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A class of nonlinear delay integral inequalities for two-variable functions and their applications in Volterra integral equations

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Abstract

In this paper, we establish some new nonlinear integral inequalities with delay in two independent variables which generalize some known inequalities recently obtained. These results can be used as handy tools to study the boundedness of solutions of Volterra integral equations. An application is given to illustrate how our results can be applied to study the boundedness of the solutions of certain Volterra equations.

Keywords: Delay integral inequality, two-variable functions, properties of solutions, Volterra integral equation.

2020 MSC: 26D10, 26D15, 45D05, 45G10.

1. Introduction

Gronwall-Bellman inequalities [5, 10] and their various generalizations [4, 12, 15, 16] that give explicit bounds on unknown functions provide a very useful and important device in the study of many qualitative as well as quantitative properties of solutions of differential equations, integral equations and integro-differential equations.

Recently, many new results on the nonlinear retarded integral inequalities can be found in [3, 7, 9, 11, 14, 17] and the references given therein.

Theorem 1.1 ([18])**.** *Let* $\varphi, \varphi', \alpha \in C^1(I, I)$ *be increasing functions with* $\varphi'(t) \leqslant k, \alpha(t) \leqslant t, \alpha(0) = 0, \forall t \in I$; k, u_0 *be positive constants, we assume that* $u(t)$ *and* $f(t)$ *are nonnegative real-valued continuous functions defined on* I *and satisfy the inequality*

$$u(t) \leqslant u_0 + \left[\int_0^{\alpha(t)} f(s)\varphi(u(s))ds\right]^2 + \int_0^{\alpha(t)} f(s)\varphi(u(s)) \left[\varphi(u(s)) + 2\int_0^s f(\tau)\varphi(u(\tau))d\tau\right] ds,$$

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for all $t \in I$. *If* $u_0^{-1} - k\int_0^{\alpha(t)} f(s)\exp\left(4\int_0^s f(\tau)d\tau\right)ds > 0$, *then*

$$u(t) \leqslant \Phi^{-1}\left(\Phi(u_0 + \int_0^{\alpha(t)} f(s)B_3(s)ds\right), \forall t \in I,$$

where

$$\Phi(x) = \int_1^x \frac{ds}{\varphi(s)}, \forall x > 0,$$

$$B_3(t) = \exp\left(4\int_0^{\alpha(t)} f(s)ds\right)\left(\varphi(u_0)^{-1} - k\int_0^{\alpha(t)} f(s)\exp\left(4\int_0^s f(\tau)d\tau\right)ds\right)^{-1}.$$

Theorem 1.2 ([1]). *Let* $u(t), g(t), f(t) \in C(I, I), \alpha(t) \in C^1(I, I)$ *be nondecreasing with* $\alpha(t) \leqslant t$ *on* I. *If the inequality*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)\left[u^{(2-p)}(s) + \int_0^s g(\lambda)u^q(\lambda)d\lambda\right]^p ds, \forall t \in I,$$

holds, where $u_0 > 0$ *and* $0 < p \leqslant 1, 0 \leqslant q < 1$ *are constants, then*

$$u(t) \leqslant u_0 + \int_0^{\alpha(t)} f(s)k_1\left(\alpha^{-1}(s)\right)\exp\left(p(2-p)\int_0^s f(\lambda)d\lambda\right)ds, \forall t \in I,$$

where

$$k_1(t) = \left[u_0^{(1-q)(2-p)} + (1-q)\int_0^{\alpha(t)} g(s)\exp\left(-(1-q)(2-p)\int_0^s f(\lambda)d\lambda\right)ds\right]^{\frac{p}{1-q}}.$$

Theorem 1.3 ([6]). *Let* $u, f, g \in C(\Delta, R_+)$, *and* $c(x,y) \in C(\Delta, R_+^*)$ *be nondecreasing with respect to* $(x,y) \in \Delta$, *let* $\alpha \in C^1(I_1, I_1), \beta \in C^1(I_2, I_2)$ *be nondecreasing with* $\alpha(x) \leqslant x$ *on* I_1, $\beta(y) \leqslant y$ *on* I_2. *Further* $\psi, \varphi \in C(R_+, R_+)$ *be nondecreasing functions with* $\{\psi, \varphi\}(u) > 0$ *for* $u > 0$, *and* $\lim_{u\to+\infty}\psi(u) = +\infty$. *If* u *satisfies*

$$\begin{aligned}\psi(u(x,y)) \leqslant a(x,y) &+ \left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t))\,dtds\right)^2 \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t))\left(\int_0^s g(\tau,t)\varphi(u(\tau,t))\,d\tau\right)dtds,\end{aligned}$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leqslant \psi^{-1}\left\{H^{-1}\left[H(a(x,y)) + B(x,y) + \left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2\right]\right\},$$

for $0 \leqslant x \leqslant x_1, 0 \leqslant y \leqslant y_1$, *where*

$$B(x,y) = \int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\left(\int_0^s g(\tau,t)d\tau\right)dtds,$$

$$H(r) = \int_{r_0}^r \frac{ds}{\left(\varphi\circ\psi^{-1}\right)^2(s)}, r \geqslant r_0 > 0, \qquad H(+\infty) = \int_{r_0}^{+\infty} \frac{ds}{\left(\varphi\circ\psi^{-1}\right)^2(s)} = +\infty,$$

and $(x_1, y_1) \in \Delta$ *is chosen so that* $\left(H(a(x,y)) + B(x,y) + \left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2\right) \in \mathrm{Dom}\left(H^{-1}\right)$.

Motivated by the results above and the inequalities obtained very recently in [8, 13] we give a generalization of nonlinear retarded integral inequalities in two independent variables of Gronwall-Bellman type which can be used as a handy tool to study the boundedness of solutions of Volterra integral equations.

2. Main results

In what follows, R denotes the set of real numbers, $R_+ = [0, +\infty)$, $R_+^* = (0, +\infty)$, $I_1 = [0, M]$, $I_2 = [0, N]$ are the given subsets of R, and $\Delta = I_1 \times I_2$. $C^i(A, B)$ denotes the class of all i times continuously differentiable functions defined on set A with range in set B ($i = 1, 2, \dots$) and $C^0(A, B) = C(A, B)$.

Theorem 2.1. *Let $u(x,y), f(x,y) \in C(\Delta, R_+)$, and $c(x,y) \in C(\Delta, R_+^*)$ be nondecreasing with respect to $(x,y) \in \Delta$, let $\alpha \in C^1(I_1, I_1), \beta \in C^1(I_2, I_2)$ be nondecreasing with $\alpha(x) \leqslant x$ on I_1, $\beta(y) \leqslant y$ on I_2. Further $\psi, \varphi \in C(R_+, R_+)$ be nondecreasing functions with $\{\psi, \varphi\}(u) > 0$ for $u > 0$, and $\lim\limits_{u \to +\infty} \psi(u) = +\infty$. If $u(x,y)$ satisfies*

$$\psi(u(x,y)) \leqslant c(x,y) + \left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\varphi(u(s,t))\, dtds\right)^2 + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\varphi(u(s,t)) \left[\varphi(u(s,t)) + 2\int_0^s \int_0^t f(\tau,\sigma)\varphi(u(\tau,\sigma))\, d\sigma d\tau\right] dtds,$$

for $(x,y) \in \Delta$, and

$$F(r) = \int_{r_0}^{r} \frac{ds}{\varphi^2\left(\psi^{-1}(s)\right)}, \quad F(+\infty) = \int_{r_0}^{+\infty} \frac{ds}{\varphi^2\left(\psi^{-1}(s)\right)} = +\infty, \tag{2.1}$$

for $r \geqslant r_0 > 0$, then

$$u(x,y) \leqslant \psi^{-1}\left\{F^{-1}\left[F(c(x,y)) + \left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) dtds\right)^2 + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\left[1 + 2\int_0^s \int_0^t f(\tau,\sigma)\, d\sigma d\tau\right] dtds\right]\right\}, \tag{2.2}$$

for $0 \leqslant x \leqslant x_1, 0 \leqslant y \leqslant y_1$, where $(x_1, y_1) \in \Delta$ is chosen so that

$$F(c(x,y)) + \left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) dtds\right)^2 + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\left[1 + 2\int_0^s \int_0^t f(\tau,\sigma)\, d\sigma d\tau\right] dtds \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Fixing an arbitrary $(X,Y) \in \Delta$, we define a positive and nondecreasing function $z(x,y)$ by

$$z(x,y) = c(X,Y) + \left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\varphi(u(s,t))\, dtds\right)^2 + \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\varphi(u(s,t)) \left[\varphi(u(s,t)) + 2\int_0^s \int_0^t f(\tau,\sigma)\varphi(u(\tau,\sigma))\, d\sigma d\tau\right] dtds$$

for $0 \leqslant x \leqslant X \leqslant x_1, 0 \leqslant y \leqslant Y \leqslant y_1$, then $z(0,y) = z(x,0) = c(X,Y)$ and

$$u(x,y) \leqslant \psi^{-1}(z(x,y)), \tag{2.3}$$

then we have

$$\frac{\partial z}{\partial x} \leqslant 2\left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}(z(s,t))\right) dtds\right) \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}(z(\alpha(x),t))\right) dt + \alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}(z(\alpha(x),t))\right)\left[\varphi\left(\psi^{-1}(z(\alpha(x),t))\right)\right.$$

$$
\begin{aligned}
&+2\int_0^{\alpha(x)}\int_0^{t} f(\tau,\sigma)\,\varphi\left(\psi^{-1}\left(z(\tau,\sigma)\right)\right)d\sigma d\tau\Big]\,dt\\
&\leqslant \varphi^2\left(\psi^{-1}\left(z(\alpha(x),\beta(y))\right)\right)\left\{2\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)dt\right.\\
&\left.+\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)\left[1+2\int_0^{\alpha(x)}\int_0^{t} f(\tau,\sigma)\,d\sigma d\tau\right]dt\right\},
\end{aligned}
$$

or

$$
\frac{\frac{\partial z(x,y)}{\partial x}}{\varphi^2\left(\psi^{-1}\left(z(x,y)\right)\right)} \leqslant \frac{\partial}{\partial x}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2+\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)\left[1+2\int_0^{\alpha(x)}\int_0^{t} f(\tau,\sigma)\,d\sigma d\tau\right]dt.
$$

In the last inequality we replace x by s, keeping y fixed, integrating the last inequality with respect to s from 0 to x, making the change of variable $s=\alpha(x)$ and using (2.1), we get

$$
F\left(z(x,y)\right)\leqslant F\left(c\left(X,Y\right)\right)+\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2+\int_0^{\alpha(x)}\int_0^{\beta(y)} f\left(s,t\right)\left[1+2\int_0^{s}\int_0^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds.
$$

Since $(X,Y)\in\Delta$ is chosen arbitrary, then

$$
\begin{aligned}
z(x,y)\leqslant F^{-1}\Bigg[&F\left(c\left(x,y\right)\right)+\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2\\
&+\int_0^{\alpha(x)}\int_0^{\beta(y)} f\left(s,t\right)\left[1+2\int_0^{s}\int_0^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds\Bigg].
\end{aligned}
\tag{2.4}
$$

From (2.3) and (2.4) we obtain the desired inequality (2.2). □

Remark 2.2. If $c\left(x,y\right)=u_0>0$ is a constant, $\psi\left(u\right)=\varphi(u)=u$, $\alpha(x)=x$, $\beta(y)=y$, and y is fixed, then Theorem 2.1 reduces to Theorem 2.1 in [2].

Remark 2.3. If $c\left(x,y\right)=u_0>0$ is a constant, $\psi\left(u\right)=u$, and y is fixed, then Theorem 2.1 reduces to Theorem 1.1.

Corollary 2.4. *Let* $u(x,y), f(x,y), c(x,y), \alpha, \beta$ *be as in Theorem 2.1 and* $p\in(0,1)$ *is a constant. If* $u(x,y)$ *satisfies*

$$
\begin{aligned}
u^{p+1}(x,y)\leqslant c\left(x,y\right)&+\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)u^p(s,t)dtds\right)^2\\
&+2\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)u^p(s,t)\left[u^p(s,t)+\int_0^{s}\int_0^{t} f\left(\tau,\sigma\right)u^p(\tau,\sigma)d\sigma d\tau\right]dtds,
\end{aligned}
$$

for $(x,y)\in\Delta$, *then*

$$
\begin{aligned}
z(x,y)\leqslant\Bigg[&\left(c(x,y)\right)^{\frac{1-p}{1+p}}+\frac{1-p}{1+p}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2\\
&+\frac{(1-p)}{1+p}\int_0^{\alpha(x)}\int_0^{\beta(y)} f\left(s,t\right)\left[1+2\int_0^{s}\int_0^{t} f\left(\tau,\sigma\right)d\sigma d\tau\right]dtds\Bigg]^{\frac{1}{1-p}}.
\end{aligned}
\tag{2.5}
$$

Proof. An application of Theorem 2.1 with

$$\psi^{-1}(u)=u^{\frac{1}{1+p}}, F(r)=\frac{1+p}{1-p}\left[r^{\frac{1-p}{1+p}}-r_0^{\frac{1-p}{1+p}}\right], F^{-1}(r)=\left[r_0^{\frac{1-p}{1+p}}+\frac{1-p}{1+p}r\right]^{\frac{1+p}{1-p}}, r\geqslant r_0>0,$$

yields the desired inequality in (2.5). □

Remark 2.5. In special cases the corollary 2.4 reduces to Theorem 3.4 in [2].

Corollary 2.6. *Let* $u(x,y), f(x,y), c(x,y), \alpha$, *and* β *be as in Theorem* 2.1, $p>2q>0$ *are constants. If* $u(x,y)$ *satisfies*

$$\begin{aligned} u^p(x,y)\leqslant c(x,y)&+\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)u^q(s,t)dtds\right)^2\\ &+\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)u^q(s,t)\left[u^q(s,t)+\int_0^s\int_0^t f(\tau,\sigma)u^q(\tau,\sigma)d\sigma d\tau\right]dtds,\end{aligned}$$

for $(x,y)\in\Delta$, *then*

$$\begin{aligned} u(x,y)\leqslant\Bigg\{&(c(x,y))^{\frac{p-2q}{p}}+\frac{p-2q}{p}\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2\\ &+\frac{p-2q}{p}\int_0^{\alpha(x)}\int_{0)}^{\beta(y)} f(s,t)\left[1+\int_0^s\int_0^t f(\tau,\sigma)d\sigma d\tau\right]dtds\Bigg\}^{\frac{1}{p-2q}}.\end{aligned} \tag{2.6}$$

Proof. An application of Theorem 2.1 with

$$\psi^{-1}(u)=u^{\frac{1}{p}}, F(r)=\frac{p}{p-2q}\left[r^{\frac{p-2q}{p}}-r_0^{\frac{p-2q}{p}}\right], F^{-1}(r)=\left[r_0^{\frac{p-2q}{p}}+\frac{p-2q}{p}r\right]^{\frac{p}{p-2q}},$$

for $r\geqslant r_0>0$, yields the desired inequality in (2.6). □

Theorem 2.7. *Let* $u(x,y), f(x,y), c(x,y), \alpha, \beta, \varphi$, *and* ψ *be as in Theorem* 2.1. *Let* $d(x,y), h(x,y), g(x,y)\in C(\Delta, R_+)$. *If* $u(x,y)$ *satisfies*

$$\begin{aligned} \psi(u(x,y))\leqslant c(x,y)&+\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi(u(s,t))dtds\right)^2\\ &+\int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\varphi(u(s,t))\left[d(s,t)+\int_0^s h(\tau,t)\varphi(u(\tau,t))d\tau\right]dtds,\end{aligned} \tag{2.7}$$

for $(x,y)\in\Delta$, *and*

$$\Phi(r)=\int_{r_0}^r\frac{ds}{\varphi(\psi^{-1}(s))},\quad \Phi(+\infty)=\int_{r_0}^{+\infty}\frac{ds}{\varphi(\psi^{-1}(s))}=+\infty, \tag{2.8}$$

$$\Omega(r)=\int_{r_0}^r\frac{ds}{\varphi\left(\psi^{-1}(\Phi^{-1}(s))\right)},\quad \Omega(+\infty)=\int_{r_0}^{+\infty}\frac{ds}{\varphi\left(\psi^{-1}(\Phi^{-1}(s))\right)}=+\infty, \tag{2.9}$$

for $r\geqslant r_0>0$, *then*

$$\begin{aligned} u(x,y)\leqslant\psi^{-1}\Bigg\{\Phi^{-1}\Bigg[\Omega^{-1}\Bigg(&\Omega(A(x,y))+2\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2\\ &+\int_{0)}^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\left(\int_0^s h(\tau,t)d\tau\right)dtds\Bigg)\Bigg]\Bigg\},\end{aligned} \tag{2.10}$$

where

$$A(x,y) = \Phi\big(c(x,y)\big) + \int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)d\,(s,t)\,dtds,$$

for $0 \leqslant x \leqslant x_1, 0 \leqslant y \leqslant y_1$, *and* $(x_1,y_1) \in \Delta$ *is chosen so that*

$$\left(\Omega\,(A\,(x,y)) + 2\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2 + \int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\left(\int_0^{s} h\,(\tau,t)\,d\tau\right)dtds\right) \in \mathrm{Dom}\left(\Omega^{-1}\right).$$

Proof. Fixing an arbitrary $(X,Y) \in \Delta$, define a positive and nondecreasing function $z(x,y)$ by

$$\begin{aligned} z\,(x,y) = c\,(X,Y) &+ \left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi\,(u(s,t))\,dtds\right)^2 \\ &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\varphi\,(u(s,t))\left[d(s,t) + \int_0^{s} h\,(\tau,t)\,\varphi\,(u(\tau,t))\,d\tau\right]dtds, \end{aligned}$$

for $0 \leqslant x \leqslant X \leqslant x_1, 0 \leqslant y \leqslant Y \leqslant y_1$, then $z\,(0,y) = z\,(x,0) = c\,(X,Y)$ and

$$u\,(x,y) \leqslant \psi^{-1}\,(z\,(x,y))\,, \tag{2.11}$$

then we have

$$\begin{aligned} \frac{\partial z}{\partial x} \leqslant{} & 2\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\,(z(s,t))\right)dtds\right)\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)\varphi\left(\psi^{-1}\,(z(\alpha(x),t))\right)dt \\ &+ \alpha'(x)\int_0^{\beta(y)} g(\alpha(x),t)\varphi\left(\psi^{-1}\,(z(\alpha(x),t))\right)\left[d\,(\alpha(x),t) + \int_0^{\alpha(x)} h\,(\tau,t)\,\varphi\left(\psi^{-1}\,(z(\tau,t))\right)d\tau\right]dt, \end{aligned}$$

or

$$\begin{aligned} \frac{\frac{\partial z(x,y)}{\partial x}}{\varphi\,(\psi^{-1}\,(z(x,y)))} \leqslant{} & 2\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\,(z(s,t))\right)dtds\right)\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)dt \\ &+ \alpha'(x)\int_0^{\beta(y)} g(\alpha(x),t)\left[d\,(\alpha(x),t) + \int_0^{\alpha(x)} h\,(\tau,t)\,\varphi\left(\psi^{-1}\,(z(\tau,t))\right)d\tau\right]dt. \end{aligned}$$

Since $\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\varphi\left(\psi^{-1}\,(z(s,t))\right)dtds$ is a nonegative and nondecreasing function with respect to $(x,y) \in \Delta$, then we get

$$\begin{aligned} \frac{\frac{\partial z(x,y)}{\partial x}}{\varphi\,(\psi^{-1}\,(z(x,y)))} \leqslant{} & 2\left(\int_0^{\alpha(X)}\int_0^{\beta(Y)} f(s,t)\varphi\left(\psi^{-1}\,(z(s,t))\right)dtds\right)\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)dt \\ &+ \alpha'(x)\int_0^{\beta(y)} g(\alpha(x),t)\left[d\,(\alpha(x),t) + \int_0^{\alpha(x)} h\,(\tau,t)\,\varphi\left(\psi^{-1}\,(z(s,t))\right)d\tau\right]dt, \end{aligned}$$

for $0 \leqslant x \leqslant X \leqslant x_1, 0 \leqslant y \leqslant Y \leqslant y_1$. Keeping y fixed, replace x by s and integrating the last inequality with respect to s from 0 to x, making the change of variable $s = \alpha(x)$ and using (2.8) we get

$$\begin{aligned} \Phi\,(z(x,y)) \leqslant \Phi\,(c\,(X,Y)) &+ \int_0^{\alpha(x)}\int_0^{\beta(y)} g(s,t)\left[d\,(s,t) + \int_0^{s} h\,(\tau,t)\,\varphi\left(\psi^{-1}\,(z(\tau,t))\right)d\tau\right]dtds \\ &+ 2\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\left(\int_0^{\alpha(X)}\int_0^{\beta(Y)} f(s,t)\varphi\left(\psi^{-1}\,(z(s,t))\right)dtds\right). \end{aligned}$$

Since $(X, Y) \in \Delta$ is chosen arbitrary, then

$$\begin{aligned}\Phi\left(z(x,y)\right) \leqslant \Phi\left(c\left(x,y\right)\right) &+ \int_0^{\alpha(x)} \int_0^{\beta(y)} g(s,t) d\left(s,t\right) dtds \\ &+ \int_0^{\alpha(x)} \int_0^{\beta(y)} g(s,t) \left(\int_0^s h\left(\tau,t\right) \varphi\left(\psi^{-1}\left(z(\tau,t)\right)\right) d\tau\right) dtds \\ &+ 2\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) dtds \left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) \varphi\left(\psi^{-1}\left(z(s,t)\right)\right) dtds\right),\end{aligned}$$

or

$$\begin{aligned}\Phi\left(z(x,y)\right) \leqslant A\left(x,y\right) &+ \int_0^{\alpha(x)} \int_0^{\beta(y)} g(s,t) \left(\int_0^s h\left(\tau,t\right) \varphi\left(\psi^{-1}\left(z(\tau,t)\right)\right) d\tau\right) dtds \\ &+ k(x,y) \int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) \varphi\left(\psi^{-1}\left(z(s,t)\right)\right) dtds,\end{aligned}$$

where $k\left(x,y\right) = 2\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) dtds$ is a nonnegative and nondecreasing function with respect to $(x,y) \in \Delta$, then for $0 \leqslant x \leqslant X \leqslant x_1, 0 \leqslant y \leqslant Y \leqslant y_1$ we get

$$\begin{aligned}\Phi\left(z(x,y)\right) \leqslant A\left(X,Y\right) + k\left(X,Y\right) &\left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) \varphi\left(\psi^{-1}\left(z(s,t)\right)\right) dtds\right) \\ &+ \int_0^{\alpha(x)} \int_0^{\beta(y)} g(s,t) \left(\int_0^s h\left(\tau,t\right) \varphi\left(\psi^{-1}\left(z(\tau,t)\right)\right) d\tau\right) dtds.\end{aligned}$$

Define a positive and nondecreasing function $\nu\left(x,y\right)$ by

$$\begin{aligned}\nu\left(x,y\right) = A\left(X,Y\right) + k\left(X,Y\right) &\left(\int_0^{\alpha(x)} \int_0^{\beta(y)} f(s,t) \varphi\left(\psi^{-1}\left(z(s,t)\right)\right) dtds\right) \\ &+ \int_0^{\alpha(x)} \int_0^{\beta(y)} g(s,t) \left(\int_0^s h\left(\tau,t\right) \varphi\left(\psi^{-1}\left(z(\tau,t)\right)\right) d\tau\right) dtds,\end{aligned}$$

for $0 \leqslant x \leqslant X \leqslant x_1, 0 \leqslant y \leqslant Y \leqslant y_1$, then $\nu\left(0,y\right) = A\left(X,Y\right)$ and

$$z(x,y) \leqslant \Phi^{-1}\left(\nu\left(x,y\right)\right), \tag{2.12}$$

and

$$\begin{aligned}\frac{\partial \nu}{\partial x} &\leqslant k\left(X,Y\right) \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),t) \varphi\left(\psi^{-1}\left(\Phi^{-1}\left(\nu\left(\alpha(x),t\right)\right)\right)\right) dt \\ &\quad + \alpha'(x) \int_0^{\beta(y)} g(\alpha(x),t) \left(\int_0^{\alpha(x)} h\left(\tau,t\right) \varphi\left(\psi^{-1}\left(\Phi^{-1}\left(\nu\left(\tau,t\right)\right)\right)\right) d\tau\right) dt, \\ &\leqslant \varphi\left(\psi^{-1}\left(\Phi^{-1}\left(\nu\left(\alpha(x),\beta(y)\right)\right)\right)\right) \left\{k\left(X,Y\right) \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),t) dt \right. \\ &\quad \left. + \alpha'(x) \int_0^{\beta(y)} g(\alpha(x),t) \left(\int_0^{\alpha(x)} h\left(\tau,t\right) d\tau\right) dt\right\},\end{aligned}$$

or

$$\frac{\frac{\partial \nu}{\partial x}}{\varphi\left(\psi^{-1}\left(\Phi^{-1}\left(\nu\left(x,y\right)\right)\right)\right)} \leqslant k\left(X,Y\right) \alpha'(x) \int_0^{\beta(y)} f(\alpha(x),t) dt + \alpha'(x) \int_0^{\beta(y)} g(\alpha(x),t) \left(\int_0^{\alpha(x)} h\left(\tau,t\right) d\tau\right) dt.$$

Keeping y fixed, replace x by s and integrating the last inequality with respect to s from 0 to x, making the change of variable $s = \alpha(x)$ and using (2.9) we get

$$\Omega\left(v\left(x,y\right)\right) \leqslant \Omega\left(A\left(X,Y\right)\right) + k\left(X,Y\right)\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)dtds + \int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} g(s,t)\left(\int_{0}^{s} h\left(\tau,t\right)d\tau\right)dtds.$$

Since $(X,Y) \in \Delta$ is chosen arbitrary, then we get

$$v\left(x,y\right) \leqslant \Omega^{-1}\left(\Omega\left(A\left(x,y\right)\right) + 2\left(\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)dtds\right)^{2} + \int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} g(s,t)\left(\int_{0}^{s} h\left(\tau,t\right)d\tau\right)dtds\right). \tag{2.13}$$

From (2.11)-(2.13) we obtain (2.10). The proof is complete. □

Remark 2.8. If $d(x,y) = 0$ for all $(x,y) \in \Delta$, then Theorem 2.7 reduces to Theorem 1.3.

Corollary 2.9. *Let* $u, f, c, \alpha,$ *and* β *be as in Theorem* 2.1, $k \in C(\Delta, R_{+}^{*})$ *be nondecreasing with respect to* $(x,y) \in \Delta$, $p > 2q > 0$ *are constants. If* $u(x,y)$ *satisfies*

$$u^{p}(x,y) \leqslant c(x,y) + \left(\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)u^{q}(s,t)dtds\right)^{2} + k(x,y)\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)u^{q}(s,t)dtds,$$

for $(x,y) \in \Delta$, *then*

$$u(x,y) \leqslant \left\{\left(S\left(x,y\right)\right)^{\frac{p-2q}{p}} + \frac{2\left(p-2q\right)}{p}\left(\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)dtds\right)^{2}\right\}^{\frac{1}{p-2q}}, \tag{2.14}$$

where

$$S(x,y) = \left(c\left(x,y\right)\right)^{\frac{p-q}{p}} + \frac{p-q}{p}k(x,y)\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)dtds.$$

Proof. By the same steps given in the proof of Theorem 2.7 with $\psi\left(u\right) = u^{p}, \varphi\left(u\right) = u^{q}, g(x,y) = f(x,y), d(x,y) = 1, h(x,y) = 0$, and

$$u(x,y) \leqslant \left(z\left(x,y\right)\right)^{\frac{1}{p}}, \tag{2.15}$$
$$\varphi\left(\psi^{-1}\left(z\right)\right) = z^{\frac{q}{p}}, \Phi\left(z(x,y)\right) = \left(z(x,y)\right)^{\frac{p-q}{p}},$$

we obtain

$$\begin{aligned}\left(z(x,y)\right)^{\frac{p-q}{p}} \leqslant{}& \left(c\left(x,y\right)\right)^{\frac{p-q}{p}} + \frac{p-q}{p}k(x,y)\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)dtds\\ &+ 2\frac{p-q}{p}\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)dtds\left(\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)\left(z(x,y)\right)^{\frac{q}{p}}dtds\right),\end{aligned}$$

or

$$\left(z(x,y)\right)^{\frac{p-q}{p}} \leqslant S\left(x,y\right) + l(x,y)\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)\left(z(x,y)\right)^{\frac{q}{p}}dtds,$$

where $l\left(x,y\right) = 2\left(\frac{p-q}{p}\right)\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)dtds$ is a nonegative and nondecreasing function with respect to $(x,y) \in \Delta$, then for $0 \leqslant x \leqslant X \leqslant x_1, 0 \leqslant y \leqslant Y \leqslant y_1$ we get

$$\left(z(x,y)\right)^{\frac{p-q}{p}} \leqslant S(X,Y) + l(X,Y)\left(\int_{0}^{\alpha(x)}\int_{0}^{\beta(y)} f(s,t)\left(z(x,y)\right)^{\frac{q}{p}}dtds\right).$$

Define a positive and nondecreasing function $v(x,y)$ by

$$v(x,y) = S(X,Y) + l(X,Y)\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)\,(z(x,y))^{\frac{q}{p}}\,dtds\right),$$

for $0 \leqslant x \leqslant X \leqslant x_1, 0 \leqslant y \leqslant Y \leqslant y_1$, then $v(0,y) = S(X,Y)$ and

$$z(x,y) \leqslant (v(x,y))^{\frac{p}{p-q}}, \tag{2.16}$$

and

$$\begin{aligned}\frac{\partial v}{\partial x} &\leqslant l(X,Y)\,\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)\,(v(\alpha(x),t))^{\frac{q}{p-q}}\,dt\\ &\leqslant (v(\alpha(x),\beta(y)))^{\frac{q}{p-q}}\,l(X,Y)\,\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)dt,\end{aligned}$$

or

$$\frac{\frac{\partial v}{\partial x}}{(v(x,y))^{\frac{q}{p-q}}} \leqslant l(X,Y)\,\alpha'(x)\int_0^{\beta(y)} f(\alpha(x),t)dt.$$

Integrating the last inequality with respect to s from 0 to x, making the change of variable $s = \alpha(x)$ we get

$$(v(x,y))^{\frac{p-2q}{p-q}} \leqslant \Big(S(X,Y)\Big)^{\frac{p-2q}{p-q}} + \frac{p-2q}{p-q}\,l(X,Y)\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds.$$

Since $(X,Y) \in \Delta$ is chosen arbitrary, then we get

$$v(x,y) \leqslant \left(\Big(S(x,y)\Big)^{\frac{p-2q}{p-q}} + 2\Big(\frac{p-2q}{p}\Big)\left(\int_0^{\alpha(x)}\int_0^{\beta(y)} f(s,t)dtds\right)^2\right)^{\frac{p-q}{p-2q}}. \tag{2.17}$$

From (2.15)-(2.17) we obtain (2.14). □

3. Applications

We shall in this section illustrate how our results can be applied to study the boundedness of the solutions of certain Volterra equation.

Consider the following Volterra integral equation in two independent variables.

$$z^{\frac{p}{2}}(x,y) = h^p(x) + g^q(y) + \int_0^x\int_0^y F\Big(s,t,z\big(\alpha(s),\beta(t)\big)\Big)dtds, \tag{3.1}$$

where $p > 2q > 0$, $F \in C(\Delta \times R, R)$, $h \in C(I_1, R)$, $g \in C(I_2, R)$, and $\alpha \in C^1(I_1, R_+), \beta \in C^1(I_2, R_+)$ are nondecreasing functions such that $\alpha(x) \leqslant x$ on I_1, $\beta(y) \leqslant y$ on I_2.

Our result deals with the boundedness of solutions.

Theorem 3.1. *Consider the problem (3.1). If*

$$|F(x,y,v)| \leqslant b(x,y)\,|v|^q, \tag{3.2}$$

and

$$|h^p(x) + g^q(y)| \leqslant a(x,y), \tag{3.3}$$

where $b \in C(\Delta, R_+)$ *and* $a \in C(\Delta, R_+^*)$ *are nondecreasing with respect to* $(x, y) \in \Delta$*, then all solutions* $z(x, y)$ *of* (3.1) *satisfy*

$$|z(x,y)| \leqslant \left\{ \Big(S(x,y)\Big)^{\frac{p-2q}{p}} + \frac{2(p-2q)}{p} \left(\int_0^{\alpha(x)} \int_0^{\beta(y)} B(s,t)dtds \right)^2 \right\}^{\frac{1}{p-2q}}, \tag{3.4}$$

where

$$S(x,y) = \Big(a(x,y)\Big)^{\frac{2(p-q)}{p}} + \frac{2(p-q)}{p} a(x,y) \int_0^{\alpha(x)} \int_0^{\beta(y)} B(s,t)dtds, \tag{3.5}$$

and

$$B(x,y) = \frac{b\Big(\alpha^{-1}(x), \beta^{-1}(y)\Big)}{\alpha'\left(\alpha^{-1}(x)\right)\beta'\left(\beta^{-1}(y)\right)}. \tag{3.6}$$

Proof. From (3.1) and (3.3), we obtain

$$|z(x,y)|^p \leqslant (a(x,y))^2 + \left| \int_0^x \int_0^y F\Big(s,t,z(\alpha(s),\beta(t))\Big)dtds \right|^2 + 2a(x,y) \left| \int_0^x \int_0^y F\Big(s,t,z(\alpha(s),\beta(t))\Big)dtds \right|.$$

Hence by (3.2) we have

$$\begin{aligned} |z(x,y)|^p \leqslant (a(x,y))^2 &+ \left[\int_0^x \int_0^y b(s,t) \left|z\big(\alpha(s),\beta(t)\big)\right|^q dtds \right]^2 \\ &+ 2a(x,y) \int_0^x \int_0^y b(s,t) \left|z\big(\alpha(s),\beta(t)\big)\right|^q dtds, \end{aligned} \tag{3.7}$$

by a change of variables $\sigma = \alpha(s)$, $\tau = \beta(t)$, in (3.7) we have

$$\begin{aligned} |z(x,y)|^p \leqslant (a(x,y))^2 &+ \left[\int_0^{\alpha(x)} \int_0^{\beta(y)} \frac{b\Big(\alpha^{-1}(\sigma), \beta^{-1}(\tau)\Big)}{\alpha'\left(\alpha^{-1}(\sigma)\right)\beta'\left(\beta^{-1}(\tau)\right)} |z(\sigma,\tau)|^q d\tau d\sigma \right]^2 \\ &+ 2a(x,y) \int_0^{\alpha(x)} \int_0^{\beta(y)} \frac{b\Big(\alpha^{-1}(\sigma), \beta^{-1}(\tau)\Big)}{\alpha'\left(\alpha^{-1}(\sigma)\right)\beta'\left(\beta^{-1}(\tau)\right)} |z(\sigma,\tau)|^q d\tau d\sigma, \end{aligned}$$

or

$$|z(x,y)|^p \leqslant (a(x,y))^2 + \left[\int_0^{\alpha(x)} \int_0^{\beta(y)} B(\sigma,\tau) |z(\sigma,\tau)|^q d\tau d\sigma \right]^2 + 2a(x,y) \int_0^{\alpha(x)} \int_0^{\beta(y)} B(\sigma,\tau) |z(\sigma,\tau)|^q d\tau d\sigma,$$

where B is defined in (3.6). Now an application of corollary 2.9 to the last inequality gives the desired estimate of $|z(x,y)|$ in (3.4). □

Corollary 3.2. *Consider the problem* (3.1)*. If*

$$|F(x,y,v)| \leqslant b(x,y) |v|^p, \tag{3.8}$$

and

$$|h^p(x) + g^q(y)| \leqslant \frac{1}{2}, \tag{3.9}$$

then all solutions $z(x,y)$ *of* (3.1) *satisfy*

$$|z(x,y)| \leqslant \Big\{ \exp\Big(C(x,y)\Big) + A(x,y) \Big\}^{-\frac{1}{p}}, \tag{3.10}$$

where

$$A(x,y)=2\left(\int_0^{\alpha(x)}\int_0^{\beta(y)}B(s,t)dtds\right)^2,\quad C(x,y)=\log\frac{1}{4}+\int_0^{\alpha(x)}\int_0^{\beta(y)}B(s,t)dtds,$$

where B *is defined in* (3.6)*.*

Proof. By (3.1) and (3.9), we obtain

$$|z(x,y)|^p\leqslant\frac{1}{4}+\left|\int_0^x\int_0^y F\Big(s,t,z\big(\alpha(s),\beta(t)\big)\Big)dtds\right|^2+\left|\int_0^x\int_0^y F\Big(s,t,z\big(\alpha(s),\beta(t)\big)\Big)dtds\right|.$$

Hence by (3.8) we have

$$|z(x,y)|^p\leqslant\frac{1}{4}+\left[\int_0^x\int_0^y b(s,t)\left|z\big(\alpha(s),\beta(t)\big)\right|^p dtds\right]^2+\left|\int_0^x\int_0^y b(s,t)\left|z\big(\alpha(s),\beta(t)\big)\right|^p dtds\right|, \tag{3.11}$$

by a change of variables $\sigma=\alpha(s)$, $\tau=\beta(t)$, in (3.11) we have

$$\begin{aligned}|z(x,y)|^p&\leqslant\frac{1}{4}+\left[\int_0^{\alpha(x)}\int_0^{\beta(y)}\frac{b\Big(\alpha^{-1}(x),\beta^{-1}(y)\Big)}{\alpha'\big(\alpha^{-1}(x)\big)\beta'\big(\beta^{-1}(y)\big)}|z(\sigma,\tau)|^p\,d\sigma d\tau\right]^2\\&\quad+\int_0^{\alpha(x)}\int_0^{\beta(y)}\frac{b\Big(\alpha^{-1}(x),\beta^{-1}(y)\Big)}{\alpha'\big(\alpha^{-1}(x)\big)\beta'\big(\beta^{-1}(y)\big)}|z(\sigma,\tau)|^p\,d\sigma d\tau,\\&\leqslant\frac{1}{4}+\left[\int_0^{\alpha(x)}\int_0^{\beta(y)}B(\sigma,\tau)\,|z(\sigma,\tau)|^p\,d\sigma d\tau\right]^2+\int_0^{\alpha(x)}\int_0^{\beta(y)}B(\sigma,\tau)\,|z(\sigma,\tau)|^p\,d\sigma d\tau.\end{aligned}$$

An application of Theorem 2.7 (with $\varphi(u)=\psi(u)=u^p$, $g(s,t)=f(s,t)$, $d(s,t)=1$, $h(s,t)=0$, and $c(x,y)=\frac{1}{4}$) to the last inequality, now gives the assertion immediately in (3.10). In particular, if B is bounded on Δ, then every solution $z(x,y)$ of (3.1) is bounded on Δ. □

Remark 3.3. Our results in this work can be also applied to study the uniqueness and continuous dependence of the solutions of certain initial boundary value problems for hyperbolic partial differential equations given in (3.1).

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