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Bi-univalent functions of order ζ connected with $(\mathfrak{m},\mathfrak{n})$-Lucas polynomials

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Abstract

With the aid of the q-binomial coefficients and utilizing the convolution, we define a new q-convolution operator that helps us introduce two new families of bi-univalent functions. These classes are connected by subordination with a function $\mathcal{G}_{\mathfrak{m},\mathfrak{n}}$. We give upper bounds for the coefficients estimate $|a_j|$ $(j = 2, 3)$ of the functions that belong to these families, followed by some special cases. Moreover, we found estimates for the Fekete-Szegö inequality for both of these families, followed by simple particular results. We emphasize that the defined convolution q-difference operator generalizes some other operators given by several authors. As an application of this study, Fekete-Szegö inequalities for these classes of functions defined by Pascal distribution are investigated.

Keywords: Bi-univalent function, q-exponential function, $(\mathfrak{m},\mathfrak{n})$-Lucas polynomials, Fekete-Szegő inequality, Pascal distribution.

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1. Introduction and motivation

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j, \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ made up of all functions that are univalent in $\mathbb{U}$ will also be defined by $\mathcal{S}$.

By the Koebe *One Quarter Theorem* [14], a function $f \in \mathcal{S}$ has an inverse function f^{-1} which is defined by

$$f^{-1}\left(f(z)\right) = z,\ z \in \mathbb{U} \quad \text{and} \quad f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f),\ r_0(f) \geqslant \frac{1}{4}\right),$$

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where

$$f^{-1}(w) = w + \sum_{j=2}^{\infty} g_j w^j.$$

Since

$$w = f\left(f^{-1}(w)\right) = w + (g_2 + a_2)\, w^2 + \left(g_3 - 2a_2^2 + a_3\right) w^3 + \left(g_4 + 5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots,$$

we get $g_2 = -a_2$, $g_3 = 2a_2^2 - a_3$, and $g_4 = -5a_2^3 + 5a_2a_3 - a_4$, then we obtain

$$f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots .$$

Assume that the inverse function f^{-1} has an analytic extension in $\mathbb{U}$. In $\mathbb{U}$, if f and f^{-1} are univalent, then f is considered to be bi-univalent in $\mathbb{U}$, and we express the class of bi-univalent functions by $\Sigma_{\mathbb{U}}$. For more explanation, the following functions belong to the class $\Sigma_{\mathbb{U}}$:

$$\frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \quad \frac{z}{1-z}, \quad -\log(1-z),$$

with the following associated inverse functions

$$\frac{e^{2w}-1}{e^{2w}+1}, \quad \frac{w}{1+w}, \quad \frac{e^{w}-1}{e^{w}}.$$

Many investigators have concerned in examining of bi-univalent functions and investigated the bounds of the coefficients estimate $|a_n|$ (see [2, 6, 8, 9, 11, 36, 38]).

The Bieberbach conjecture-related Fekete-Szegö problem centers on an inequality for the coefficients of univalent functions f in (1.1), that have been discovered by Fekete and Szegö [18]. The inequality consists of determining an estimate of the maximum value of the coefficient functional $\left|a_3 - \xi a_2^2\right|$ (see [10, 12, 15, 21, 25, 29, 31, 32]).

For any analytic functions f and g in $\mathbb{U}$, if there is a Schwarz function ϕ, that is, ϕ is analytic in $\mathbb{U}$, with $\phi(z) = 0$, $|\phi(z)| < 1$, such that $f(z) = g(\phi(z))$, then f is considered a *subordinate* to g, indicated by $f(z) \prec g(z)$. If g is univalent in $\mathbb{U}$, then the next relationship holds:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Lately, many investigators have concentrated on the scope of q-calculus, motivated by the success of q-calculus applications in physics, mathematics and other fields, such as differential equations, special functions, combinatorics, special polynomials, operator theory, and other related fields. In 1908, Jackson ([24]), was the first to introduce the concepts of q-analysis (derivative and integral). Also, the geometries for q-analysis have been explored in numerous works introduced on quantum groups. Recently, numerous studies have examined certain classes analytic functions in $\mathbb{U}$ utilizing fractional q-calculus operators. In the present paper, some parts of the q-calculus theory will be employed to reach our results: see the following papers for more information on the notions of q-calculus ([4, 17, 20, 22, 23, 34, 37]).

Recently, some investigators have developed the q-calculus using the parameter p, such that $0 < q < p \leqslant 1$. Subsequently, Srivastava [35] has demonstrated that the value of p does not contribute to developing q-calculus, and it is a trivial and inconsequential value. Therefore, we support the study of the q-calculus and its applications. We are now going to provide the concepts of q-calculus that we need to comprehend this work.

Definition 1.1 ([24]).

(i) Let $0 < q < 1$ and $[j]_q!$ indicates the q-factorial, which is expressed as follows:

$$[j]_q! = \begin{cases} [j]_q\,[j-1]_q \cdots [2]_q\,[1]_q, & \text{if } j = 1, 2, 3, \ldots, \\ 1, & \text{if } j = 0, \end{cases}$$

where

$$[j]_q := \frac{1-q^j}{1-q} \quad \text{and} \quad [0]_q := 0.$$

(ii) For $0 < q < 1$, the q-derivative operator is established by

$$D_q f(z) := \frac{f(qz) - f(z)}{(q-1)z},\ z \neq 0.$$

We let the q-exponential function e_q defined by (see [27])

$$e_q(z) := \sum_{j=0}^{\infty} \frac{z^j}{[j]_q!},\ (z \in \mathbb{U}).$$

Furthermore, we define the q-binomial series (see [13])

$$(1-\eta)_q^r := \sum_{s=0}^{r} \binom{r}{s}_q (-1)^s \eta^s,\ r \in \mathbb{N},\ s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where

$$\binom{r}{s}_q := \frac{[r]_q!}{[s]_q!\,[r-s]_q!},$$

represents the q-binomial coefficients (or Gaussian binomial coefficients). We use these q-binomial coefficients to introduce the differential operator $\mathcal{L}_\eta^{r,q,k} : \mathcal{A} \to \mathcal{A}$ by

$$\begin{aligned} &\mathcal{L}_\eta^{r,q,0} f(z) := f(z), \\ &\mathcal{L}_\eta^{r,q,1} f(z) := (1-\eta)_q^r\, f(z) + \left(1 - (1-\eta)_q^r\right) z D_q f(z), \\ &\quad\vdots \\ &\mathcal{L}_\eta^{r,q,k} f(z) := \mathcal{L}_\eta^{r,q,1} f(z) \left(\mathcal{L}_\eta^{r,q,k-1} f(z)\right),\ k \geqslant 2. \end{aligned} \tag{1.2}$$

If f is given by (1.1), from (1.2) we obtain

$$\mathcal{L}_\eta^{r,q,k} f(z) = z + \sum_{j=2}^{\infty} \left(1 + ([j]_q - 1)\chi_q^r(\eta)\right)^k a_j z^j,\ z \in \mathbb{U}, \tag{1.3}$$

where $\chi_q^r(\eta) := \sum_{s=0}^{r} \binom{r}{s}_q (-1)^{s+1} \eta^s$.

Remark 1.2. Note the following operators of particular cases of the operator $\mathcal{L}_\eta^{r,q,k}$ previously determined by other authors.

1. Taking $q \to 1-$, we acquire the operator $D_{m,\lambda}^{\zeta}$ introduced by Frasin ([3, 19]).
2. For $q \to 1-$ and $\mathcal{L}_\eta^{1,k}$, this last operator reduces to the operator defined by Al-Oboudi [5].
3. If $q \to 1-$ and $\mathcal{L}_1^{1,k}$, we obtain the operator of Sălăgean [33].

Definition 1.3. For a function $f \in \mathcal{A}$, by utilizing the convolution principle, we define a q-convolution operator $\mathcal{Q}_q^{\eta,r,k} f(z) : \mathcal{A} \to \mathcal{A}$ given as follows:

$$\mathcal{Q}_q^{\eta,r,k} f(z) = \mathcal{L}_\eta^{r,q,k} f(z) * e_q.$$

From the above definition, it follows that

$$\mathcal{Q}_q^{\eta,r,k}f(z) = z + \sum_{j=2}^{\infty} \Upsilon(j) a_j z^j, \ (z \in \mathbb{U}), \tag{1.4}$$

with

$$\Upsilon(j) := \frac{\left(1 + ([j]_q - 1)\chi_q^r(\eta)\right)^k}{[j]_q!}. \tag{1.5}$$

Moreover, from (1.3) it is easy to show the next relationship:

$$\chi_q^r(\eta) z D_q \left(\mathcal{L}_\eta^{r,q,k} f(z)\right) = \mathcal{L}_\eta^{r,q,k+1} f(z) - \left(1 - \chi_q^r(\eta)\right) \mathcal{L}_\eta^{r,q,k} f(z), \ z \in \mathbb{U}.$$

In 2012, Lee and Asci [26] established the (m,n)-Lucas polynomial $\mathcal{L}_{m,n,j}(x)$ based on the frequency relationship between the polynomials $m(x)$ and $n(x)$ with real coefficients, as follows:

$$\begin{aligned} &\mathcal{L}_{m,n,0}(x) = 2, \quad \mathcal{L}_{m,n,1}(x) = m(x), \text{ and } \mathcal{L}_{m,n,2}(x) = m^2(x) + 2n(x), \\ &\mathcal{L}_{m,n,j}(x) = m(x)\mathcal{L}_{m,n,j-1}(x) + n(x)\mathcal{L}_{m,n,j-2}(x), \ j \geqslant 2, \end{aligned} \tag{1.6}$$

and the generating function of the (m,n)-Lucas Polynomial is given by

$$\mathcal{G}_{m,n}(z) := \sum_{j=2}^{\infty} \mathcal{L}_{m,n,j}(x) z^j = \frac{2 - m(x)z}{1 - m(x)z - n(x)z^2}.$$

Depending on the values of m and n, some special polynomials can be obtained from the (m,n)-Lucas polynomial like the Lucas polynomials $\mathcal{L}_{x,1,j}(x)$, the Pell-Lucas polynomials $\mathcal{L}_{2x,1,j}(x)$, the Jacobsthal polynomials $\mathcal{L}_{1,2x,j}(x)$, the Fermat-Lucas polynomials $\mathcal{L}_{3x,-2,j}(x)$, and the first-kind Chebyshev polynomials $\mathcal{L}_{2x,-1,j}(x)$. Many authors have provided studies on the (m,n)-Lucas polynomial ([1, 7, 39–41]).

This work aims to define a q-convolution operator that contains the q-binomial series and q-exponential function, and to determine upper bounds for several families of analytic functions defined by subordination. After that, our findings deal with the Fekete-Szegö inequality for (m,n)-Lucas polynomials. Finally, we determine the upper bounds of Fekete-Szegö inequality by using Pascal distribution.

2. Initial coefficient bounds for some families connected with the (m,n)-Lucas polynomials

This section begins by introducing the concept of new families $\mathcal{FK}_q(\eta,r,k;\rho,\zeta)$ and $\mathcal{JF}_q(\eta,r,k;\rho,\zeta)$. To demonstrate our first results, it needs to mention the following lemma.

Lemma 2.1 ([30], p. 172)**.** *Let* $\omega(z) = \sum_{n=1}^{\infty} \omega_n z^n$ *be an analytic function in* $\mathbb{U}$ *such that* $|\omega(z)| < 1$ *for all* $z \in \mathbb{U}$. *Then,*

$$|\omega_1| \leqslant 1, \quad |\omega_n| \leqslant 1 - |\omega_1|^2, \ n = 2, 3, \ldots.$$

Definition 2.2. A function $f \in \Sigma_{\mathbb{U}}$ is said to be in the family $\mathcal{FK}_q(\eta,r,k;\rho,\zeta)$ if the below conditions hold:

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)' + \rho z^2\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)''}{(1-\rho)\mathcal{Q}_q^{\eta,r,k}f(z) + \rho z\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)'} - 1\right) \prec \mathcal{G}_{m,n}(z) - 1$$

and

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k}f^{-1}(w)\right)' + \rho z^2\left(\mathcal{Q}_q^{\eta,r,k}f^{-1}(w)\right)''}{(1-\rho)\mathcal{Q}_q^{\eta,r,k}f^{-1}(w) + \rho z\left(\mathcal{Q}_q^{\eta,r,k}f^{-1}(w)\right)'} - 1\right) \prec \mathcal{G}_{m,n}(w) - 1,$$

where $\rho \in [0,1]$, $\zeta \in \mathbb{C} \setminus \{0\}$ and $\mathcal{Q}_q^{\eta,r,k}f(z)$ defined in (1.4).

Example 2.3.

(i) For $k=0$, we get the subfamily $\mathcal{FK}_q(\eta,r,s,0;\rho,\zeta)=\mathcal{FK}_q(\eta,r,s;\rho,\zeta)$ defined by

$$1+\frac{1}{\zeta}\left(\frac{z\big(\mathcal{Q}_q f(z)\big)'+\rho z^2\big(\mathcal{Q}_q f(z)\big)''}{(1-\rho)\mathcal{Q}_q f(z)+\rho z\big(\mathcal{Q}_q f(z)\big)'}-1\right)\prec \mathcal{G}_{m,n}(z)-1$$

and

$$1+\frac{1}{\zeta}\left(\frac{z\big(\mathcal{Q}_q f^{-1}(w)\big)'+\rho z^2\big(\mathcal{Q}_q f^{-1}(w)\big)''}{(1-\rho)\mathcal{Q}_q f^{-1}(w)+\rho z\big(\mathcal{Q}_q f^{-1}(w)\big)'}-1\right)\prec \mathcal{G}_{m,n}(w)-1,$$

where $\rho\in[0,1]$, $\zeta\in\mathbb{C}\setminus\{0\}$ and $\Upsilon(j):=\frac{1}{[j]_q!}=\sigma_j$.

(ii) For $\rho=1$, we get the subfamily $\mathcal{FK}_q(\eta,r,k;1,\zeta)=\mathcal{FK}_q(\eta,r,k;\zeta)$ defined by

$$1+\frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)''}{\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)'}\right)\prec \mathcal{G}_{m,n}(z)-1 \quad\text{and}\quad 1+\frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k}f^{-1}(w)\right)''}{\left(\mathcal{Q}_q^{\eta,r,k}f^{-1}(w)\right)'}\right)\prec \mathcal{G}_{m,n}(w)-1,$$

where $\zeta\in\mathbb{C}\setminus\{0\}$ and $\mathcal{Q}_q^{\eta,r,k}f(z)$ defined in (1.4).

Definition 2.4. A function $f\in\Sigma_{\mathbb{U}}$ is said to be in the family $\mathcal{JF}_q(\eta,r,k;\rho,\zeta)$ if the next conditions hold:

$$1+\frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)'}{(1-\rho)\mathcal{Q}_q^{\eta,r,k}f(z)+\rho z\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)'}-1\right)\prec \mathcal{G}_{m,n}(z)-1$$

and

$$1+\frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,m}f^{-1}(w)\right)'}{(1-\rho)\mathcal{Q}_q^{\eta,r,m}f^{-1}(w)+\rho z\left(\mathcal{Q}_q^{\eta,r,m}f^{-1}(w)\right)'}-1\right)\prec \mathcal{G}_{m,n}(w)-1,$$

where $\rho\in[0,1)$, $\zeta\in\mathbb{C}\setminus\{0\}$ and $\mathcal{Q}_q^{\eta,r,k}f(z)$ defined in (1.4).

Example 2.5. For $\rho=0$, we get the subfamily $\mathcal{JF}_q(\eta,r,k;0,\zeta)$ defined by

$$1+\frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k}f(z)\right)'}{\mathcal{Q}_q^{\eta,r,k}f(z)}-1\right)\prec \mathcal{G}_{m,n}(z)-1 \quad\text{and}\quad 1+\frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k}f^{-1}(w)\right)'}{\mathcal{Q}_q^{\eta,r,k}f^{-1}(w)}-1\right)\prec \mathcal{G}_{m,n}(w)-1,$$

where $\zeta\in\mathbb{C}\setminus\{0\}$ and $\mathcal{Q}_q^{\eta,r,k}f(z)$ defined in (1.4).

Theorem 2.6. *If $f\in\mathcal{FK}_q(\eta,r,k;\rho,\zeta)$ has the form (1.1), then*

$$|a_2|\leqslant\min\{A;B\},\quad |a_3|\leqslant\frac{m^2(x)\,|\zeta|}{(1+\rho)^2\,|\Upsilon^2(2)|}+\frac{|m(x)|\,|\zeta|}{2(1+2\rho)\,|\Upsilon(3)|},$$

where

$$A:=\frac{|m(x)|\,\sqrt{|m(x)|\,|\zeta|}}{\sqrt{2\left|m^2(x)\Big((1+2\rho)\Upsilon(3)-(1+\rho)^2\Upsilon^2(2)\Big)-2n(x)(1+\rho)^2\Upsilon^2(2)\right|}},\qquad B:=\frac{|\zeta|\,m^2(x)}{(1+\rho)^2\,|\Upsilon^2(2)|},$$

$\rho\in[0,1]$, $\zeta\in\mathbb{C}\setminus\{0\}$, *and Υ is defined by* (1.5).

Proof. If we suppose that $f \in \mathcal{FK}_q(\eta, r, k; \rho, \zeta)$, then there exist two analytic functions ϕ and φ, with $\phi(0) = \varphi(0) = 0$, $|\phi(z)| < 1$, $|\varphi(w)| < 1$, for all $z, w \in \mathbb{U}$, such that

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k} f(z)\right)' + \rho z^2\left(\mathcal{Q}_q^{\eta,r,k} f(z)\right)''}{(1-\rho)\mathcal{Q}_q^{\eta,r,k} f(z) + \rho z\left(\mathcal{Q}_q^{\eta,r,k} f(z)\right)'} - 1\right) = \mathcal{G}_{m,n}\left(\phi(z)\right) - 1,\ z \in \mathbb{U}, \tag{2.1}$$

and

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k} f^{-1}(w)\right)' + \rho z^2\left(\mathcal{Q}_q^{\eta,r,k} f^{-1}(w)\right)'}{(1-\rho)\mathcal{Q}_q^{\eta,r,k} f^{-1}(w) + \rho z\left(\mathcal{Q}_q^{\eta,r,k} f^{-1}(w)\right)'} - 1\right) = \mathcal{G}_{m,n}\left(\varphi(w)\right) - 1,\ w \in \mathbb{U}, \tag{2.2}$$

where ϕ and φ have the form

$$\phi(z) = \sigma_1 z + \sigma_2 z^2 + \sigma_3 z^3 + \cdots,\ z \in \mathbb{U}, \qquad \varphi(w) = l_1 w + l_2 w^2 + l_3 w^3 + \cdots,\ w \in \mathbb{U}. \tag{2.3}$$

Since

$$\mathcal{G}_{m,n}\left(\phi(z)\right) - 1 = -1 + \mathcal{L}_{m,n,0}(x) + \mathcal{L}_{m,n,1}(x)\phi(z) + \mathcal{L}_{m,n,2}(x)\phi^2(z) + \cdots,\ z \in \mathbb{U}, \tag{2.4}$$

$$\mathcal{G}_{m,n}\left(\varphi(w)\right) - 1 = -1 + \mathcal{L}_{m,n,0}(x) + \mathcal{L}_{m,n,1}(x)\varphi(w) + \mathcal{L}_{m,n,2}(x)\varphi^2(w) + \cdots,\ w \in \mathbb{U}, \tag{2.5}$$

equating the first two corresponding coefficients of the equalities (2.1) and (2.2), we get

$$\frac{(1+\rho)\Upsilon(2)a_2}{\zeta} = \mathcal{L}_{m,n,1}(x)\sigma_1, \tag{2.6}$$

$$\frac{2(1+2\rho)\Upsilon(3)a_3 - (1+\rho)^2\Upsilon^2(2)a_2^2}{\zeta} = \mathcal{L}_{m,n,1}(x)\sigma_2 + \mathcal{L}_{m,n,2}(x)\sigma_1^2, \tag{2.7}$$

$$-\frac{(1+\rho)\Upsilon(2)a_2}{\zeta} = \mathcal{L}_{m,n,1}(x)l_1, \tag{2.8}$$

$$\frac{2(1+2\rho)\Upsilon(3)\left(2a_2^2 - a_3\right) - (1+\rho)^2\Upsilon^2(2)a_2^2}{\zeta} = \mathcal{L}_{m,n,1}(x)l_2 + \mathcal{L}_{m,n,2}(x)l_1^2. \tag{2.9}$$

From (2.6) and (2.8) we get

$$\sigma_1 = -l_1, \tag{2.10}$$

and

$$\frac{2(1+\rho)^2\Upsilon^2(2)a_2^2}{\zeta} = \mathcal{L}_{m,n,1}^2(x)\left(\sigma_1^2 + l_1^2\right). \tag{2.11}$$

Adding (2.7) and (2.9) we have

$$\frac{2\left[2(1+2\rho)\Upsilon(3) - (1+\rho)^2\Upsilon^2(2)\right]}{\zeta} a_2^2 = \mathcal{L}_{m,n,1}(x)(\sigma_2 + l_2) + \mathcal{L}_{m,n,2}(x)\left(\sigma_1^2 + l_1^2\right), \tag{2.12}$$

and if we substitute the value of $\left(\sigma_1^2 + l_1^2\right)$ from (2.11) into the relation (2.12), then

$$a_2^2 = \frac{\zeta\mathcal{L}_{m,n,1}^3(x)(\sigma_2 + l_2)}{2\left[2\mathcal{L}_{m,n,1}^2(x)\left((1+2\rho)\Upsilon(3) - (1+\rho)^2\Upsilon^2(2)\right) - 2\mathcal{L}_{m,n,2}(x)(1+\rho)^2\Upsilon^2(2)\right]}. \tag{2.13}$$

Using (1.6) and the first inequality of Lemma 2.1, from (2.13) it follows that

$$|a_2| = \sqrt{\left|\frac{\zeta\mathcal{L}_{m,n,1}^3(x)(\sigma_2 + l_2)}{2\left[2\mathcal{L}_{m,n,1}^2(x)\left((1+2\rho)\Upsilon(3) - (1+\rho)^2\Upsilon^2(2)\right) - 2\mathcal{L}_{m,n,2}(x)(1+\rho)^2\Upsilon^2(2)\right]}\right|},$$

therefore

$$|a_2| \leqslant \frac{|m(x)|\,\sqrt{|m(x)|\,|\zeta|}}{\sqrt{2\left|m^2(x)\Big((1+2\rho)\Upsilon(3)-(1+\rho)^2\Upsilon^2(2)\Big)-2n(x)(1+\rho)^2\Upsilon^2(2)\right|}} = A.$$

Another upper bound for $|a_2|$ could be obtained from (2.11), because it implies

$$a_2^2 = \frac{\zeta\mathcal{L}^2_{m,n,1}(x)\left(\sigma_1^2+l_1^2\right)}{2(1+\rho)^2\Upsilon^2(2)},$$

hence

$$|a_2| = \sqrt{\left|\frac{\zeta\mathcal{L}^2_{m,n,1}(x)\left(\sigma_1^2+l_1^2\right)}{2(1+\rho)^2\Upsilon^2(2)}\right|} \leqslant \frac{|\zeta|\,m^2(x)}{(1+\rho)^2\,|\Upsilon^2(2)|} = B.$$

Now, from the two upper bound values of $|a_2|$ we conclude that

$$|a_2| \leqslant \min\{A; B\}.$$

To get a majorant for $|a_3|$, using (2.10) and subtracting (2.9) from (2.7) we get

$$\frac{4(1+2\rho)\Upsilon(3)}{\zeta}\left(a_3-a_2^2\right) = \mathcal{L}_{m,n,1}(x)\left(\sigma_2-l_2\right)+\mathcal{L}_{m,n,2}(x)\left(\sigma_1^2-l_1^2\right). \tag{2.14}$$

Based on (2.10) and (2.14) it follows that

$$a_3 = \frac{\mathcal{L}^2_{m,n,1}(x)\zeta\left(\sigma_1^2+l_1^2\right)}{2(1+\rho)^2\Upsilon^2(2)} + \frac{\mathcal{L}_{m,n,1}(x)\zeta\left(\sigma_2-l_2\right)}{4(1+2\rho)\Upsilon(3)},$$

and

$$\begin{aligned} |a_3| &\leqslant \left|\frac{\mathcal{L}^2_{m,n,1}(x)\zeta\left(\sigma_1^2+l_1^2\right)}{2(1+\rho)^2\Upsilon^2(2)}\right| + \left|\frac{\mathcal{L}_{m,n,1}(x)\zeta\left(\sigma_2-l_2\right)}{4(1+2\rho)\Upsilon(3)}\right| \\ &\leqslant \left|\frac{\mathcal{L}^2_{m,n,1}(x)\zeta\cdot 2}{2(1+\rho)^2\Upsilon^2(2)}\right| + \left|\frac{\mathcal{L}_{m,n,1}(x)\zeta\cdot 2}{4(1+2\rho)\Upsilon(3)}\right| = \left|\frac{\mathcal{L}^2_{m,n,1}(x)\zeta}{(1+\rho)^2\Upsilon^2(2)}\right| + \left|\frac{\mathcal{L}_{m,n,1}(x)\zeta}{2(1+2\rho)\Upsilon(3)}\right|. \end{aligned}$$

Hence

$$|a_3| \leqslant \frac{m^2(x)\,|\zeta|}{(1+\rho)^2\,|\Upsilon^2(2)|} + \frac{|m(x)|\,|\zeta|}{2(1+2\rho)\,|\Upsilon(3)|}.$$

□

For the particular case $k=0$, Theorem 2.6 gives the following.

Corollary 2.7. *If* $f \in \mathcal{FK}_q(\eta, r, s, 0; \rho, \zeta)$ *has the form* (1.1)*, then*

$$|a_2| \leqslant \min\left\{\widehat{A}_1; \widehat{B}_1\right\}, \quad |a_3| \leqslant \frac{m^2(x)\,|\zeta|}{(1+\rho)^2\left|\sigma_2^2\right|} + \frac{|m(x)|\,|\zeta|}{2(1+2\rho)\,|\sigma_3|},$$

where

$$\widehat{A}_1 := \frac{|m(x)|\,\sqrt{|m(x)|\,|\zeta|}}{\sqrt{2\left|m^2(x)\Big((1+2\rho)\sigma_3-(1+\rho)^2\sigma_2^2\Big)-2n(x)(1+\rho)^2\sigma_2^2\right|}}, \qquad \widehat{B}_1 := \frac{|\zeta|\,m^2(x)}{(1+\rho)^2\left|\sigma_2^2\right|},$$

$\rho \in [0,1]$, $\zeta \in \mathbb{C}\setminus\{0\}$, *and* $\sigma_j := \dfrac{1}{[j]_q!}$, $j = 2, 3$.

If we take $\rho = 1$ in Theorem 2.6, we deduce the next special case.

Corollary 2.8. *If* $f \in \mathcal{FK}_q(\eta, r, k; \zeta)$ *has the form* (1.1)*, then*

$$|a_2| \leqslant \min\left\{\widehat{A}_2; \widehat{B}_2\right\}, \quad |a_3| \leqslant \frac{m^2(x)\,|\zeta|}{4\,|\Upsilon^2(2)|} + \frac{|m(x)|\,|\zeta|}{6\,|\Upsilon(3)|},$$

where

$$\widehat{A}_2 := \frac{|m(x)|\,\sqrt{|m(x)|\,|\zeta|}}{\sqrt{2\left|m^2(x)\left(3\Upsilon(3) - 2\Upsilon^2(2)\right) - 4n(x)\Upsilon^2(2)\right|}}, \qquad \widehat{B}_2 := \frac{|\zeta|\,m^2(x)}{4\,|\Upsilon^2(2)|},$$

$\zeta \in \mathbb{C} \setminus \{0\}$*, and* Υ *is defined by* (1.5)*.*

Theorem 2.9. *If* $f \in \mathcal{JF}_q(\eta, r, k; \rho, \zeta)$ *has the form* (1.1)*, then*

$$|a_2| \leqslant \min\{C; D\}, \quad |a_3| \leqslant \frac{m^2(x)\,|\zeta|}{(1-\rho)^2\,|\Upsilon^2(2)|} + \frac{|m(x)|\,|\zeta|}{2(1-\rho)\,|\Upsilon(3)|},$$

where

$$C := \frac{|m(x)|\,\sqrt{|m(x)|\,|\zeta|}}{\sqrt{2\left|m^2(x)\left((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\right) - 2n(x)(1-\rho)^2\Upsilon^2(2)\right|}}, \qquad D := \frac{|\zeta|\,m^2(x)}{(1-\rho)^2\,|\Upsilon^2(2)|},$$

$\rho \in [0, 1)$, $\zeta \in \mathbb{C} \setminus \{0\}$*, and* Υ *is defined by* (1.5)*.*

Proof. If we assume that $f \in \mathcal{JF}_q(\eta, r, k; \rho, \zeta)$, then there exist two analytic functions ϕ and φ, with $\phi(0) = \varphi(0) = 0$, $|\phi(z)| < 1$, $|\varphi(w)| < 1$, for all $z, w \in \mathbb{U}$, such that

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k} f(z)\right)'}{(1-\rho)\mathcal{Q}_q^{\eta,r,k} f(z) + \rho z\left(\mathcal{Q}_q^{\eta,r,k} f(z)\right)'} - 1\right) \prec \mathcal{G}_{m,n}\left(\phi(z)\right) - 1, \ z \in \mathbb{U}, \tag{2.15}$$

and

$$1 + \frac{1}{\zeta}\left(\frac{z\left(\mathcal{Q}_q^{\eta,r,k} f^{-1}(w)\right)'}{(1-\rho)\mathcal{Q}_q^{\eta,r,k} f^{-1}(w) + \rho z\left(\mathcal{Q}_q^{\eta,r,k} f^{-1}(w)\right)'} - 1\right) \prec \mathcal{G}_{m,n}\left(\varphi(w)\right) - 1, \ w \in \mathbb{U}, \tag{2.16}$$

where ϕ and φ have the form (2.3).

Since $\mathcal{G}_{m,n}\left(\phi(z)\right)$ and $\mathcal{G}_{m,n}\left(\varphi(w)\right)$ have the form (2.4) and (2.5), respectively, equating the first two corresponding coefficients of the equalities (2.15) and (2.16), we have

$$\frac{(1-\rho)\Upsilon(2)a_2}{\zeta} = \mathcal{L}_{m,n,1}(x)\sigma_1, \tag{2.17}$$

$$\frac{2(1-\rho)\Upsilon(3)a_3 - (1-\rho)^2\Upsilon^2(2)a_2^2}{\zeta} = \mathcal{L}_{m,n,1}(x)\sigma_2 + \mathcal{L}_{m,n,2}(x)\sigma_1^2, \tag{2.18}$$

$$-\frac{(1-\rho)\Upsilon(2)a_2}{\zeta} = \mathcal{L}_{m,n,1}(x)l_1, \tag{2.19}$$

$$\frac{2(1-\rho)\Upsilon(3)\left(2a_2^2 - a_3\right) - (1-\rho)^2\Upsilon^2(2)a_2^2}{\zeta} = \mathcal{L}_{m,n,1}(x)l_2 + \mathcal{L}_{m,n,2}(x)l_1^2. \tag{2.20}$$

From (2.17) and (2.19) we get

$$\sigma_1 = -l_1 \tag{2.21}$$

and

$$\frac{2(1-\rho)^2\Upsilon^2(2)a_2^2}{\zeta} = \mathcal{L}^2_{m,n,1}(x)\left(\sigma_1^2 + l_1^2\right). \tag{2.22}$$

Adding (2.18) to (2.20) we have

$$\frac{2\left[2(1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\right]}{\zeta} a_2^2 = \mathcal{L}_{m,n,1}(x)(\sigma_2 + l_2) + \mathcal{L}_{m,n,2}(x)\left(\sigma_1^2 + l_1^2\right), \tag{2.23}$$

and substituting the value of $\left(\sigma_1^2 + l_1^2\right)$ from (2.22) into the equation (2.23) it follows that

$$a_2^2 = \frac{\zeta\mathcal{L}^3_{m,n,1}(x)(\sigma_2 + l_2)}{2\left[2\mathcal{L}^2_{m,n,1}(x)\Big((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\Big) - 2\mathcal{L}_{m,n,2}(x)(1-\rho)^2\Upsilon^2(2)\right]}. \tag{2.24}$$

Using (1.6) and the first inequality of Lemma 2.1, from (2.24) it follows that

$$|a_2| = \sqrt{\left|\frac{\zeta\mathcal{L}^3_{m,n,1}(x)(\sigma_2 + l_2)}{2\left[2\mathcal{L}^2_{m,n,1}(x)\Big((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\Big) - 2\mathcal{L}_{m,n,2}(x)(1-\rho)^2\Upsilon^2(2)\right]}\right|},$$

and

$$|a_2| \leqslant \frac{|m(x)|\ \sqrt{|m(x)|\ |\zeta|}}{\sqrt{2\left|m^2(x)\Big((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\Big) - 2n(x)(1-\rho)^2\Upsilon^2(2)\right|}} = C.$$

Another upper bound for $|a_2|$ could be obtained from (2.22), since this relation implies

$$a_2^2 = \frac{\zeta\mathcal{L}^2_{m,n,1}(x)\left(\sigma_1^2 + l_1^2\right)}{2(1-\rho)^2\Upsilon^2(2)},$$

thus

$$|a_2| = \sqrt{\left|\frac{\zeta\mathcal{L}^2_{m,n,1}(x)\left(\sigma_1^2 + l_1^2\right)}{2(1-\rho)^2\Upsilon^2(2)}\right|} \leqslant \frac{|\zeta|\ m^2(x)|}{(1-\rho)^2\ |\Upsilon^2(2)|} = D,$$

and we conclude that

$$|a_2| \leqslant \min\{C; D\}.$$

To get a majorant for $|a_3|$, using (2.21) and subtracting (2.20) from (2.18) we get

$$\frac{4(1-\rho)\Upsilon(3)}{\zeta}(a_3 - a_2^2) = \mathcal{L}_{m,n,1}(x)\left(\sigma_2 - l_2\right) + \mathcal{L}_{m,n,2}(x)\left(\sigma_1^2 - l_1^2\right). \tag{2.25}$$

From (2.21) and (2.25) it follows that

$$a_3 = \frac{\mathcal{L}^2_{m,n,1}(x)\zeta\left(\sigma_1^2 + l_1^2\right)}{2(1-\rho)^2\Upsilon^2(2)} + \frac{\mathcal{L}_{m,n,1}(x)\zeta\left(\sigma_2 - l_2\right)}{4(1-\rho)\Upsilon(3)},$$

then

$$|a_3| \leqslant \left|\frac{\mathcal{L}^2_{m,n,1}(x)\zeta\left(\sigma_1^2 + l_1^2\right)}{2(1-\rho)^2\Upsilon^2(2)}\right| + \left|\frac{\mathcal{L}_{m,n,1}(x)\zeta\left(\sigma_2 - l_2\right)}{4(1-\rho)\Upsilon(3)}\right|$$

$$\leqslant \left|\frac{\mathcal{L}^2_{m,n,1}(x)\zeta\cdot 2}{2(1-\rho)^2\Upsilon^2(2)}\right| + \left|\frac{\mathcal{L}_{m,n,1}(x)\zeta\cdot 2}{4(1-\rho)\Upsilon(3)}\right| = \left|\frac{\mathcal{L}^2_{m,n,1}(x)\zeta}{(1-\rho)^2\Upsilon^2(2)}\right| + \left|\frac{\mathcal{L}_{m,n,1}(x)\zeta}{2(1-\rho)\Upsilon(3)}\right|,$$

that is

$$|a_3| \leqslant \frac{m^2(x)\,|\zeta|}{(1-\rho)^2\,|\Upsilon^2(2)|} + \frac{|m(x)|\,|\zeta|}{2(1-\rho)\,|\Upsilon(3)|}.$$

□

If we take $\rho = 0$ in Theorem 2.9, we obtain the following special case.

Corollary 2.10. *If $f \in \mathcal{JF}_q(\eta, r, k; 0, \zeta)$ has the form (1.1), then*

$$|a_2| \leqslant \min\left\{\widetilde{C}; \widetilde{D}\right\}, \quad |a_3| \leqslant \frac{m^2(x)\,|\zeta|}{|\Upsilon^2(2)|} + \frac{|m(x)|\,|\zeta|}{2\,|\Upsilon(3)|},$$

where

$$\widetilde{C} := \frac{|m(x)|\,\sqrt{|m(x)|\,|\zeta|}}{\sqrt{2\left|m^2(x)\left(\Upsilon(3)-\Upsilon^2(2)\right) - 2n(x)\Upsilon^2(2)\right|}}, \qquad \widetilde{D} := \frac{|\zeta|\,m^2(x)}{|\Upsilon^2(2)|},$$

$\zeta \in \mathbb{C}\setminus\{0\}$ and Υ defined by (1.5).

2.1. *Fekete-Szegö problem for the families $\mathcal{FK}_q(\eta, r, k; \rho, \zeta)$ and $\mathcal{JF}_q(\eta, r, k; \rho, \zeta)$*

This part determines upper bounds for the Fekete-Szegö functional $\left|a_3 - \xi a_2^2\right|$ for the classes $\mathcal{FK}_q(\eta, r, k; \rho, \zeta)$ and $\mathcal{JF}_q(\eta, r, k; \rho, \zeta)$.

Theorem 2.11. *Let $f \in \mathcal{FK}_q(\eta, r, k; \rho, \zeta)$ and $\xi \in \mathbb{R}$. Then,*

$$\left|a_3 - \xi a_2^2\right| \leqslant \begin{cases} \dfrac{|m(x)|}{2(1+2\rho)\Upsilon(3)}, & \text{if } |\xi - 1| \leqslant \dfrac{1}{(1+2\rho)\Upsilon(3)}\left|\Omega^{\rho,1}_{m,n}\right|, \\[2ex] \dfrac{|1-\xi|\,|m^3(x)|}{2\left|m^2(x)\left((1+2\rho)\Upsilon(3)-(1+\rho)^2\Upsilon^2(2)\right) - 2n(x)(1+\rho)^2\Upsilon^2(2)\right|}, & \text{if } |\xi - 1| \geqslant \dfrac{1}{(1+2\rho)\Upsilon(3)}\left|\Omega^{\rho,1}_{m,n}\right|, \end{cases}$$

where

$$\Omega^{\rho,1}_{m,n} := 2\left((1+2\rho)\Upsilon(3) - (1+\rho)^2\Upsilon^2(2)\right) - 2\frac{n(x)}{m^2(x)}(1+\rho)^2\Upsilon^2(2),$$

$\rho \in [0,1]$, $\zeta \in \mathbb{C}\setminus\{0\}$, and Υ is defined by (1.5).

Proof. If $f \in \mathcal{FK}_q(\eta, r, k; \rho, \zeta)$, then from the relations (2.13) and (2.14) it follows that

$$\begin{aligned} a_3 - \xi a_2^2 &= \frac{\zeta\mathcal{L}^3_{m,n,1}(x)(1-\xi)(\sigma_2 + l_2)}{2\left[2\mathcal{L}^2_{m,n,1}(x)\left((1+2\rho)\Upsilon(3)-(1+\rho)^2\Upsilon^2(2)\right) - 2\mathcal{L}_{m,n,2}(x)(1+\rho)^2\Upsilon^2(2)\right]} - \frac{\mathcal{L}_{m,n,1}(x)\zeta\,(\sigma_2 - l_2)}{4(1+2\rho)\Upsilon(3)} \\ &= \mathcal{L}_{m,n,1}(x)\left\{\left(\Phi^{\xi}_1(x) + \frac{1}{4(1+2\rho)\Upsilon(3)}\right)\sigma_2 + \left(\Phi^{\xi}_1(x) - \frac{1}{4(1+2\rho)\Upsilon(3)}\right) l_2\right\}, \end{aligned}$$

where

$$\Phi^{\xi}_1(x) := \frac{\mathcal{L}^2_{m,n,1}(x)\zeta(1-\xi)}{2\left[2\mathcal{L}^2_{m,n,1}(x)\left((1+2\rho)\Upsilon(3)-(1+\rho)^2\Upsilon^2(2)\right) - 2\mathcal{L}_{m,n,2}(x)(1+\rho)^2\Upsilon^2(2)\right]\,a_2^2}.$$

Thus, according to the relation (2.8) and using (1.6) we conclude that

$$\left|a_3 - \xi a_2^2\right| \leqslant \begin{cases} \dfrac{|m(x)|}{2(1+2\rho)\Upsilon(3)}, & \text{if } \left|\Phi^{\xi}_1(x)\right| \leqslant \dfrac{1}{4(1+2\rho)\Upsilon(3)}, \\[2ex] 2\,|m(x)|\left|\Phi^{\xi}_1(x)\right|, & \text{if } \left|\Phi^{\xi}_1(x)\right| \geqslant \dfrac{1}{4(1+2\rho)\Upsilon(3)}, \end{cases}$$

that is equivalent to our result. □

Theorem 2.12. *Let* $f \in \mathcal{JF}_q(\eta, r, k; \rho, \zeta)$ *and* $\xi \in \mathbb{R}$. *Then,*

$$\left|a_3 - \xi a_2^2\right| \leqslant \begin{cases} \dfrac{|m(x)|}{2(1-\rho)\Upsilon(3)}, & \text{if } |\xi - 1| \leqslant \dfrac{1}{(1-\rho)\Upsilon(3)}\left|\Omega_{m,n}^{\rho,2}\right|, \\ \dfrac{|1-\xi|\,|m^3(x)|}{2\left|m^2(x)\left((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\right) - 2n(x)(1-\rho)^2\Upsilon^2(2)\right|}, & \text{if } |\xi - 1| \geqslant \dfrac{1}{(1-\rho)\Upsilon(3)}\left|\Omega_{m,n}^{\rho,2}\right|, \end{cases}$$

where

$$\Omega_{m,n}^{\rho,2} := 2\left((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\right) - 2\frac{n(x)}{m^2(x)}(1-\rho)^2\Upsilon^2(2),$$

$\rho \in [0, 1)$, $\zeta \in \mathbb{C} \setminus \{0\}$, *and* Υ *is defined by* (1.5).

Proof. If $f \in \mathcal{JF}_q(\eta, r, k; \rho, \zeta)$, then using the relations (2.24) and (2.25) we get

$$\begin{aligned} a_3 - \xi a_2^2 &= \frac{\zeta \mathcal{L}_{m,n,1}^3(x)(1-\xi)(\sigma_2 + l_2)}{2\left[2\mathcal{L}_{m,n,1}^2(x)\left((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\right) - 2\mathcal{L}_{m,n,2}(x)(1-\rho)^2\Upsilon^2(2)\right]} - \frac{\mathcal{L}_{m,n,1}(x)\zeta(\sigma_2 - l_2)}{4(1-\rho)\Upsilon(3)} \\ &= \mathcal{L}_{m,n,1}(x)\left\{\left(\Phi_2^{\xi}(x) + \frac{1}{4(1-\rho)\Upsilon(3)}\right)\sigma_2 + \left(\Phi_2^{\xi}(x) - \frac{1}{4(1-\rho)\Upsilon(3)}\right) l_2\right\}, \end{aligned}$$

where

$$\Phi_2^{\xi}(x) := \frac{\mathcal{L}_{m,n,1}^2(x)\zeta(1-\xi)}{2\left[2\mathcal{L}_{m,n,1}^2(x)\left((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\right) - 2\mathcal{L}_{m,n,2}(x)(1-\rho)^2\Upsilon^2(2)\right] a_2^2}.$$

Using now the relations (2.8) and (1.6) we obtain that

$$|a_3 - \xi a_2^2| \leqslant \begin{cases} \dfrac{|m(x)|}{2(1-\rho)\Upsilon(3)}, & \text{if } \left|\Phi_2^{\xi}(x)\right| \leqslant \dfrac{1}{4(1-\rho)\Upsilon(3)}, \\ 2\,|m(x)|\left|\Phi_2^{\xi}(x)\right|, & \text{if } \left|\Phi_2^{\xi}(x)\right| \geqslant \dfrac{1}{4(1-\rho)\Upsilon(3)}, \end{cases}$$

that represents our result. □

If we take $k = 0$ in Theorem 2.11 and $\rho = 0$ in Theorem 2.12, then we obtain the next special cases, respectively.

Corollary 2.13. *If* $f \in \mathcal{FK}_q(\eta, r, s, 0; \rho, \zeta)$ *and* $\xi \in \mathbb{R}$, *then*

$$|a_3 - \xi a_2^2| \leqslant \begin{cases} \dfrac{|m(x)|}{2(1+2\rho)\sigma_3}, & \text{if } |\xi - 1| \leqslant \dfrac{1}{(1+2\rho)\sigma_3}\left|\Omega_{m,n}^{\rho,3}\right|, \\ \dfrac{|1-\xi|\,|m^3(x)|}{2\left|m^2(x)\left((1+2\rho)\sigma_3 - (1+\rho)^2\sigma_2^2\right) - 2n(x)(1+\rho)^2\sigma_2^2\right|}, & \text{if } |\xi - 1| \geqslant \dfrac{1}{(1+2\rho)\sigma_3}\left|\Omega_{m,n}^{\rho,3}\right|, \end{cases}$$

where

$$\Omega_{m,n}^{\rho,3} := 2\left((1+2\rho)\sigma_3 - (1+\rho)^2\sigma_2^2\right) - 2\frac{n(x)}{m^2(x)}(1+\rho)^2\sigma_2^2,$$

$\rho \in [0, 1]$, $\zeta \in \mathbb{C} \setminus \{0\}$, *and* $\sigma_j := \dfrac{1}{[j]_q!}$, $j = 2, 3$.

Corollary 2.14. *If* $f \in \mathcal{JF}_q(\eta, r, k; 0, \zeta)$ *and* $\xi \in \mathbb{R}$, *then*

$$|a_3 - \xi a_2^2| \leqslant \begin{cases} \dfrac{|m(x)|}{2\Upsilon(3)}, & \text{if } |\xi - 1| \leqslant \dfrac{1}{\Upsilon(3)}\left|\Omega_{m,n}^{\rho,4}\right|, \\ \dfrac{|1-\xi|\,|m^3(x)|}{2\left|m^2(x)\left(\Upsilon(3) - \Upsilon^2(2)\right) - 2n(x)\Upsilon^2(2)\right|}, & \text{if } |\xi - 1| \geqslant \dfrac{1}{\Upsilon(3)}\left|\Omega_{m,n}^{\rho,4}\right|, \end{cases}$$

where

$$\Omega^{\rho,4}_{m,n} := 2\Big(\Upsilon(3) - \Upsilon^2(2)\Big) - 2\frac{n(x)}{m^2(x)}\Upsilon^2(2),$$

and Υ *is defined by* (1.5).

2.2. Application of functions defined by Pascal distribution

The power series whose coefficients are probabilities of the Pascal distribution have been studied by El-Deeb et al. [16]. Using the function

$$\Phi^{\mathfrak{r}}_{\theta}(z) := z + \sum_{j=2}^{\infty}\binom{j+\mathfrak{r}-2}{\mathfrak{r}-1}\theta^{j-1}(1-\theta)^{\mathfrak{r}}z^j, \quad (\mathfrak{r} \geqslant 1,\ 0 \leqslant \theta \leqslant 1\ z \in \mathbb{U}),$$

and motivated by this function, Murugusundaramoorthy et al. [28] investigated the convolution operator $\mathcal{W}^{\mathfrak{r}}_{\theta} : \mathcal{S} \to \mathcal{S}$ formulated by

$$\mathcal{W}^{\mathfrak{r}}_{\theta}f(z) := \Phi^{\mathfrak{r}}_{\theta}(z) * f(z) = z + \sum_{j=2}^{\infty}\binom{j+\mathfrak{r}-2}{\mathfrak{r}-1}\theta^{j-1}(1-\theta)^{\mathfrak{r}}a_j z^j, \quad (z \in \mathbb{U}),$$

where $f(z)$ in the formula (1.1). Since

$$\mathcal{W}^{\mathfrak{r}}_{\theta}f(z) = z + \sum_{j=2}^{\infty}\binom{j+\mathfrak{r}-2}{\mathfrak{r}-1}\theta^{j-1}(1-\theta)^{\mathfrak{r}}a_j z^j = z + \mathfrak{r}\,\theta(1-\theta)^{\mathfrak{r}}a_2z^2 + \frac{\mathfrak{r}(\mathfrak{r}+1)}{2}\,\theta^2(1-\theta)^{\mathfrak{r}}a_3z^3 + \cdots \ (z \in \mathbb{U}),$$

if we replace in the Theorems 2.11, 2.12, and in Corollaries 2.13 and 2.14 the function f by $\mathcal{W}^{\mathfrak{r}}_{\theta}f$ defined above, then we obtain the subsequent outcomes.

Theorem 2.15. *Let* $\mathcal{W}^{\mathfrak{r}}_{\theta}f \in \mathcal{FK}_q(\eta, r, k; \rho, \zeta)$ *and* $\xi \in \mathbb{R}$, *where* $f \in \mathcal{S}$ *has the form* (1.1), *then*

$$\left|(\mathfrak{r}+1)\,\theta a_3 - 2\xi\,\mathfrak{r}(1-\theta)^{\mathfrak{r}}\,a_2^2\right|$$
$$\leqslant \frac{2}{\mathfrak{r}\,\theta\,(1-\theta)^{\mathfrak{r}}}\cdot\begin{cases}\dfrac{|m(x)|}{2(1+2\rho)\Upsilon(3)}, & \text{if } |\xi-1| \leqslant \dfrac{1}{(1+2\rho)\Upsilon(3)}\left|\Omega^{\rho,1}_{m,n}\right|,\\[2ex] \dfrac{|1-\xi|\,\left|m^3(x)\right|}{2\left|m^2(x)\Big((1+2\rho)\Upsilon(3)-(1+\rho)^2\Upsilon^2(2)\Big)-2n(x)(1+\rho)^2\Upsilon^2(2)\right|}, & \text{if } |\xi-1| \geqslant \dfrac{1}{(1+2\rho)\Upsilon(3)}\left|\Omega^{\rho,1}_{m,n}\right|,\end{cases}$$

where

$$\Omega^{\rho,1}_{m,n} := 2\Big((1+2\rho)\Upsilon(3) - (1+\rho)^2\Upsilon^2(2)\Big) - 2\frac{n(x)}{m^2(x)}(1+\rho)^2\Upsilon^2(2),$$

Υ *is defined by* (1.5), $\rho \in [0,1]$, $\zeta \in \mathbb{C}\setminus\{0\}$, *and* $0 \leqslant \theta < 1$.

Theorem 2.16. *Let* $\mathcal{W}^{\mathfrak{r}}_{\theta}f \in \mathcal{JF}_q(\eta, r, k; \rho, \zeta)$ *and* $\xi \in \mathbb{R}$, *where* $f \in \mathcal{S}$ *has the form* (1.1). *Then,*

$$\left|(\mathfrak{r}+1)\,\theta a_3 - 2\xi\,\mathfrak{r}(1-\theta)^{\mathfrak{r}}\,a_2^2\right|$$
$$\leqslant \frac{2}{\mathfrak{r}\,\theta\,(1-\theta)^{\mathfrak{r}}}\cdot\begin{cases}\dfrac{|m(x)|}{2(1-\rho)\Upsilon(3)}, & \text{if } |\xi-1| \leqslant \dfrac{1}{(1-\rho)\Upsilon(3)}\left|\Omega^{\rho,2}_{m,n}\right|,\\[2ex] \dfrac{|1-\xi|\,\left|m^3(x)\right|}{2\left|m^2(x)\Big((1-\rho)\Upsilon(3)-(1-\rho)^2\Upsilon^2(2)\Big)-2n(x)(1-\rho)^2\Upsilon^2(2)\right|}, & \text{if } |\xi-1| \geqslant \dfrac{1}{(1-\rho)\Upsilon(3)}\left|\Omega^{\rho,2}_{m,n}\right|,\end{cases}$$

where

$$\Omega^{\rho,2}_{m,n} := 2\Big((1-\rho)\Upsilon(3) - (1-\rho)^2\Upsilon^2(2)\Big) - 2\frac{n(x)}{m^2(x)}(1-\rho)^2\Upsilon^2(2),$$

Υ *is defined by* (1.5), $\rho \in [0,1)$, $\zeta \in \mathbb{C}\setminus\{0\}$, *and* $0 \leqslant \theta < 1$.

Corollary 2.17. *If* $\mathcal{W}_\theta^{\mathfrak{r}} \in \mathcal{FK}_q(\eta, r, s, 0; \rho, \zeta)$ *and* $\xi \in \mathbb{R}$, *where* $f \in \mathcal{S}$ *has the form* (1.1), *then*

$$\left|(\mathfrak{r}+1)\,\theta a_3 - 2\xi\,\mathfrak{r}(1-\theta)^{\mathfrak{r}}\,a_2^2\right|$$
$$\leqslant \frac{2}{\mathfrak{r}\,\theta\,(1-\theta)^{\mathfrak{r}}}\cdot\begin{cases}\dfrac{|m(x)|}{2(1+2\rho)\sigma_3}, & \text{if } |\xi-1| \leqslant \dfrac{1}{(1+2\rho)\sigma_3}\left|\Omega_{m,n}^{\rho,3}\right|,\\[2ex] \dfrac{|1-\xi|\left|m^3(x)\right|}{2\left|m^2(x)\left((1+2\rho)\sigma_3-(1+\rho)^2\sigma_2^2\right)-2n(x)(1+\rho)^2\sigma_2^2\right|}, & \text{if } |\xi-1| \geqslant \dfrac{1}{(1+2\rho)\sigma_3}\left|\Omega_{m,n}^{\rho,3}\right|,\end{cases}$$

where

$$\Omega_{m,n}^{\rho,3} := 2\left((1+2\rho)\sigma_3-(1+\rho)^2\sigma_2^2\right)-2\frac{n(x)}{m^2(x)}(1+\rho)^2\sigma_2^2,$$

$\sigma_j := \dfrac{1}{[j]_q!}$, $j = 2, 3$, $\zeta \in \mathbb{C}\setminus\{0\}$, *and* $0 \leqslant \theta < 1$.

Corollary 2.18. *If* $\mathcal{W}_\theta^{\mathfrak{r}} \in \mathcal{JF}_q(\eta, r, k; 0, \zeta)$ *and* $\xi \in \mathbb{R}$, *where* $f \in \mathcal{S}$ *has the form* (1.1), *then*

$$\left|(\mathfrak{r}+1)\,\theta a_3 - 2\xi\,\mathfrak{r}(1-\theta)^{\mathfrak{r}}\,a_2^2\right|$$
$$\leqslant \frac{2}{\mathfrak{r}\,\theta\,(1-\theta)^{\mathfrak{r}}}\cdot\begin{cases}\dfrac{|m(x)|}{2\Upsilon(3)}, & \text{if } |\xi-1| \leqslant \dfrac{1}{\Upsilon(3)}\left|\Omega_{m,n}^{\rho,4}\right|,\\[2ex] \dfrac{|1-\xi|\left|m^3(x)\right|}{2\left|m^2(x)\left(\Upsilon(3)-\Upsilon^2(2)\right)-2n(x)\Upsilon^2(2)\right|}, & \text{if } |\xi-1| \geqslant \dfrac{1}{\Upsilon(3)}\left|\Omega_{m,n}^{\rho,4}\right|,\end{cases}$$

where

$$\Omega_{m,n}^{\rho,4} := 2\left(\Upsilon(3)-\Upsilon^2(2)\right)-2\frac{n(x)}{m^2(x)}\Upsilon^2(2),$$

Υ *is defined by* (1.5), $\zeta \in \mathbb{C}\setminus\{0\}$, *and* $0 \leqslant \theta < 1$.

Remark 2.19. Setting $q \to 1-$, $r = k = s = \eta = 1$, and replacing $\mathcal{G}_{m,n}(z) - 1$ by $\Phi(z)$, such that $\Phi(0) = 1$ and $\Phi'(0) \geqslant 0$, in Theorems 2.6 and 2.11, then we deduce Theorems 2.3 and 2.4, that have been investigated by Kazımoğlu and Deniz [25].

3. Conclusion

Recently, many studies have focused on introducing new concepts by using q-calculus, due to its wide applications in physics, mathematics and other fields. We used the q-calculus to define a new q-differential operator that involves the q-binomial series, which is a generalization to an operator provided by Frasin [19]. With the aid of this operator and by using the concept of subordination, we defined two families of functions $\mathcal{FK}_q(\eta, r, k; \rho, \zeta)$ and $\mathcal{JF}_q(\eta, r, k; \rho, \zeta)$. These subordinations are connected with the function $\mathcal{G}_{m,n}$, where some special polynomials can be obtained from the (m, n)-Lucas Polynomial like the Lucas polynomials.

We found estimates for the first two coefficients for the functions of classes, followed by some special cases obtained for special values of the parameters. Also, we determined estimates for the Fekete-Szegö inequality for these families and some special cases of the main results are given.

The importance of these results consists in the next facts: the new differential operator $\mathcal{L}_\eta^{r,q,k}$ connected with the q-binomial coefficients extend some operators previously defined by other authors, and the subordinations used to define the new classes which are connected to the well-known (m, n)-Lucas polynomial like the Lucas polynomials. The results we obtained are simple to use, giving an upper bound for the modules of the first two coefficients, and for the Fekete-Szegö functional for these new classes.

We consider that those who are interested in the topics of Geometric Function Theory of one variable functions connected with special functions could use these results for some possible studies.

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