\forall \upsilon, \iota \in \mathfrak{U}$;

(vi) *if* $\varpi_i \leqslant \beta_i$, $\forall i \in \{1,2,\dots,k\}$, *then* $\sum_{i=1}^{k} \varpi_i \leqslant \sum_{i=1}^{k} \beta_i$;

(vii) $\varpi(n) \leqslant (\upsilon\varpi)(\upsilon n)$, $\forall \upsilon \in \mathfrak{U}$ *and* $n \in G$;

(viii) $(\forall \upsilon \in \mathfrak{U})(\varpi(n) \leqslant \gamma(\upsilon n)$, $\forall n \in G$ *if and only if* $\upsilon\varpi \leqslant \gamma)$;

(ix) *if* $\varpi_i \in FS(G)$ *and has the same tip*, $\forall i \in \{1,2,\dots,k\}$, *then* $\varpi_i \leqslant \sum_{i=1}^{k} \varpi_i$;

(x) $(\upsilon\varpi + \iota\beta)(\upsilon n + \iota m) \geqslant \min\{\varpi(n), \beta(m)\}$, $\forall \upsilon, \iota \in \mathfrak{U}$ *and* $n, m \in G$;

(xi) $(\upsilon\varpi + \iota\beta)(\upsilon n - \iota m) \geqslant \min\{\varpi(n), \beta(-m)\}$, $\forall \upsilon, \iota \in \mathfrak{U}$ *and* $n, m \in G$, *in particular, if* (FBCCSM-2) *holds, then* $(\upsilon\varpi + \iota\beta)(\upsilon n - \iota m) \geqslant \min\{\varpi(n), \beta(m)\}$, $\forall \upsilon, \iota \in \mathfrak{U}$ *and* $n, m \in G$;

(xii) *if* $\gamma \geqslant \upsilon\varpi + \iota\beta$ *for* $\upsilon, \iota \in \mathfrak{U}$, *then* $\gamma(\upsilon n - \iota m) \geqslant \min\{\varpi(n), \beta(-m)\}$, $\forall n, m \in G$, *in particular, if* (FBCCSM-2) *holds, then* $\gamma(\upsilon n - \iota m) \geqslant \min\{\varpi(n), \beta(m)\}$, $\forall n, m \in G$.

Proof.

(i) Let $n \in G$. Then

$$(1\varpi)(0) = \sup_{0=1n}\{\varpi(n)\} = \sup_{n\in G}\{\varpi(n)\} \geqslant \varpi(n).$$

Hence, $(1\varpi)(0) \geqslant \varpi(n)$, $\forall n \in G$.

(ii) Assume that G is unitary. Let $n \in G$. Then

$$\begin{aligned}(0\varpi)(n) &= \sup_{n=0m}\{\varpi(m)\}\\ &= \sup_{m=n}\{\varpi(m)\} && \text{(Unitary)}\\ &= \varpi(n).\end{aligned}$$

Hence, $0\varpi = \varpi$.

(iii) Assume that $\varpi \leqslant \beta$ and let $\upsilon \in \mathcal{U}$. Let $n \in G$. Then

$$(\upsilon\varpi)(n) = \sup_{n=\upsilon m}\{\varpi(m)\} \leqslant \sup_{n=\upsilon m}\{\beta(m)\} = (\upsilon\beta)(n).$$

Hence, $\upsilon\varpi \leqslant \upsilon\beta$.

(iv) Assume that G is unitary and $0\varpi \leqslant 0\beta$. By (ii), we have $\varpi \leqslant \beta$. By (iii), we have $\upsilon\varpi \leqslant \upsilon\beta$, $\forall \upsilon \in \mathcal{U}$.

(v) Let $\upsilon, \iota \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned}((\upsilon \wedge \iota)\varpi)(n) &= \sup_{n=(\upsilon\wedge\iota)m}\{\varpi(m)\}\\ &= \sup_{n=\upsilon(\iota m)}\{\varpi(m)\} && \text{(BCCM-1)}\\ &= \sup_{n=\upsilon k}\{\sup_{k=\iota m}\{\varpi(m)\}\} = \sup_{n=\upsilon k}\{(\iota\varpi)(k)\} = (\upsilon(\iota\varpi))(n).\end{aligned}$$

Hence, $(\upsilon \wedge \iota)\varpi = \upsilon(\iota\varpi)$.

(vi) Assume that $\varpi_i \leqslant \beta_i$, $\forall i \in \{1, 2, \ldots, k\}$. Let $n \in G$. Then

$$(\sum_{i=1}^{k}\varpi_i)(n) = \sup_{n=\sum_{i=1}^{k} a_i}\{\min_{i=1}^{k}\{\varpi_i(a_i)\}\} \leqslant \sup_{n=\sum_{i=1}^{k} a_i}\{\min_{i=1}^{k}\{\beta_i(a_i)\}\} = (\sum_{i=1}^{k}\beta_i)(n).$$

Hence, $\sum_{i=1}^{k}\varpi_i \leqslant \sum_{i=1}^{k}\beta_i$.

(vii) Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned}(\upsilon\varpi)(\upsilon n) &= \sup_{\upsilon n=\upsilon m}\{\varpi(m)\}\\ &= \sup_{\upsilon n=\upsilon m}\{\varpi(n)\} && n = m\\ &\geqslant \varpi(n).\end{aligned}$$

Hence, $\varpi(n) \leqslant (\upsilon\varpi)(\upsilon n)$, $\forall \upsilon \in \mathcal{U}$ and $n \in G$.

(viii) Let $\upsilon \in \mathcal{U}$ and $\varpi(n) \leqslant \gamma(\upsilon n)$, $\forall n \in G$. Let $n \in G$. Then

$$(\upsilon\varpi)(n) = \sup_{n=\upsilon m}\{\varpi(m)\} \leqslant \sup_{n=\upsilon m}\{\gamma(\upsilon m)\} = \gamma(n).$$

Hence, $\upsilon\varpi \leqslant \gamma$.

Conversely, let $\upsilon \in \mathcal{U}$ and $\upsilon\varpi \leqslant \gamma$. By (vii), we have $\varpi(n) \leqslant (\upsilon\varpi)(\upsilon n) \leqslant \gamma(\upsilon n)$, $\forall n \in G$.

(ix) Assume that $\varpi_i \in FS(G)$ and ϖ_i has the same tip, $\forall i \in \{1, 2, \ldots, k\}$. Let $i \in \{1, 2, \ldots, k\}$ and $n \in G$. Then

$$\begin{aligned}\varpi_i(n) &= \min\{\varpi_i(0), \varpi_i(n)\} && (3.1)\\ &= \min_{j=1, j\neq i}^{k}\{\varpi_j(0), \varpi_i(n)\} \leqslant \sup_{n=\sum_{i=1}^{k} a_i}\{\min_{i=1}^{k}\{\varpi_i(a_i)\}\} = (\sum_{i=1}^{k}\varpi_i)(n).\end{aligned}$$

Hence, $\varpi_i \leqslant \sum_{i=1}^{k}\varpi_i$.

(x) Let $\upsilon, \iota \in \mathcal{U}$ and $n, m \in G$. Then

$$\begin{aligned}(\upsilon\varpi + \iota\beta)(\upsilon n + \iota m) &= \sup_{\upsilon n + \iota m = a_1 + a_2} \{\min\{(\upsilon\varpi)(a_1), (\iota\beta)(a_2)\}\} \\ &\geqslant \min\{(\upsilon\varpi)(\upsilon n), (\iota\beta)(\iota m)\} \\ &\geqslant \min\{\varpi(n), \beta(m)\}. && \text{Proposition 4.3 (vii)}\end{aligned}$$

(xi) Let $\upsilon, \iota \in \mathcal{U}$ and $n, m \in G$. Then

$$\begin{aligned}(\upsilon\varpi + \iota\beta)(\upsilon n - \iota m) &= (\upsilon\varpi + \iota\beta)(\upsilon n + (-\iota m)) \\ &= (\upsilon\varpi + \iota\beta)(\upsilon n + \iota(-m)) && \text{Proposition 2.5 (v)} \\ &\geqslant \min\{\varpi(n), \beta(-m)\}. && \text{Proposition 4.3 (x)}\end{aligned}$$

In particular, if (FBCCSM-2) holds, then $(\upsilon\varpi + \iota\beta)(\upsilon n - \iota m) \geqslant \min\{\varpi(n), \beta(m)\}$, $\forall \upsilon, \iota \in \mathcal{U}$ and $n, m \in G$.
(xii) It is straightforward by (xi). □

Theorem 4.4. *If* $\varpi \in FS(G)$, *then it satisfies* (FBCCSM-2) *and*

$$(\forall \upsilon \in \mathcal{U})(\upsilon\varpi \leqslant \varpi), \tag{4.1}$$

$$\varpi + \varpi \leqslant \varpi. \tag{4.2}$$

Proof. Assume that $\varpi \in FS(G)$. Then (FBCCSM-2) holds. Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned}(\upsilon\varpi)(n) &= \sup_{n=\upsilon m} \{\varpi(m)\} \\ &\leqslant \sup_{n=\upsilon m} \{\varpi(\upsilon m)\} && \text{(FBCCSM-3)} \\ &= \varpi(n)\end{aligned}$$

and

$$\begin{aligned}(\varpi + \varpi)(n) &= \sup_{n=a+b} \{\min\{\varpi(a), \varpi(b)\}\} \\ &\leqslant \sup_{n=a+b} \{\varpi(a+b)\} && \text{(FBCCSM-1)} \\ &= \varpi(n),\end{aligned}$$

this means that $\upsilon\varpi \leqslant \varpi$ and $\varpi + \varpi \leqslant \varpi$. □

Corollary 4.5. *If* $k \in \mathbb{N}$ *and* $\varpi \in FS(G)$, *then*

$$k\varpi \leqslant \varpi. \tag{4.3}$$

Proof. Assume that $k \in \mathbb{N}$ and $\varpi \in FS(G)$. Let $n = \sum_{i=1}^{k} n_i \in G$. By (FBCCSM-1), we have

$$\varpi(n) = \varpi(\sum_{i=1}^{k} n_i) \geqslant \min\{\varpi(n_1), \varpi(n_2), \ldots, \varpi(n_k)\}.$$

Thus $\varpi(n)$ is an upper bound of $\{\min\{\varpi(n_1), \varpi(n_2), \ldots, \varpi(n_k)\} \mid n = \sum_{i=1}^{k} n_i\}$. That is,

$$\varpi(n) \geqslant \sup_{n=\sum_{i=1}^{k} n_i} \{\min\{\varpi(n_1), \varpi(n_2), \ldots, \varpi(n_k)\}\} = (k\varpi)(n).$$

Therefore, $k\varpi \leqslant \varpi$. □

Theorem 4.6. *Let* G *be unitary and* $\varpi \in F(G)$. *If* ϖ *satisfies* (FBCCSM-2), (4.1), *and* (4.2), *then* $\varpi \in FS(G)$.

Proof. Assume that ϖ satisfies (FBCCSM-2), (4.1), and (4.2). By (4.1) and using Proposition 4.3 (viii), we have (FBCCSM-3) holds. Let $n_1, n_2 \in G$. Then

$$\begin{aligned}
\varpi(n_1 - n_2) &\geqslant (\varpi + \varpi)(n_1 - n_2) && (4.2)\\
&= (0\varpi + 0\varpi)(0n_1 - 0n_2) && \text{Proposition 4.3 (ii), (Unitary)}\\
&\geqslant \min\{\varpi(n_1), \varpi(n_2)\}, && \text{Proposition 4.3 (xi)}
\end{aligned}$$

this means that (3.2) holds. By Theorem 3.4, we have ϖ is a FBCCSM of G, that is, $\varpi \in FS(G)$. □

Theorem 4.7. *Let* G *be unitary and* $\varpi \in F(G)$. *If* ϖ *satisfies* (3.1), (FBCCSM-2), *and*

$$(\forall \upsilon, \iota \in \mathcal{U})(\upsilon\varpi + \iota\varpi \leqslant \varpi), \tag{4.4}$$

then $\varpi \in FS(G)$.

Proof. Assume that ϖ satisfies (3.1), (FBCCSM-2), and (4.4). Let $\upsilon, \iota \in \mathcal{U}$ and $n, m \in G$. Then

$$\begin{aligned}
\varpi(\upsilon n - \iota m) &\geqslant (\upsilon\varpi + \iota\varpi)(\upsilon n - \iota m) && (4.4)\\
&\geqslant \min\{\varpi(n), \varpi(m)\}, && \text{Proposition 4.3 (xi)}
\end{aligned}$$

this means that (3.3) holds. By Theorem 3.5, we have ϖ is a FBCCSM of G, that is, $\varpi \in FS(G)$. □

Theorem 4.8. *Let* $\varpi \in FS(G)$. *Then*

(i) $-\varpi \in FS(G)$;

(ii) *if* G *is unitary,* $\mathcal{U}$ *is meet-commutative, and* $\upsilon\varpi$ *satisfies* (4.2) *for* $\upsilon \in \mathcal{U}$, *that is,* $\upsilon\varpi + \upsilon\varpi \leqslant \upsilon\varpi$, *then* $\upsilon\varpi \in FS(G)$.

Proof.

(i) Let $\upsilon \in \mathcal{U}$ and $n, m \in G$. Then

$$\begin{aligned}
(-\varpi)(n + m) &= \varpi(-(n + m))\\
&= \varpi((-n) + (-m))\\
&\geqslant \min\{\varpi(-n), \varpi(-m)\} && \text{(FBCCSM-1)}\\
&\geqslant \min\{(-\varpi)(n), (-\varpi)(m)\},\\
(-\varpi)(-n) &= \varpi(-(-n))\\
&= \varpi(-n) && \text{(FBCCSM-2)}\\
&= (-\varpi)(n),
\end{aligned}$$

and

$$\begin{aligned}
(-\varpi)(\upsilon n) &= \varpi(-(\upsilon n))\\
&= \varpi(\upsilon(-n)) && \text{Proposition 2.5 (v)}\\
&\geqslant \varpi(-n) && \text{(FBCCSM-3)}\\
&= (-\varpi)(n),
\end{aligned}$$

this means that (FBCCSM-1), (FBCCSM-2), and (FBCCSM-3) hold. Hence, $-\varpi \in FS(G)$.

(ii) Assume that G is unitary, $\mathcal{U}$ is meet-commutative, and $\upsilon\varpi$ satisfies (4.2) for $\upsilon \in \mathcal{U}$. Let $\iota, \sigma \in \mathcal{U}$ and $n \in G$. Since $\varpi \in FS(G)$, it follows from (4.1) that $\iota\varpi \leqslant \varpi$ and $\sigma\varpi \leqslant \varpi$. By Proposition 4.3 (iii) and (vi), we have

$$\upsilon(\iota\varpi) + \upsilon(\sigma\varpi) \leqslant \upsilon\varpi + \upsilon\varpi. \tag{4.5}$$

Thus

$$\begin{aligned}
\iota(\upsilon\varpi) + \sigma(\upsilon\varpi) &= (\iota \wedge \upsilon)\varpi + (\sigma \wedge \upsilon)\varpi && \text{Proposition 4.3 (v)}\\
&= (\upsilon \wedge \iota)\varpi + (\upsilon \wedge \sigma)\varpi && \text{(Meet-commutative)}\\
&= \upsilon(\iota\varpi) + \upsilon(\sigma\varpi) && \text{Proposition 4.3 (v)}\\
&\leqslant \upsilon\varpi + \upsilon\varpi && (4.5)\\
&\leqslant \upsilon\varpi, && (4.2)\\
(\upsilon\varpi)(-n) &= \sup_{-n=\upsilon m} \{\varpi(m)\}\\
&= \sup_{n=-\upsilon m} \{\varpi(m)\} && -(-n) = n\\
&= \sup_{n=\upsilon(-m)} \{\varpi(m)\} && \text{Proposition 2.5 (v)}\\
&= \sup_{n=\upsilon(-m)} \{\varpi(-m)\} && \text{(FBCCSM-2)}\\
&= (\upsilon\varpi)(n),
\end{aligned}$$

and

$$\begin{aligned}
(\upsilon\varpi)(0) &= \sup_{0=\upsilon m} \{\varpi(m)\}\\
&\geqslant \varpi(0) && \text{Proposition 2.5 (ii)}\\
&\geqslant \sup_{n=\upsilon m} \{\varpi(m)\} && (3.1)\\
&= (\upsilon\varpi)(n),
\end{aligned}$$

this means that (4.4), (FBCCSM-2), and (3.1) hold. Hence, by Theorem 4.7, we have $\upsilon\varpi \in FS(G)$. □

Definition 4.9. Let $\{\varpi_i \mid i \in \Lambda\} \subseteq F(G)$. We define FSs $\bigcap_{i\in\Lambda} \varpi_i$ and $\bigcup_{i\in\Lambda} \varpi_i$ in G as follows:

$$(\forall n \in G)((\bigcap_{i\in\Lambda} \varpi_i)(n) = \inf_{i\in\Lambda}\{\varpi_i(n)\}), \qquad (\forall n \in G)((\bigcup_{i\in\Lambda} \varpi_i)(n) = \sup_{i\in\Lambda}\{\varpi_i(n)\}).$$

Lemma 4.10. *Let* $\beta, \varpi_i \in F(G)$. *Then*

(i) *if* $\beta \leqslant \varpi_i$, $\forall i \in \Lambda$, *then* $\beta \leqslant \bigcap_{i\in\Lambda} \varpi_i$;

(ii) *if* $\varpi_i \leqslant \beta$, $\forall i \in \Lambda$, *then* $\bigcup_{i\in\Lambda} \varpi_i \leqslant \beta$.

Proof.

(i) Assume that $\beta, \varpi_i \in F(G)$ such that $\beta \leqslant \varpi_i$, $\forall i \in \Lambda$. Let $n \in G$. Then $\beta(n) \leqslant \varpi_i(n)$, $\forall i \in \Lambda$. Thus $\beta(n)$ is a lower bound of $\{\varpi_i(n) \mid i \in \Lambda\}$. Hence, $\beta(n) \leqslant \inf_{i\in\Lambda}\{\varpi_i(n)\}$, that is, $\beta \leqslant \bigcap_{i\in\Lambda} \varpi_i$.

(ii) Assume that $\beta, \varpi_i \in F(G)$ such that $\varpi_i \leqslant \beta$, $\forall i \in \Lambda$. Let $n \in G$. Then $\varpi_i(n) \leqslant \beta(n)$, $\forall i \in \Lambda$. Thus $\beta(n)$ is an upper bound of $\{\varpi_i(n) \mid i \in \Lambda\}$. Hence, $\sup_{i\in\Lambda}\{\varpi_i(n)\} \leqslant \beta(n)$, that is, $\bigcup_{i\in\Lambda} \varpi_i \leqslant \beta$. □

We may immediately establish the following theorem using Lemma 4.10 (i) and (ii).

Theorem 4.11. $(\mathrm{F}(\mathrm{G}), \cup, \cap)$ *is a complete lattice.*

Theorem 4.12. *If* $\varpi_i \in \mathrm{FS}(\mathrm{G})$, $\forall i \in \Lambda$, *then* $\bigcap_{i\in\Lambda} \varpi_i \in \mathrm{FS}(\mathrm{G})$.

Proof. Let $\upsilon \in \mathcal{U}$ and $n, m \in G$. Then

$$\begin{aligned}
(\bigcap_{i\in\Lambda} \varpi_i)(n+m) &= \inf_{i\in\Lambda}\{\varpi_i(n+m)\} \\
&\geqslant \inf_{i\in\Lambda}\{\min\{\varpi_i(n), \varpi_i(m)\}\} && \text{(FBCCSM-1)} \\
&= \min\{\inf_{i\in\Lambda}\{\varpi_i(n)\}, \inf_{i\in\Lambda}\{\varpi_i(m)\}\} \\
&= \min\{(\bigcap_{i\in\Lambda} \varpi_i)(n), (\bigcap_{i\in\Lambda} \varpi_i)(m)\}, \\
(\bigcap_{i\in\Lambda} \varpi_i)(-n) &= \inf_{i\in\Lambda}\{\varpi_i(-n)\} \\
&= \inf_{i\in\Lambda}\{\varpi_i(n)\} && \text{(FBCCSM-2)} \\
&= (\bigcap_{i\in\Lambda} \varpi_i)(n),
\end{aligned}$$

and

$$\begin{aligned}
(\bigcap_{i\in\Lambda} \varpi_i)(\upsilon n) &= \inf_{i\in\Lambda}\{\varpi_i(\upsilon n)\} \\
&\geqslant \inf_{i\in\Lambda}\{\varpi_i(n)\} && \text{(FBCCSM-3)} \\
&= (\bigcap_{i\in\Lambda} \varpi_i)(n),
\end{aligned}$$

this means that (FBCCSM-1), (FBCCSM-2), and (FBCCSM-3) hold. Hence, $\bigcap_{i\in\Lambda} \varpi_i$ is a FBCCSM of G, that is, $\bigcap_{i\in\Lambda} \varpi_i \in \mathrm{FS}(\mathrm{G})$. □

Theorem 4.13. *If* $\varpi_i \in \mathrm{F}(\mathrm{G})$ *satisfies* (FBCCSM-2) *and* (FBCCSM-3), $\forall i \in \Lambda$, *then* $\bigcup_{i\in\Lambda} \varpi_i$ *satisfies* (FBCCSM-2) *and* (FBCCSM-3), *respectively.*

Proof. Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned}
(\bigcup_{i\in\Lambda} \varpi_i)(-n) &= \sup_{i\in\Lambda}\{\varpi_i(-n)\} \\
&= \sup_{i\in\Lambda}\{\varpi_i(n)\} && \text{(FBCCSM-2)} \\
&= (\bigcup_{i\in\Lambda} \varpi_i)(n)
\end{aligned}$$

and

$$\begin{aligned}
(\bigcup_{i\in\Lambda} \varpi_i)(\upsilon n) &= \sup_{i\in\Lambda}\{\varpi_i(\upsilon n)\} \\
&\geqslant \sup_{i\in\Lambda}\{\varpi_i(n)\} && \text{(FBCCSM-3)} \\
&= (\bigcup_{i\in\Lambda} \varpi_i)(n),
\end{aligned}$$

this means that (FBCCSM-2) and (FBCCSM-3) hold. □

Definition 4.14. Let Y be a subset of a set $\mathcal{U}$. The characteristic function of Y is defined as follows:

$$(\forall \upsilon \in \mathcal{U}) \left(\chi_Y(\upsilon) = \begin{cases} 1, & \text{if } \upsilon \in Y, \\ 0, & \text{otherwise,} \end{cases} \right).$$

In particular, $\chi_\emptyset = 0_\mathcal{U}$ and $\chi_\mathcal{U} = 1_\mathcal{U}$.

Definition 4.15. Let ι be an element of a set $\mathcal{U}$ and $\wp \in [0,1]$. The fuzzy point $\iota_\wp$ in $\mathcal{U}$ is defined as follows:

$$(\forall \upsilon \in \mathcal{U}) \left(\iota_\wp(\upsilon) = \begin{cases} \wp, & \text{if } \upsilon = \iota, \\ 0, & \text{otherwise,} \end{cases} \right).$$

Note 4.16. Let $\mathcal{U}$ be a set, $\iota \in \mathcal{U}$, and $s, \wp \in [0,1]$. Then

(i) $\iota_0 = 0_\mathcal{U}$;

(ii) if $s \leqslant \wp$, then $\iota_s \leqslant \iota_\wp$.

5. Fuzzy BCC-submodule generated by a set

In this section, we define and study the FBCCSM generated by a set of FSs in BCC-modules, as well as provide for their properties and their relation to FBCCSMs.

Definition 5.1. Let $A \subseteq F(G)$. The intersection of all FBCCSMs of G greater than all FSs in A is called the *FBCCSM generated by* A, denoted by $\langle A \rangle$. By Theorem 4.12, we get $\langle A \rangle$ is the least FBCCSM of G greater than all FSs in A. If $A = \{\varpi_1, \varpi_2, \dots, \varpi_k\}$, then we write $\langle A \rangle = \langle \varpi_1, \varpi_2, \dots, \varpi_k \rangle$. If A is finite and $\varpi = \langle A \rangle$, then we say that ϖ is finitely generated. In particular, if $\varpi = \langle \varpi_\wp \rangle$, then we say that ϖ is *cyclic*.

Note 5.2.

(i) $\langle \emptyset \rangle = 0_G$;

(ii) $\langle F(G) \rangle = 1_G$;

(iii) if $\varpi \in FS(G)$, then $\langle \varpi \rangle = \varpi$.

Definition 5.3. Let $N \subseteq G$. We define a subset $[N]$ of G as follows:

$$[N] = \{n \in G \mid n = \upsilon m \text{ for some } \upsilon \in \mathcal{U} \text{ and } m \in N\}.$$

Lemma 5.4. *Let* $N \subseteq G$. *Then*

(i) *if* G *is unitary, then* $N \subseteq [N]$;

(ii) *if* N *is a BCC-submodule of* G, *then* $[N] \subseteq N$;

(iii) *if* G *is unitary and* N *is a BCC-submodule of* G, *then* $N = [N]$.

Proof.
(i) It follows from (Unitary).
(ii) It follows from Theorem 2.7.
(iii) It is a direct result of (i) and (ii). □

Lemma 5.5. *Let* $\varpi, \beta \in F(G)$. *Then*

(i) *if* ϖ *satisfies* (FBCCSM-1), *then* $U(\varpi; s) + U(\varpi; \wp) \subseteq U(\varpi; \min\{s, \wp\})$, $\forall s, \wp \in [0,1]$;

(ii) *if ϖ satisfies* (FBCCSM-1) *and* (FBCCSM-3), *then* $[U(\varpi;s)] + [U(\varpi;\wp)] \subseteq U(\varpi;\min\{s,\wp\})$, $\forall s,\wp \in [0,1]$;

(iii) *if* $\varpi \leqslant \beta$, *then* $U(\varpi;\wp) \subseteq U(\beta;\wp)$, $\forall \wp \in [0,1]$;

(iv) *if* $\varpi \leqslant \beta$ *and* β *satisfies* (FBCCSM-3), *then* $[U(\varpi;\wp)] \subseteq U(\beta;\wp)$, $\forall \wp \in [0,1]$.

Proof.

(i) Assume that ϖ satisfies (FBCCSM-1). Let $s,\wp \in [0,1]$ and let $n \in U(\varpi;s) + U(\varpi;\wp)$. Then $n = n_s + n_\wp$ for some $n_s \in U(\varpi;s)$ and $n_\wp \in U(\varpi;\wp)$. Thus $\varpi(n_s) \geqslant s$ and $\varpi(n_\wp) \geqslant \wp$. By (FBCCSM-1), we have $\varpi(n) = \varpi(n_s + n_\wp) \geqslant \min\{\varpi(n_s), \varpi(n_\wp)\} \geqslant \min\{s,\wp\}$. Thus $n \in U(\varpi;\min\{s,\wp\})$, so $U(\varpi;s) + U(\varpi;\wp) \subseteq U(\varpi;\min\{s,\wp\})$.

(ii) Assume that ϖ satisfies (FBCCSM-1) and (FBCCSM-3). Let $s,\wp \in [0,1]$ and let $n \in [U(\varpi;s)] + [U(\varpi;\wp)]$. Then $n = \upsilon_s n_s + \upsilon_\wp n_\wp$ for some $\upsilon_s, \upsilon_\wp \in \mathcal{U}, n_s \in U(\varpi;s)$, and $n_\wp \in U(\varpi;\wp)$. Thus $\varpi(n_s) \geqslant s$ and $\varpi(n_\wp) \geqslant \wp$. By (FBCCSM-1) and (FBCCSM-3), we have $\varpi(n) = \varpi(\upsilon_s n_s + \upsilon_\wp n_\wp) \geqslant \min\{\varpi(\upsilon_s n_s), \varpi(\upsilon_\wp n_\wp)\} \geqslant \min\{\varpi(n_s), \varpi(n_\wp)\} \geqslant \min\{s,\wp\}$. Thus $n \in U(\varpi;\min\{s,\wp\})$, so $[U(\varpi;s)] + [U(\varpi;\wp)] \subseteq U(\varpi;\min\{s,\wp\})$.

(iii) Assume that $\varpi \leqslant \beta$. Let $\wp \in [0,1]$ and let $n \in U(\varpi;\wp)$. Then $\varpi(n) \geqslant \wp$. Thus $\beta(n) \geqslant \varpi(n) \geqslant \wp$. Hence, $n \in U(\beta;\wp)$, so $U(\varpi;\wp) \subseteq U(\beta;\wp)$.

(iv) Assume that $\varpi \leqslant \beta$ and β satisfies (FBCCSM-3). Let $\wp \in [0,1]$ and let $n \in [U(\varpi;\wp)]$. Then $n = \upsilon m$ for some $\upsilon \in \mathcal{U}$ and $m \in U(\varpi;\wp)$, that is, $\varpi(m) \geqslant \wp$. By assumption and (FBCCSM-3), we have $\beta(n) = \beta(\upsilon m) \geqslant \beta(m) \geqslant \varpi(m) \geqslant \wp$. Thus $n \in U(\beta;\wp)$, so $[U(\varpi;\wp)] \subseteq U(\beta;\wp)$. □

Corollary 5.6. *Let* G *be unitary and* $\varpi, \beta \in F(G)$. *Then*

(i) *if ϖ satisfies* (FBCCSM-1), *then* $U(\varpi;s) + U(\varpi;\wp) \subseteq U(\varpi;\min\{s,\wp\}) \subseteq [U(\varpi;\min\{s,\wp\})]$, $\forall s,\wp \in [0,1]$;

(ii) *if ϖ satisfies* (FBCCSM-1) *and* (FBCCSM-3), *then*

$$U(\varpi;s) + U(\varpi;\wp) \subseteq [U(\varpi;s)] + [U(\varpi;\wp)] \subseteq U(\varpi;\min\{s,\wp\}) \subseteq [U(\varpi;\min\{s,\wp\})],\ \forall s,\wp \in [0,1];$$

(iii) *if* $\varpi \leqslant \beta$, *then* $U(\varpi;\wp) \subseteq U(\beta;\wp) \subseteq [U(\beta;\wp)]$, $\forall \wp \in [0,1]$;

(iv) *if* $\varpi \leqslant \beta$ *and* β *satisfies* (FBCCSM-3), *then* $U(\varpi;\wp) \subseteq [U(\varpi;\wp)] \subseteq U(\beta;\wp) \subseteq [U(\beta;\wp)]$, $\forall \wp \in [0,1]$.

Proof. By Lemmas Lemmas 5.4 (i) and 5.5, it is simple. □

Lemma 5.7. *If* $\varpi \in F(G)$ *satisfies* (FBCCSM-3), *then*

$$(\forall \upsilon \in \mathcal{U})(\sup\{\wp \in [0,1] \mid \upsilon n \in U(\varpi;\wp)\} \geqslant \sup\{\wp \in [0,1] \mid n \in U(\varpi;\wp)\}). \tag{5.1}$$

Proof. Assume that ϖ satisfies (FBCCSM-3) and let $\upsilon \in \mathcal{U}$. Let $\wp_0 \in \{\wp \in [0,1] \mid n \in U(\varpi;\wp)\}$. Then $\varpi(n) \geqslant \wp_0$. By (FBCCSM-3), we have $\varpi(\upsilon n) \geqslant \varpi(n) \geqslant \wp_0$. Thus $\wp_0 \in \{\wp \in [0,1] \mid \upsilon n \in U(\varpi;\wp)\}$, that is, $\{\wp \in [0,1] \mid n \in U(\varpi;\wp)\} \subseteq \{\wp \in [0,1] \mid \upsilon n \in U(\varpi;\wp)\}$. Hence, $\sup\{\wp \in [0,1] \mid \upsilon n \in U(\varpi;\wp)\} \geqslant \sup\{\wp \in [0,1] \mid n \in U(\varpi;\wp)\}$. □

Definition 5.8. Let $f \in F(\mathcal{U})$ and $\beta \in F(G)$. The composition $f \circ \beta$ and the product $f\beta$ of f and β are defined as follows:

$$(\forall n \in G)((f \circ \beta)(n) = \sup_{n=\upsilon m} \{\min\{f(\upsilon), \beta(m)\}\}),$$
$$(\forall n \in G)((f\beta)(n) = \sup_{n=\sum_{i=1}^{k} \upsilon_i n_i} \{\min\{f(\upsilon_1), \dots, f(\upsilon_k), \beta(n_1), \dots, \beta(n_k)\}\}).$$

From Definition 5.8, we know that

$$(\forall f \in F(\mathcal{U}), \forall \beta \in F(G))(f \circ \beta \leqslant f\beta). \tag{5.2}$$

Theorem 5.9. *Let* $A = \{\varpi_i \mid i \in I\} \subseteq FS(G)$. *Then*

(i) *if* $\bigcup_{i\in I} \varpi_i$ *satisfies* (FBCCSM-1), *then*

$$(\forall n \in G)(\langle A\rangle(n) = \sup\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\});$$

(ii) *if* G *is separability, then* $a_\wp = 1_{\mathcal{U}} \circ a_\wp$, $\forall a \in G$ *and* $\wp \in [0,1]$;

(iii) $\langle 0_\wp \rangle = 0_\wp$, $\forall \wp \in [0,1]$;

(iv) *if* G *is separability, then* $\bigcup_{a_\wp \leqslant \varpi}(1_{\mathcal{U}} \circ a_\wp) \leqslant \varpi$, $\forall \varpi \in F(G)$.

Proof.

(i) Let λ be a FS in G defined by

$$(\forall n \in G)(\lambda(n) = \sup\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\}).$$

We shall show that λ is a FBCCSM of G. By Theorem 4.13, we have $\bigcup_{i\in I} \varpi_i$ satisfies (FBCCSM-2) and (FBCCSM-3). Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned}\lambda(\upsilon n) &= \sup\{\wp \in [0,1] \mid \upsilon n \in U(\bigcup_{i\in I} \varpi_i; \wp)\}\\ &\geqslant \sup\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\} && (5.1)\\ &= \lambda(n),\end{aligned}$$

this means that (FBCCSM-3) holds. Let $n_1, n_2 \in G$ and let $\min\{\lambda(n_1), \lambda(n_2)\} = e \in [0,1]$ and $s = e - \frac{1}{m}$, where m is the least positive integer such that $m > \frac{1}{e}$ (by Archimedean property and well-ordering principle). Then $\lambda(n_1) \geqslant e > s$ and $\lambda(n_2) \geqslant e > s$.

We shall show that $n_1 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_1))$ and $n_2 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_2))$. Since $\lambda(n_1) = \sup\{\wp \in [0,1] \mid n_1 \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$, we have $(\bigcup_{i\in I} \varpi_i)(n_1) \geqslant \wp$, $\forall \wp \in \{\wp \in [0,1] \mid n_1 \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$. Thus $(\bigcup_{i\in I} \varpi_i)(n_1)$ is an upper bound of $\{\wp \in [0,1] \mid n_1 \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$, so $(\bigcup_{i\in I} \varpi_i)(n_1) \geqslant \sup\{\wp \in [0,1] \mid n_1 \in U(\bigcup_{i\in I} \varpi_i; \wp)\} = \lambda(n_1)$. Hence, $n_1 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_1))$. Likewise, we can prove that $n_2 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_2))$.

We shall show that $-n_2 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_2))$. Since $n_2 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_2))$, we have $(\bigcup_{i\in I} \varpi_i)(n_2) \geqslant \lambda(n_2)$. Since $\bigcup_{i\in I} \varpi_i$ satisfies (FBCCSM-2), we have $(\bigcup_{i\in I} \varpi_i)(-n_2) = (\bigcup_{i\in I} \varpi_i)(n_2) \geqslant \lambda(n_2)$. Thus $-n_2 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_2))$.

Next, we shall show that $n_1 - n_2 \in U(\bigcup_{i\in I} \varpi_i; \min\{\lambda(n_1), \lambda(n_2)\}$. Since $n_1 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_1)$ and $n_2 \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_2)$ and $\bigcup_{i\in I} \varpi_i$ satisfies (FBCCSM-1), it follows from Lemma 5.5 (i) that

$$n_1 - n_2 = n_1 + (-n_2) \in U(\bigcup_{i\in I} \varpi_i; \lambda(n_1)) + U(\bigcup_{i\in I} \varpi_i; \lambda(n_2)) \subseteq U(\bigcup_{i\in I} \varpi_i; \min\{\lambda(n_1), \lambda(n_2)\}) = U(\bigcup_{i\in I} \varpi_i; e).$$

This shows that $e \in \{\wp \in [0,1] \mid n_1 - m_2 \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$. Thus $\lambda(n_1 - n_2) = \sup\{\wp \in [0,1] \mid n_1 - m_2 \in U(\bigcup_{i\in I} \varpi_i; \wp)\} \geqslant e = \min\{\lambda(n_1), \lambda(n_2)\}$, this means that (3.2) holds. By Theorem 3.4, we have λ is a FBCCSM of G.

Let $n \in G$ and $\varpi_i(n) = \wp_i$ for some $\wp_i \in [0,1]$. Since $\varpi_i \leqslant \bigcup_{i\in I} \varpi_i$, $\forall i \in I$, it follows from Lemma 5.5 (iii) that

$$n \in U(\varpi_i; \wp_i) \subseteq U(\bigcup_{i\in I} \varpi_i; \wp_i).$$

Thus $\wp_i \in \{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$, so $\lambda(n) = \sup\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\} \geqslant \wp_i = \varpi_i(n)$, $\forall i \in I$. This shows that $\varpi_i \leqslant \lambda$, $\forall i \in I$. Hence, λ is a FBCCSM of G greater than ϖ_i, $\forall i \in I$.

Let γ be a FBCCSM of G greater than ϖ_i, $\forall i \in I$. Then $\gamma(n) \geqslant \sup\{\varpi_i(n) \mid i \in I\} = (\bigcup_{i\in I} \varpi_i)(n)$, $\forall n \in G$. Since $\lambda(n) = \sup\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$, we have $\gamma(n) \geqslant (\bigcup_{i\in I} \varpi_i)(n) \geqslant \wp$, $\forall \wp \in \{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$. Thus $\gamma(n)$ is an upper bound of $\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\}$. Hence, $\gamma(n) \geqslant \sup\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\} = \lambda(n)$, this means that $\lambda \leqslant \gamma$. Therefore, λ is the least FBCCSM of G greater than ϖ_i, $\forall i \in I$, that is,

$$(\forall n \in G)(\langle A\rangle(n) = \sup\{\wp \in [0,1] \mid n \in U(\bigcup_{i\in I} \varpi_i; \wp)\}).$$

(ii) Assume that G is separability. Let $a, n \in G$ and $\wp \in [0,1]$. Then

$$\begin{aligned}(1_{\mathcal{U}} \circ a_\wp)(n) &= \sup_{n=\upsilon n}\{\min\{1_{\mathcal{U}}(\upsilon), a_\wp(n)\}\} && \text{(Separability)}\\ &= \sup_{n=\upsilon n}\{\min\{1, a_\wp(n)\}\} = \sup_{n=\upsilon n}\{a_\wp(n)\} = a_\wp(n).\end{aligned}$$

Hence, $a_\wp = 1_{\mathcal{U}} \circ a_\wp$.

(iii) Let $\wp \in [0,1]$. Then we can show that $0_\wp$ satisfies (FBCCSM-1). Let $n \in G$. If $n \neq 0$, then

$$\begin{aligned}\langle 0_\wp\rangle(n) &= \sup\{s \in [0,1] \mid n \in U(0_\wp; s)\} && \text{(i)}\\ &= \sup\{s \in [0,1] \mid 0_\wp(n) \geqslant s\} = \sup\{s \in [0,1] \mid 0 \geqslant s\} = \sup\{s \in [0,1] \mid s = 0\} = 0 = 0_\wp(n).\end{aligned}$$

If $n = 0$, then

$$\begin{aligned}\langle 0_\wp\rangle(0) &= \sup\{s \in [0,1] \mid 0 \in U(0_\wp; s)\} && \text{(i)}\\ &= \sup\{s \in [0,1] \mid 0_\wp(0) \geqslant s\} = \sup\{s \in [0,1] \mid \wp \geqslant s\} = \wp = 0_\wp(0).\end{aligned}$$

Hence, $\langle 0_\wp\rangle = 0_\wp$.

(iv) Assume that G is separability and $\varpi \in F(G)$. Let $a \in G$ and $\wp \in [0,1]$ be such that $a_\wp \leqslant \varpi$. By (ii), we have $1_{\mathcal{U}} \circ a_\wp = a_\wp \leqslant \varpi$. Hence, $\bigcup_{a_\wp \leqslant \varpi}(1_{\mathcal{U}} \circ a_\wp) \leqslant \varpi$. □

Theorem 5.10. *Let $A = \{\varpi_i \mid i \in \{1,2,\dots,k\}\} \subseteq FS(G)$ with the same tip. Then $\bigcup_{i=1}^k \varpi_i \leqslant \sum_{i=1}^k \varpi_i$. Moreover, if $\sum_{i=1}^k \varpi_i$ is a FBCCSM of G, then $\langle \bigcup_{i=1}^k \varpi_i\rangle = \sum_{i=1}^k \varpi_i$.*

Proof. By Proposition 4.3 (ix), we have $\varpi_i \leqslant \sum_{i=1}^k \varpi_i$, $\forall i \in \{1,2,\dots,k\}$. By Lemma (ii), we have

$$\bigcup_{i=1}^k \varpi_i \leqslant \sum_{i=1}^k \varpi_i.$$

Next, assume that $\sum_{i=1}^k \varpi_i$ is a FBCCSM of G. Let γ be a FBCCSM of G such that $\bigcup_{i=1}^k \varpi_i \leqslant \gamma$. Then $\varpi_i \leqslant \gamma$, $\forall i \in \{1,2,\dots,k\}$. By Proposition 4.3 (vi) and (4.3), we have

$$\sum_{i=1}^k \varpi_i \leqslant \sum_{i=1}^k \gamma \leqslant \gamma.$$

Hence, $\sum_{i=1}^k \varpi_i$ is the least FBCCSM of G greater than $\bigcup_{i=1}^k \varpi_i$, that is, hence $\langle \bigcup_{i=1}^k \varpi_i\rangle = \sum_{i=1}^k \varpi_i$. □

Lemma 5.11. *Let Λ_i be an index set, $\forall i = 1, 2, 3, \ldots, m$. Then*

$$\min\{\sup_{\varpi_1\in\Lambda_1}\{\wp_{\varpi_1}\}, \sup_{\varpi_2\in\Lambda_2}\{\wp_{\varpi_2}\}, \ldots, \sup_{\varpi_m\in\Lambda_m}\{\wp_{\varpi_m}\}\} \geqslant \sup_{(\varpi_1,\varpi_2,\ldots,\varpi_m)\in\Lambda_1\times\Lambda_2\times\ldots\times\Lambda_m}\{\min\{\wp_{\varpi_1}, \wp_{\varpi_2}, \ldots, \wp_{\varpi_m}\}\}.$$

Proof. Since $\sup_{\varpi_i\in\Lambda_i}\{\wp_{\varpi_i}\} \geqslant \wp_{\varpi_i}$, $\forall i \in \{1, 2, 3, \ldots, m\}$, we have $\min\{\sup_{\varpi_1\in\Lambda_1}\{\wp_{\varpi_1}\}, \sup_{\varpi_2\in\Lambda_2}\{\wp_{\varpi_2}\}, \ldots, \sup_{\varpi_m\in\Lambda_m}\{\wp_{\varpi_m}\}\} \geqslant \min\{\wp_{\varpi_1}, \wp_{\varpi_2}, \ldots, \wp_{\varpi_m}\}$. Thus $\min\{\sup_{\varpi_1\in\Lambda_1}\{\wp_{\varpi_1}\}, \sup_{\varpi_2\in\Lambda_2}\{\wp_{\varpi_2}\}, \ldots, \sup_{\varpi_m\in\Lambda_m}\{\wp_{\varpi_m}\}\}$ is an upper bound of $\{\min\{\wp_{\varpi_1}, \wp_{\varpi_2}, \ldots, \wp_{\varpi_m}\} \mid (\varpi_1, \varpi_2, \ldots, \varpi_m) \in \Lambda_1 \times \Lambda_2 \times \ldots \times \Lambda_m\}$. Hence,

$$\min\{\sup_{\varpi_1\in\Lambda_1}\{\wp_{\varpi_1}\}, \sup_{\varpi_2\in\Lambda_2}\{\wp_{\varpi_2}\}, \ldots, \sup_{\varpi_m\in\Lambda_m}\{\wp_{\varpi_m}\}\} \geqslant \sup_{(\varpi_1,\varpi_2,\ldots,\varpi_m)\in\Lambda_1\times\Lambda_2\times\ldots\times\Lambda_m}\{\min\{\wp_{\varpi_1}, \wp_{\varpi_2}, \ldots, \wp_{\varpi_m}\}\}.$$

□

Theorem 5.12. *Let $\varpi, \beta, \gamma \in F(G)$. If ϖ satisfies (FBCCSM-1), then*

$$\varpi \cap (\beta + \gamma) \geqslant (\varpi \cap \beta) + (\varpi \cap \gamma).$$

Proof. Assume that ϖ satisfies (FBCCSM-1). Let $n \in G$. Then

$$\begin{aligned}
\varpi \cap (\beta+\gamma)(n) &= \min\{\varpi(n), (\beta+\gamma)(n)\} \\
&\geqslant \min\{\min_{n=m+k}\{\varpi(m), \varpi(k)\}, \sup_{n=m+k}\{\min\{\beta(m), \gamma(k)\}\} && \text{(FBCCSM-1)} \\
&= \min\{\sup_{n=m+k}\{\min\{\varpi(m), \varpi(k)\}\}, \sup_{n=m+k}\{\min\{\beta(m), \gamma(k)\}\} \\
&\geqslant \sup_{n=m+k}\{\min\{\min\{\varpi(m), \varpi(k)\}, \min\{\beta(m), \gamma(k)\}\}\} && \text{Lemma 5.11} \\
&= \sup_{n=m+k}\{\min\{\min\{\varpi(m), \beta(m)\}, \min\{\varpi(k), \gamma(k)\}\}\} \\
&= \sup_{n=m+k}\{\min\{(\varpi\cap\beta)(m), (\varpi\cap\gamma)(k)\}\} \\
&= ((\varpi\cap\beta) + (\varpi\cap\gamma))(n).
\end{aligned}$$

Hence, $\varpi \cap (\beta + \gamma) \geqslant (\varpi \cap \beta) + (\varpi \cap \gamma)$. □

6. Fuzzy BCC-ideals of BCC-algebras

In this section, we apply the concept of FBCCIs of BCC-algebras while providing properties. Further we find the results of the composition and the product between FBCCIs and FBCCSMs.

Definition 6.1 ([32]). A FS f in $\mathcal{U}$ is called a *fuzzy BCC-ideal* (FBCCI) of $\mathcal{U}$ if

$$(\forall \upsilon \in \mathcal{U})(f(1) \geqslant f(\upsilon)), \tag{6.1}$$
$$(\forall \upsilon, \iota, \sigma \in \mathcal{U})(f(\upsilon \cdot \sigma) \geqslant \min\{f(\upsilon \cdot (\iota \cdot \sigma)), f(\iota)\}). \tag{6.2}$$

We define $FI(\mathcal{U})$ as the set of all FBCCIs of $\mathcal{U}$.

Proposition 6.2. *If $f \in FI(\mathcal{U})$, then*

$$(\forall \upsilon, \iota \in \mathcal{U})(f(\upsilon \wedge \iota) \geqslant \max\{f(\upsilon), f(\iota)\}). \tag{6.3}$$

Proof. Assume that $f \in FI(\mathfrak{U})$. Let $\upsilon, \iota \in \mathfrak{U}$. Then

$$\begin{aligned} f(\upsilon \wedge \iota) &= f((\iota \cdot \upsilon) \cdot \upsilon) && \text{(Meet)} \\ &\geqslant \min\{f((\iota \cdot \upsilon) \cdot (\upsilon \cdot \upsilon)), f(\upsilon)\} && (6.2) \\ &= \min\{f((\iota \cdot \upsilon) \cdot 1), f(\upsilon)\} && (1.1) \\ &= \min\{f(1), f(\upsilon)\} && \text{(BCC-3)} \\ &= f(\upsilon) && (6.1) \end{aligned}$$

and

$$\begin{aligned} f(\upsilon \wedge \iota) &= f((\iota \cdot \upsilon) \cdot \upsilon) && \text{(Meet)} \\ &\geqslant \min\{f((\iota \cdot \upsilon) \cdot (\iota \cdot \upsilon)), f(\iota)\} && (6.2) \\ &= \min\{f(1), f(\iota)\} && (1.1) \\ &= f(\iota). && (6.1) \end{aligned}$$

Hence, $f(\upsilon \wedge \iota) \geqslant \max\{f(\upsilon), f(\iota)\}$. □

Definition 6.3. An $\varpi \in F(G)$ is said to be *increasing* if

$$(\forall n, m \in G)(\varpi(n + m) \geqslant \max\{\varpi(n), \varpi(m)\}). \tag{Increasing}$$

We know that every increasing FS in a BCC-module satisfies (FBCCSM-1).

Lemma 6.4. *If $\varpi \in F(G)$ satisfies (FBCCSM-3) and is increasing, then*

$$(\forall n \in G)(\varpi(n) = \varpi(0)).$$

Proof. Assume that ϖ satisfies (FBCCSM-3) and is increasing. Let $n \in G$. Then

$$\begin{aligned} \varpi(n) &= \varpi(n + 0) \\ &\geqslant \max\{\varpi(n), \varpi(0)\} && \text{(Increasing)} \\ &\geqslant \varpi(0) && (3.1) \\ &\geqslant \varpi(n). && (3.1) \end{aligned}$$

Hence, $\varpi(n) = \varpi(0)$. □

Theorem 6.5. *If G is distributive, $f \in FI(\mathfrak{U})$, and $\beta \in F(G)$ is increasing, then $f \circ \beta$ is increasing, that is, it satisfies* (FBCCSM-1).

Proof. Assume that $f \in FI(\mathfrak{U})$ and $\beta \in F(G)$ is increasing. Let $n, m \in G$. Then

$$\begin{aligned} (f \circ \beta)(n + m) &= \sup_{n+m=\upsilon k} \{\min\{f(\upsilon), \beta(k)\}\} \\ &\geqslant \sup_{n+m=\iota k_1+\sigma k_2=(\iota\wedge\sigma)(k_1+k_2)} \{\min\{f(\iota \wedge \sigma), \beta(k_1 + k_2)\}\} && \text{(Distributive)} \\ &\geqslant \sup_{n=\iota k_1} \{\min\{f(\iota), \beta(k_1)\}\} && \text{(6.3), (Increasing)} \\ &= (f \circ \beta)(n). \end{aligned}$$

Since $+$ is commutative, we have $(f \circ \beta)(n + m) = (f \circ \beta)(m + n) \geqslant (f \circ \beta)(m)$. Thus $(f \circ \beta)(n + m) \geqslant \max\{(f \circ \beta)(n), (f \circ \beta)(m)\}$. Hence, $f \circ \beta$ is increasing, that is, it satisfies (FBCCSM-1). □

Theorem 6.6. *If $f \in FI(\mathfrak{U})$ and $\beta \in F(G)$ satisfies* (FBCCSM-2)*, then $f \circ \beta$ satisfies* (FBCCSM-2), (FBCCSM-3).

Proof. Assume that $f \in FI(\mathcal{U})$ and $\beta \in F(G)$ satisfies (FBCCSM-2). Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned}(f \circ \beta)(-n) &= \sup_{-n=\upsilon k} \{\min\{f(\upsilon), \beta(k)\}\} \\ &= \sup_{n=-\upsilon k} \{\min\{f(\upsilon), \beta(k)\}\} \\ &= \sup_{n=\upsilon(-k)} \{\min\{f(\upsilon), \beta(-k)\}\} && \text{Proposition 2.5 (v), (FBCCSM-2)} \\ &= (f \circ \beta)(n)\end{aligned}$$

and

$$\begin{aligned}(f \circ \beta)(\upsilon n) &= \sup_{\upsilon n=\sigma m} \{\min\{f(\sigma), \beta(m)\}\} \\ &\geqslant \sup_{\upsilon n=(\upsilon \wedge \iota)k=\upsilon(\iota k)} \{\min\{f(\upsilon \wedge \iota), \beta(k)\}\} && \text{(BCCM-1)} \\ &\geqslant \sup_{n=\iota k} \{\min\{f(\iota), \beta(k)\}\} && (6.3) \\ &= (f \circ \beta)(n).\end{aligned}$$

Hence, $f \circ \beta$ satisfies (FBCCSM-2) and (FBCCSM-3). □

Theorem 6.7. *If* G *is distributive,* $f \in FI(\mathcal{U})$*, and* $\beta \in F(G)$ *is increasing and satisfies* (FBCCSM-2)*, then* $f \circ \beta \in FS(G)$.

Proof. Theorems 6.5 and 6.6 make it simple. □

Theorem 6.8. *If* G *is distributive,* $f \in FI(\mathcal{U})$*, and* $\beta \in F(G)$ *is increasing, then* $f\beta$ *is increasing, that is, it satisfies* (FBCCSM-1).

Proof. Assume that G is distributive, $f \in FI(\mathcal{U})$, and $\beta \in F(G)$ is increasing. Let $n, m \in G$ be such that $n = \sum_{i=1}^{r} \iota_i n_i$ and $m = \sum_{i=1}^{r} \sigma_i m_i$, where $r \in \mathbb{N}, \iota_i, \sigma_i \in \mathcal{U}$, and $n_i, m_i \in G$, $\forall i = 1, 2, \dots, r$. Then

$$\begin{aligned}n + m = \sum_{i=1}^{r} \iota_i n_i + \sum_{i=1}^{r} \sigma_i m_i &= \sum_{i=1}^{r} (\iota_i n_i + \sigma_i m_i) \\ &= \sum_{i=1}^{r} (\iota_i \wedge \sigma_i)(n_i + m_i). && \text{(Distributive)}\end{aligned}$$

Thus

$$\begin{aligned}(f\beta)(n + m) &= \sup_{n+m=\sum_{i=1}^{r} \upsilon_i k_i} \{\min_{i=1}^{r}\{f(\upsilon_i), \beta(k_i)\}\} \\ &\geqslant \sup_{n+m=\sum_{i=1}^{r}(\iota_i \wedge \sigma_i)(n_i+m_i)} \{\min_{i=1}^{r}\{f(\iota_i \wedge \sigma_i), \beta(n_i + m_i)\}\} \\ &\geqslant \sup_{n=\sum_{i=1}^{r} \iota_i n_i} \{\min_{i=1}^{r}\{f(\iota_i), \beta(n_i)\}\} && \text{(6.3), (Increasing)} \\ &= (f\beta)(n) \geqslant \min\{(f\beta)(n), (f\beta)(m)\}.\end{aligned}$$

Since $+$ is commutative, we have $(f\beta)(n + m) = (f\beta)(m + n) \geqslant (f\beta)(n)$. Thus $(f\beta)(n + m) \geqslant \max\{(f\beta)(n), (f\beta)(m)\}$. Hence, $f\beta$ is increasing, that is, it satisfies (FBCCSM-1). □

Theorem 6.9. *If* $f \in FI(\mathcal{U})$ *and* $\beta \in F(G)$ *satisfies* (FBCCSM-2)*, then* $f\beta$ *satisfies* (FBCCSM-2) *and* (FBCCSM-3).

Proof. Assume that $f \in FI(\mathcal{U})$ and $\beta \in F(G)$ satisfies (FBCCSM-2). Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$\begin{aligned}(f\beta)(-n) &= \sup_{-n=\sum_{i=1}^{r}\upsilon_i k_i}\{\min_{i=1}^{r}\{f(\upsilon_i),\beta(k_i)\}\} \\ &= \sup_{n=-\sum_{i=1}^{r}\upsilon_i k_i}\{\min_{i=1}^{r}\{f(\upsilon_i),\beta(k_i)\}\} \\ &= \sup_{n=\sum_{i=1}^{r}\upsilon_i(-k_i)}\{\min_{i=1}^{r}\{f(\upsilon_i),\beta(-k_i)\}\} && \text{Proposition 2.5 (vii), (FBCCSM-2)} \\ &= (f\beta)(n).\end{aligned}$$

Let $n \in G$ be such that $n = \sum_{i=1}^{r}\upsilon_i k_i$, where $r \in \mathbb{N}, \upsilon_i \in \mathcal{U}$, and $k_i \in G$, $\forall i = 1, 2, \ldots, r$. Then

$$\begin{aligned}\upsilon n &= \upsilon(\sum_{i=1}^{r}\upsilon_i k_i) \\ &= \sum_{i=1}^{r}\upsilon(\upsilon_i k_i) && \text{(BCCM-2)} \\ &= \sum_{i=1}^{r}(\upsilon\wedge\upsilon_i)k_i. && \text{(BCCM-1)}\end{aligned}$$

Thus

$$\begin{aligned}(f\beta)(\upsilon n) &= \sup_{\upsilon n=\sum_{i=1}^{r}\iota_i m_i}\{\min_{i=1}^{r}\{f(\iota_i),\beta(m_i)\}\} \\ &\geqslant \sup_{\upsilon n=\sum_{i=1}^{r}(\upsilon\wedge\upsilon_i)k_i}\{\min_{i=1}^{r}\{f(\upsilon\wedge\upsilon_i),\beta(k_i)\}\} \\ &\geqslant \sup_{n=\sum_{i=1}^{r}\upsilon_i k_i}\{\min_{i=1}^{r}\{f(\upsilon_i),\beta(k_i)\}\} && (6.3) \\ &= (f\beta)(n).\end{aligned}$$

Hence, $f\beta$ satisfies (FBCCSM-2) and (FBCCSM-3). □

Theorem 6.10. *If* G *is distributive,* $f \in FI(\mathcal{U})$*, and* $\beta \in F(G)$ *is increasing and satisfies* (FBCCSM-2)*, then* $f\beta \in FS(G)$.

Proof. Because of Theorems 6.8 and 6.9, it is simple. □

Proposition 6.11. *Let* $f, g \in F(\mathcal{U})$ *and* $\varpi, \beta \in F(G)$*, where* β *satisfies* (FBCCSM-1)*. Then* $f\circ\varpi \leqslant \beta$ *if and only if* $f\varpi \leqslant \beta$.

Proof. Assume that $f\circ\varpi \leqslant \beta$. Let $n \in G$. Then

$$\begin{aligned}(f\varpi)(n) &= \sup_{n=\sum_{i=1}^{k}\upsilon_i n_i}\{\min_{i=1}^{k}\{f(\upsilon_i),\varpi(n_i)\}\} \\ &= \sup_{n=\sum_{i=1}^{k}\upsilon_i n_i}\{\min_{i=1}^{k}\{\min\{f(\upsilon_i),\varpi(n_i)\}\}\} \\ &= \sup_{n=\sum_{i=1}^{k}\upsilon_i n_i}\{\min_{i=1}^{k}\{\sup\{\min\{f(\upsilon_i),\varpi(n_i)\}\}\}\}\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sup_{n=\sum_{i=1}^{k} \upsilon_i n_i} \{\min_{i=1}^{k}\{(f \circ \varpi)(\upsilon_i n_i)\}\} \\
&= \sup_{n=\sum_{i=1}^{k} \upsilon_i n_i} \{\min_{i=1}^{k}\{\beta(\upsilon_i n_i)\}\} && (f \circ \varpi \leqslant \beta) \\
&\leqslant \beta(\sum_{i=1}^{k} \upsilon_i n_i) && (\text{FBCCSM-1}) \\
&= \beta(n).
\end{aligned}
$$

Hence, $f\varpi \leqslant \beta$.

Conversely, assume that $f\varpi \leqslant \beta$. By (5.2), we have $f \circ \varpi \leqslant f\varpi \leqslant \beta$. □

Proposition 6.12. *Let* $f, g \in F(\mathcal{U})$ *and* $\varpi, \beta \in F(G)$. *Then*

(i) *if* $\varpi \leqslant \beta$, *then* $f \circ \varpi \leqslant f \circ \beta$ *and* $f\varpi \leqslant f\beta$;

(ii) *if* $f \leqslant g$, *then* $f \circ \beta \leqslant g \circ \beta$ *and* $f\beta \leqslant g\beta$.

Proof.

(i) Assume that $\varpi \leqslant \beta$. Let $n \in G$. Then

$$
\begin{aligned}
(f \circ \varpi)(n) &= \sup_{n=\upsilon m} \{\min\{f(\upsilon), \varpi(m)\}\} \\
&\leqslant \sup_{n=\upsilon m} \{\min\{f(\upsilon), \beta(m)\}\} && \text{assumption} \\
&= (f \circ \beta)(n)
\end{aligned}
$$

and

$$
\begin{aligned}
(f\varpi)(n) &= \sup_{n=\sum_{i=1}^{k} \upsilon_i n_i} \{\min\{f(\upsilon_1), f(\upsilon_2), \dots, f(\upsilon_k), \varpi(n_1), \varpi(n_2), \dots, \varpi(n_k)\}\} \\
&\leqslant \sup_{n=\sum_{i=1}^{k} \upsilon_i n_i} \{\min\{f(\upsilon_1), f(\upsilon_2), \dots, f(\upsilon_k), \beta(n_1), \beta(n_2), \dots, \beta(n_k)\}\} && \text{assumption} \\
&= (f\beta)(n).
\end{aligned}
$$

Hence, $f \circ \varpi \leqslant f \circ \beta$ and $f\varpi \leqslant f\beta$.

(ii) Assume that $f \leqslant g$. Let $n \in G$. Then

$$
\begin{aligned}
(f \circ \varpi)(n) &= \sup_{n=\upsilon m} \{\min\{f(\upsilon), \varpi(m)\}\} \\
&\leqslant \sup_{n=\upsilon m} \{\min\{g(\upsilon), \varpi(m)\}\} && \text{assumption} \\
&= (g \circ \varpi)(n)
\end{aligned}
$$

and

$$
\begin{aligned}
(f\varpi)(n) &= \sup_{n=\sum_{i=1}^{k} \upsilon_i n_i} \{\min\{f(\upsilon_1), f(\upsilon_2), \dots, f(\upsilon_k), \varpi(n_1), \varpi(n_2), \dots, \varpi(n_k)\}\} \\
&\leqslant \sup_{n=\sum_{i=1}^{k} \upsilon_i n_i} \{\min\{g(\upsilon_1), g(\upsilon_2), \dots, g(\upsilon_k), \varpi(n_1), \varpi(n_2), \dots, \varpi(n_k)\}\} && \text{assumption} \\
&= (g\varpi)(n).
\end{aligned}
$$

Hence $f \circ \beta \leqslant g \circ \beta$ and $f\beta \leqslant g\beta$. □

Lemma 6.13. *Let* $f \in F(\mathcal{U})$ *and* $\beta \in F(G)$. *Then*

$$(\forall \upsilon \in \mathcal{U}, \forall n \in G)((f\beta)(0) \geqslant (f \circ \beta)(0) \geqslant \max\{\min\{f(\upsilon), \beta(0)\}, \min\{f(1), \beta(n)\}\}).$$

Proof. Let $\upsilon \in \mathcal{U}$ and $n \in G$. Then

$$(f \circ \beta)(0) = \sup_{0=\upsilon n} \{\min\{f(\upsilon), \beta(n)\}\} \geqslant \min\{f(\upsilon), \beta(0)\}$$

and

$$(f \circ \beta)(0) = \sup_{0=\upsilon n} \{\min\{f(\upsilon), \beta(n)\}\} \geqslant \min\{f(1), \beta(n)\}.$$

Hence, $(f \circ \beta)(0) \geqslant \max\{\min\{f(\upsilon), \beta(0)\}, \min\{f(1), \beta(n)\}\}$. By (5.2), we have

$$(f\beta)(0) \geqslant (f \circ \beta)(0) \geqslant \max\{\min\{f(\upsilon), \beta(0)\}, \min\{f(1), \beta(n)\}\}.$$

□

Lemma 6.14. *Let* $\upsilon_s, \upsilon_\wp, f \in F(\mathcal{U})$ *and* $a_s, a_\wp, \beta \in F(G)$. *Then*

(i) $f \circ a_{\min\{\wp,s\}} \leqslant (f \circ a_s) \cap (f \circ a_\wp)$;

(ii) $fa_{\min\{\wp,s\}} \leqslant (fa_s) \cap (fa_\wp)$;

(iii) $\upsilon_{\min\{\wp,s\}} \circ \beta \leqslant (\upsilon_s \circ \beta) \cap (\upsilon_\wp \circ \beta)$;

(iv) $\upsilon_{\min\{\wp,s\}}\beta \leqslant (\upsilon_s\beta) \cap (\upsilon_\wp\beta)$.

Proof. By Note 4.16 (ii) and Proposition 6.12, it is established. □

Lemma 6.15. *Let* $f \in F(\mathcal{U})$ *and* $a_\wp, b_s, \beta \in F(G)$, *where* β *satisfies* (FBCCSM-1). *If* $f \circ a_\wp \leqslant \beta$ *and* $f \circ b_s \leqslant \beta$, *then*

$$(\forall \upsilon \in \mathcal{U})((f \circ (a+b)_{\min\{\wp,s\}})(\upsilon(a+b)) \leqslant \beta(\upsilon(a+b))).$$

Proof. Assume that $f \circ a_\wp \leqslant \beta$ and $f \circ b_s \leqslant \beta$. Let $\upsilon \in \mathcal{U}$. Then

$$\beta(\upsilon a) \geqslant (f \circ a_\wp)(\upsilon a) = \sup_{\upsilon a=\iota n} \{\min\{f(\iota), a_\wp(n)\}\} \geqslant \min\{f(\upsilon), a_\wp(a)\} = \min\{f(\upsilon), \wp\} \tag{6.4}$$

and

$$\beta(\upsilon b) \geqslant (f \circ b_s)(\upsilon b) = \sup_{\upsilon b=\iota n} \{\min\{f(\iota), b_s(n)\}\} \geqslant \min\{f(\upsilon), b_s(b)\} = \min\{f(\upsilon), s\}. \tag{6.5}$$

Thus

$$\begin{aligned}
(f \circ (a+b)_{\min\{\wp,s\}})(\upsilon(a+b)) &= \sup_{\upsilon(a+b)=\iota m} \{\min\{f(\iota), (a+b)_{\min\{\wp,s\}}(m)\}\} \\
&= \sup_{\upsilon(a+b)=\iota(a+b)} \{\min\{f(\iota), (a+b)_{\min\{\wp,s\}}(a+b)\}\} \\
&= \sup_{\upsilon(a+b)=\iota(a+b)} \{\min\{f(\iota), \min\{\wp, s\}\}\} \\
&= \sup_{\upsilon(a+b)=\iota(a+b)} \{\min\{\min\{f(\iota), \wp\}, \min\{f(\iota), s\}\}\} \\
&\leqslant \sup_{\upsilon(a+b)=\iota(a+b)} \{\min\{\beta(\iota a), \beta(\iota b)\}\} && (6.4), (6.5) \\
&\leqslant \beta(\iota a + \iota b) && \text{(FBCCSM-1)} \\
&= \beta(\iota(a+b)) \\
&= \beta(\upsilon(a+b)). && \upsilon(a+b) = \iota(a+b)
\end{aligned}$$

□

Lemma 6.16. *Let* $f \in F(\mathcal{U})$ *and* $a_\wp, b_s, \beta \in F(G)$, *where* β *satisfies* (FBCCSM-1) *and* (FBCCSM-2). *If* $f \circ a_\wp \leqslant \beta$ *and* $f \circ b_s \leqslant \beta$, *then*

$$(\forall \upsilon \in \mathcal{U})((f \circ (a-b)_{\min\{\wp,s\}})(\upsilon(a-b)) \leqslant \beta(\upsilon(a-b))).$$

Proof. Assume that $f \circ a_\wp \leqslant \beta$ and $f \circ b_s \leqslant \beta$. Let $\upsilon \in \mathcal{U}$. Thus

$$\begin{aligned}
(f \circ (a-b)_{\min\{\wp,s\}})(\upsilon(a-b)) &= \sup_{\upsilon(a-b)=\iota m} \{\min\{f(\iota), (a-b)_{\min\{\wp,s\}}(m)\}\} \\
&= \sup_{\upsilon(a-b)=\iota(a-b)} \{\min\{f(\iota), (a-b)_{\min\{\wp,s\}}(a-b)\}\} \\
&= \sup_{\upsilon(a-b)=\iota(a-b)} \{\min\{f(\iota), \min\{\wp, s\}\}\} \\
&= \sup_{\upsilon(a-b)=\iota(a-b)} \{\min\{\min\{f(\iota), \wp\}, \min\{f(\iota), s\}\}\} \\
&\leqslant \sup_{\upsilon(a-b)=\iota(a-b)} \{\min\{\beta(\iota a), \beta(\iota b)\}\} && (6.4), (6.5) \\
&= \sup_{\upsilon(a-b)=\iota(a-b)} \{\min\{\beta(\iota a), \beta(-\iota b)\}\} && \text{(FBCCSM-2)} \\
&\leqslant \beta(\iota a - \iota b) && \text{(FBCCSM-1)} \\
&= \beta(\iota(a-b)) \\
&= \beta(\upsilon(a-b)). && \upsilon(a-b) = \iota(a-b)
\end{aligned}$$

□

Proposition 6.17. *Let* $f \in FI(\mathcal{U})$ *and* $a_\wp \in F(G)$. *Then*

$$(\forall \upsilon, \iota \in \mathcal{U})(f \circ (\upsilon a)_\wp)(\iota(\upsilon a)) \leqslant (f \circ a_\wp)(\iota(\upsilon a)).$$

Proof. Let $\upsilon, \iota \in \mathcal{U}$. Then

$$\begin{aligned}
(f \circ (\upsilon a)_\wp)(\iota(\upsilon a)) &= \sup_{\iota(\upsilon a)=\sigma m} \{\min\{f(\sigma), (\upsilon a)_\wp(m)\}\} \\
&= \sup_{\iota(\upsilon a)=\iota'(\upsilon a)} \{\min\{f(\iota'), (\upsilon a)_\wp(\upsilon a)\}\} \\
&\leqslant \sup_{(\iota \wedge \upsilon)a=(\iota' \wedge \upsilon)a} \{\min\{f(\iota' \wedge \upsilon), a_\wp(a)\}\} && \text{(BCCM-1)}, (6.3) \\
&= \sup_{(\iota \wedge \upsilon)a=\sigma m} \{\min\{f(\sigma), a_\wp(m)\}\} \\
&= (f \circ a_\wp)((\iota \wedge \upsilon)a) \\
&= (f \circ a_\wp)(\iota(\upsilon a)). && \text{(BCCM-1)}
\end{aligned}$$

□

Lemma 6.18. *Let* $\upsilon_\wp \in F(\mathcal{U})$ *and* $\varpi \in F(G)$. *If* ϖ *satisfies* (FBCCSM-3), *then*

$$(1_\wp \circ \varpi)(0) \leqslant (\upsilon_\wp \circ \varpi)(0).$$

Proof. Assume that ϖ satisfies (FBCCSM-3). Then

$$\begin{aligned}
(1_\wp \circ \varpi)(0) &= \sup_{0=\iota a}\{\min\{1_\wp(\iota), \varpi(a)\}\} \\
&= \sup_{0=1a}\{\min\{1_\wp(1), \varpi(a)\}\}
\end{aligned}$$

$$\begin{aligned}
&= \sup_{0=1a}\{\min\{\wp, \varpi(a)\}\} \\
&\leqslant \min\{\wp, \varpi(0)\} && (3.1) \\
&= \min\{\upsilon_\wp(\upsilon), \varpi(0)\}\} \\
&\leqslant \sup_{0=\iota a}\{\min\{\upsilon_\wp(\iota), \varpi(a)\}\} && \text{Proposition 2.5 (ii)} \\
&= (\upsilon_\wp \circ \varpi)(0).
\end{aligned}$$

□

Lemma 6.19. *Let* $f \in F(\mathcal{U})$ *and* $a_\wp \in F(G)$*. If* f *satisfies* (6.1)*, then*

$$(f \circ 0_\wp)(0) \leqslant (f \circ a_\wp)(0).$$

Proof. Suppose f meets (6.1). Then

$$\begin{aligned}
(f \circ 0_\wp)(0) &= \sup_{0=\iota b}\{\min\{f(\iota), 0_\wp(b)\}\} \\
&= \sup_{0=\iota 0}\{\min\{f(\iota), 0_\wp(0)\}\} \\
&= \sup_{0=\iota 0}\{\min\{f(\iota), \wp\}\} \\
&\leqslant \min\{f(1), \wp\} && (6.1) \\
&= \min\{f(1), a_\wp(a)\} \\
&\leqslant \sup_{0=\iota b}\{\min\{f(\iota), a_\wp(b)\}\} && \text{(BCCM-3)} \\
&= (f \circ a_\wp)(0).
\end{aligned}$$

□

Lemma 6.20. *Let* $f \in F(\mathcal{U})$ *and* $a_\wp, b_s, \beta \in F(G)$*, where* β *satisfies* (FBCCSM-1)*. If* $fa_\wp \leqslant \beta$ *and* $fb_s \leqslant \beta$*, then*

$$(\forall \upsilon \in \mathcal{U})((f(a+b)_{\min\{\wp,s\}})(\upsilon(a+b)) \leqslant \beta(\upsilon(a+b))).$$

Proof. Assume that $fa_\wp \leqslant \beta$ and $fb_s \leqslant \beta$. Let $\upsilon \in \mathcal{U}$. By (5.2), (6.4), and (6.5), we have

$$\beta(\upsilon a) \geqslant (fa_\wp)(\upsilon a) \geqslant (f \circ a_\wp)(\upsilon a) \geqslant \min\{f(\upsilon), \wp\} \tag{6.6}$$

and

$$\beta(\upsilon b) \geqslant (fb_s)(\upsilon b) \geqslant (f \circ b_s)(\upsilon b) \geqslant \min\{f(\upsilon), s\}. \tag{6.7}$$

Thus

$$\begin{aligned}
(f(a+b)_{\min\{\wp,s\}})(\upsilon(a+b)) &= \sup_{\upsilon(a+b)=\sum_{i=1}^k \iota_i m_i}\{\min_{i=1}^{k}\{f(\iota_i), (a+b)_{\min\{\wp,s\}}(m_i)\}\} \\
&= \sup_{\upsilon(a+b)=\sum_{i=1}^k \iota_i (a+b)}\{\min_{i=1}^{k}\{f(\iota_i), (a+b)_{\min\{\wp,s\}}(a+b)\}\} \\
&= \sup_{\upsilon(a+b)=\sum_{i=1}^k \iota_i (a+b)}\{\min_{i=1}^{k}\{f(\iota_i), \min\{\wp, s\}\}\} \\
&= \sup_{\upsilon(a+b)=\sum_{i=1}^k \iota_i (a+b)}\{\min\{\min_{i=1}^{k}\{f(\iota_i), \wp\}, \min_{i=1}^{k}\{f(\iota_i), s\}\}\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sup_{\upsilon(a+b)=\sum_{i=1}^{k}\iota_i(a+b)} \{\min_{i=1}^{k}\{\beta(\iota_i a), \beta(\iota_i b)\}\} && (6.6), (6.7)\\
&\leqslant \beta(\sum_{i=1}^{k}(\iota_i a + \iota_i b)) && (\text{FBCCSM-1})\\
&= \beta(\sum_{i=1}^{k}\iota_i(a+b))\\
&= \beta(\upsilon(a+b)). && \upsilon(a+b)=\textstyle\sum_{i=1}^{k}\iota_i(a+b)
\end{aligned}
$$

□

Lemma 6.21. *Let* $f \in F(\mathcal{U})$ *and* $a_\wp, b_s, \beta \in F(G)$*, where* β *satisfies* (FBCCSM-1) *and* (FBCCSM-2)*. If* $fa_\wp \leqslant \beta$ *and* $fb_s \leqslant \beta$*, then*

$$(\forall \upsilon \in \mathcal{U})((f(a-b)_{\min\{\wp,s\}})(\upsilon(a-b)) \leqslant \beta(\upsilon(a-b))).$$

Proof. Assume that $fa_\wp \leqslant \beta$ and $fb_s \leqslant \beta$. Let $\upsilon \in \mathcal{U}$. Thus

$$
\begin{aligned}
(f(a-b)_{\min\{\wp,s\}})(\upsilon(a-b)) &= \sup_{\upsilon(a-b)=\sum_{i=1}^{k}\iota_i m_i} \{\min_{i=1}^{k}\{f(\iota_i), (a-b)_{\min\{\wp,s\}}(m_i)\}\}\\
&= \sup_{\upsilon(a-b)=\sum_{i=1}^{k}\iota_i(a-b)} \{\min_{i=1}^{k}\{f(\iota_i), (a-b)_{\min\{\wp,s\}}(a-b)\}\}\\
&= \sup_{\upsilon(a-b)=\sum_{i=1}^{k}\iota_i(a-b)} \{\min_{i=1}^{k}\{f(\iota_i), \min\{\wp, s\}\}\}\\
&= \sup_{\upsilon(a-b)=\sum_{i=1}^{k}\iota_i(a-b)} \{\min\{\min_{i=1}^{k}\{f(\iota_i), \wp\}, \min_{i=1}^{k}\{f(\iota_i), s\}\}\}\\
&\leqslant \sup_{\upsilon(a-b)=\sum_{i=1}^{k}\iota_i(a-b)} \{\min_{i=1}^{k}\{\beta(\iota_i a), \beta(\iota_i b)\}\} && (6.6), (6.7)\\
&= \sup_{\upsilon(a-b)=\sum_{i=1}^{k}\iota_i(a-b)} \{\min_{i=1}^{k}\{\beta(\iota_i a), \beta(-\iota_i b)\}\} && (\text{FBCCSM-2})\\
&\leqslant \beta(\sum_{i=1}^{k}(\iota_i a - \iota_i b)) && (\text{FBCCSM-1})\\
&= \beta(\sum_{i=1}^{k}\iota_i(a-b))\\
&= \beta(\upsilon(a-b)). && \upsilon(a-b)=\textstyle\sum_{i=1}^{k}\iota_i(a-b)
\end{aligned}
$$

□

Proposition 6.22. *Let* $f \in FI(\mathcal{U})$ *and* $a_\wp \in F(G)$*. Then*

$$(\forall \upsilon, \iota \in \mathcal{U})(f(\upsilon a)_\wp)(\iota(\upsilon a)) \leqslant (fa_\wp)(\iota(\upsilon a)).$$

Proof. Let $\upsilon, \iota \in \mathcal{U}$. Then

$$(f(\upsilon a)_\wp)(\iota(\upsilon a)) = \sup_{\iota(\upsilon a)=\sum_{i=1}^{k}\sigma_i m_i} \{\min_{i=1}^{k}\{f(\sigma_i), (\upsilon a)_\wp(m_i)\}\}$$

$$
\begin{aligned}
&= \sup_{\iota(\upsilon a)=\sum_{i=1}^{k}\iota_i'(\upsilon a)} \{\min_{i=1}^{k}\{f(\iota_i'), (\upsilon a)_\wp(\upsilon a)\}\} \\
&\leqslant \sup_{(\iota\wedge\upsilon)a=\sum_{i=1}^{k}(\iota'\wedge\upsilon)a} \{\min_{i=1}^{k}\{f(\iota_i'\wedge\upsilon), a_\wp(a)\}\} && \text{(BCCM-1), (6.3)} \\
&= \sup_{(\iota\wedge\upsilon)a=\sum_{i=1}^{k}\sigma_i m_i} \{\min_{i=1}^{k}\{f(\sigma_i), a_\wp(m_i)\}\} \\
&= (fa_\wp)((\iota\wedge\upsilon)a) \\
&= (fa_\wp)(\iota(\upsilon a)). && \text{(BCCM-1)}
\end{aligned}
$$

□

7. Conclusions and future works

The idea of BCC-modules has been proposed in this article. The idea of FBCCSMs of BCC-modules has also been proposed, and its features and necessary and sufficient conditions have been given. We have defined FSs in BCC-modules of several patterns, identifying the properties and relationships of these FSs to the FBCCSM. We have also defined and studied the FBCCSM generated by a set of FSs in BCC-modules, as well as identifying their properties and their relationship to FBCCSMs. Finally, we have applied the concept of FBCCIs of BCC-algebras while providing properties and searched for the results of the composition and the product between FBCCIs and FBCCSMs.

In the near future, more research on the following subjects will be conducted:

(1) to investigate Fermatean fuzzy sets based on the Senapati and Yager notion [31];

(2) to present the idea of bipolar Pythagorean fuzzy soft sets using the Jana and Pal concept [13];

(3) to study generalized intuitionistic fuzzy BCC-ideals based on the Jana and Pal concept [12];

(4) to study Pythagorean fuzzy sets based on Pythagorean fuzzy points and Pythagorean fuzzy numbers according to Jana et al.'s approach [14, 15].

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