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Hybrid differential inclusion with nonlocal, infinite-point or Riemann-Stieltjes integral boundary conditions

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Abstract

Here, we investigate the existence of solutions for the initial value problem of fractional-order differential inclusion containing nonlocal infinite-point or Riemann-Stieltjes integral boundary conditions. The sufficient condition for the uniqueness of the solution will be given. The continuous dependence of the solution will be studied. Finally, an example is provided to illustrate our results.

Keywords: Functional integro-differential inclusion, fixed point theorem, Riemann-Stieltjes integral boundary conditions, infinite-point boundary conditions.

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1. Introduction

Differential and integral equation models have appeared in a variety of applications (see [2, 3, 5, 7, 8, 14]). In physical sciences and applied mathematics, boundary value problems involving fractional differential equations occur. Subsidiary conditions are imposed locally in some of these issues. Nonlocal conditions are imposed in other cases. Nonlocal conditions are frequently preferable to local conditions because the measurements required by a nonlocal condition are sometimes more precise than the measurements provided by a local condition. As a result, a number of outstanding results on fractional boundary value problems (abbreviated BVPs) with resonant requirements have been obtained. Bai [4] investigated a class of fractional differential equations with m-point boundary conditions. Using the same technique Kosmatov [16] investigated the fractional order three points BVP with resonant case. Considering the fact that the study of fractional BVPs at resonance has yielded fruitful results, it should be highlighted that problems involving Riemann-Stieltjes integrals are very scarce, as a consequence, the Riemann-Stieltjes integral has been considered as both multipoint and integral, which is more frequent; see Ahmad et al. [11].

Some authors have investigated boundary value issues with nonlocal, integral, and infinite points boundary conditions (see,[7, 18–22]).

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In this paper, we study the existence of solutions for a hybrid differential inclusion of the form

$$^{c}D^{\eta}\left(\frac{\mu(\mathfrak{r})-\mu(0)}{\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r}))}\right)\in\Phi_1(\mathfrak{r},I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi_2(\mathfrak{r})))),\ \alpha\in(0,1),\ \mathfrak{r}\in(0,1], \tag{1.1}$$

equipped with Riemann-Stieltjes integro boundary conditions

$$\mu(0)+\int_0^1\mu(\sigma)dh(\sigma)=\mu_\circ, h:[0,1]\to\mathbb{R}\text{is nondecreasing function} \tag{1.2}$$

or the infinite-point boundary conditions given by with the nonlocal condition

$$\mu(0)+\sum_{k=1}^{\infty}a_k\mu(\tau_k)=\mu_\circ,\ a_k>0,\ \tau_k\in(0,1], \tag{1.3}$$

where $\mathfrak{r}\in I=[0,1]$,$0<\sigma\leqslant 1$, and $^{c}D^{\sigma}$ is the Caputo fractional derivative, and $\Phi_1:[0,1]\times\mathbb{R}^+\to P(\mathbb{R})$ is a set-valued mapping and $P(\mathbb{R})$ denotes the family of nonempty subsets of $\mathbb{R}$ under a set of several suitable assumptions on the function Φ_1. Our study is based on the selections of the set-valued function Φ_1 by reformulating the functional integral inclusion into a coupled system. We first find the continuous solution of the problem (1.1) with the m-point BCs given by

$$\mu(0)+\sum_{k=1}^{m}a_k\mu(\tau_k)=\mu_\circ,\ a_k>0,\ \tau_k\in[0,1], \tag{1.4}$$

and then, by using the properties of the Riemann sum for continuous functions, we investigate the solutions of the BVP with the Riemann-Stieltjes integral given by (1.1) and (1.2) as well as the BVP with infinite points given by (1.1) and (1.3). Our approach is based on Schauder fixed point theorem, many authors use fixed point theorems to prove the existence and the uniqueness of the solution to nonlinear fractional differential equations; (see[1, 2, 8–10, 12]).

The remainder of the paper is organized as follows. Section 2 contains our main finding for the problem (1.1)-(1.4). In light of the developments result, we investigate the BVP provided by (1.1)-(1.2) and by (1.1)-(1.3). We demonstrate sufficient conditions in each for the problem (1.1) under the Riemann-Stieltjes functional integral BC (1.2) and under infinite-point BC (1.3), while Section 3 covered the continuous dependency and the uniqueness of solutions. Examples are provided in Section 4 to illustrate how our findings can be applied. The last Section 5 discusses the conclusion.

2. Main result

In this section, cases are particularized for the existence of a continuous solution to our problem. One of them is endowed with a continuous solution to (1.1) using the m-Point BCs (1.4) and Riemann-Stieltjes integral BCs, (1.2) is the way to present the other case. Finally the results have been generalized for infinite-point boundary condition (1.3).

2.1. *Non-local condition* (1.4)

Take into account the following assumptions.

(i) Let $\Phi_1:I\times\mathbb{R}\to 2^{\mathbb{R}}$, satisfy the following assumptions:
   - (a) $\Phi_1(\mathfrak{r},\mu)$ is upper semicontinuous in μ for every $\mathfrak{r}\in I$;
   - (b) $\Phi_1(\mathfrak{r},\mu)$ is measurable in $\mathfrak{r}\in I$ for every $\mu\in\mathbb{R}$;
   - (c) the Lipschitzian set-valued map $\Phi_1:I\times\mathbb{R}\to 2^{\mathbb{R}}$ has a nonempty compact convex subset of $2^{\mathbb{R}}$, utilizing the Lipschitz constant $k>0$.

Note: The set of Lipschitz selections for Φ_1 is not empty and there exists $\phi_1 \in \Phi_1$, by [3, Sect.9, Chap. 1, Theorem 1] with

$$|\phi_1(\mathfrak{r},\mu)-\phi_1(\mathfrak{r},\nu)| \leqslant k|\mu-\nu|.$$

So, it is obvious that we have

$$|\phi_1(\mathfrak{r},\mu)| \leqslant k|\mu| + \phi_1^*, \text{where} \phi_1^* = \sup_{\mathfrak{r}\in[0,T]} |\phi_1(\mathfrak{r},0)|.$$

(ii) $\vartheta : I \times R \to R$ is measurable in $\mathfrak{r}$ for any $u \in R$ and Lipschitz in u for $\mathfrak{r} \in [0,1]$, and there exists a positive constant $k_1 > 0$ such that

$$|\vartheta(\mathfrak{r},u)-\vartheta(\mathfrak{r},\nu)| \leqslant k_1|u-\nu|.$$

So, it is obvious that we have

$$|\vartheta(\mathfrak{r},u)| \leqslant k_1|u(\mathfrak{r})| + \mathcal{V}, \text{where} \mathcal{V} = \sup_{\mathfrak{r}\in I} |\vartheta(\mathfrak{r},0)|.$$

(iii) The functions $\varphi_i : I \to I$ are continuous and $\varphi_i(\mathfrak{r}) \leqslant \mathfrak{r}, i = 1,2$.

(iv) The Caratheodory requirement is satisfied for function $\phi_2 : I \times \mathbb{R} \to \mathbb{R}$, i.e., ϕ_2 is measurable in $\mathfrak{r}$ for any $\mu \in \mathbb{R}$ and continuous in μ for almost all $\mathfrak{r} \in I$. There exist a function $a(\mathfrak{r})$ that is measurable bounded and there is a positive constant $b > 0$, with

$$|\phi_2(\mathfrak{r},\mu)| \leqslant a(\mathfrak{r}) + b|\mu|, \ \forall \mathfrak{r} \in I \text{and} \mu \in \mathbb{R}.$$

(v) The statement $1-[k_1\|\mu\|+\mathcal{V}][a\sum_{k=1}^{m}|a_k|+1]\frac{k}{\Gamma(\eta+1)} < 1$, $\frac{b}{\Gamma(\sigma+1)} < 1$, and $I_c^{\gamma} a(.) \leqslant M, \forall \gamma \leqslant \sigma, c \geqslant 0$ is true.

Lemma 2.1. *For any $\mu \in C(I,\mathbb{R})$, the solution of the the boundary single-valued problem*

$$D^{\eta}\left(\frac{\mu(\mathfrak{r})-\mu(0)}{\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r})))}\right) = \phi_1(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi_2(\mathfrak{r})))), \eta,\sigma \in (0,1), \mathfrak{r} \in I, \tag{2.1}$$

supplemented with the non-local condition (1.4), is equivalence to the integral equation

$$\begin{aligned}\mu(\mathfrak{r}) = a\left(\mu_{\circ} - \sum_{k=1}^{m} a_k\vartheta(\tau_k,\mu(\varphi_1(\tau_k)))\int_0^{\tau_k} \phi_1(\sigma, I^{\sigma}\phi_2(\sigma,\mu(\varphi_2(\sigma))))d\sigma\right)\\ +\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r})))\int_0^{\mathfrak{r}} \phi_1(\sigma, I^{\sigma}\phi_2(\sigma,\mu(\varphi_2(\sigma))))d\sigma,\end{aligned} \tag{2.2}$$

where $a = (1+\sum_{k=1}^{m} a_k)^{-1}$ holds.

Proof. We begin by considering the problem (2.1) with m-point BCs in (1.4). Integrating both sides of (2.1), we obtain

$$\mu(\mathfrak{r}) = \mu(0) + \vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r})) I^{\eta}\phi_1(\mathfrak{r}, I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi_2(\mathfrak{r})))).$$

When we substitute the value of $\mu(0)$ from (1.4), we get

$$\mu(\mathfrak{r}) = \mu_{\circ} - \sum_{k=1}^{m} a_k\mu(\tau_k) + \vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r})))\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^{\sigma}\phi_2(\sigma,\mu(\varphi_2(\sigma))))d\sigma. \tag{2.3}$$

In fact, when we set $\mathfrak{r} = \tau_k \in [0,T]$ in equation (2.3), we have

$$\mu(\tau_k) = \mu_{\circ} - \sum_{k=1}^{m} a_k\mu(\tau_k) + \vartheta(\tau_k,\mu(\varphi_1(\tau_k)))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^{\sigma}\phi_2(\sigma,\mu(\varphi_2(\sigma))))d\sigma.$$

So, we have

$$\begin{aligned}\mu(\tau_k) &= \mu(\mathfrak{r}) + \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma \\ &\quad - \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma.\end{aligned} \tag{2.4}$$

Substituting (2.4) in (2.3),

$$\begin{aligned}\mu(\mathfrak{r}) = \mu_\circ - \sum_{k=1}^{m} a_k(\mu(\mathfrak{r}) &+ \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))))ds \\ &+ \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))))ds) \\ &+ \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))))d\sigma.\end{aligned}$$

As a result, we obtain

$$\begin{aligned}(1+\sum_{k=1}^{m} a_k)\mu(\mathfrak{r}) &= \mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma \\ &\quad + (1+\sum_{k=1}^{m} a_k)\vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma.\end{aligned}$$

Letting $a = (1+\sum_{k=1}^{m} a_k)^{-1}$, we conclude that an integral equation can be derived from the non-local problem (1.1)-(1.4).

$$\begin{aligned}\mu(\mathfrak{r}) &= a\left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma\right) \\ &\quad + \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-s)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma.\end{aligned}$$

Additionally, we demonstrate that equation (2.2) meets problem (2.1), when combined with the m-point BCs in (1.4) in order to complete the preceding Lemma's proof. In particular, when differentiating (2.2) with respect to $\mathfrak{r}$, we obtain

$$D^\eta\left(\frac{\mu(\mathfrak{r})-\mu(0)}{\vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r}))}\right) = \phi_1(\mathfrak{r}, I^\sigma \phi_2(\mathfrak{r}, \mu(\varphi_2(\mathfrak{r})))).$$

As a result, we obtain

$$\begin{aligned}\mu(\tau_k) &= a\left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma)))ds\right) \\ &\quad + \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma, \\ (1+\sum_{k=1}^{m} a_k)\mu(\tau_k) &= x_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi)_2(\sigma))))d\sigma \\ &\quad + (1+\sum_{k=1}^{m} a_k)\vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma,\end{aligned} \tag{2.5}$$

$$\mu(\tau_k) + \sum_{k=1}^{m} a_k \mu(\tau_k) = \mu_\circ + \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma,$$

then

$$\sum_{k=1}^{m} a_k \mu(\tau_k) = \mu_\circ - \mu(\tau_k) + \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(s))))d\sigma. \tag{2.6}$$

From (2.2) we have

$$\mu(0) = a \left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k - s)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma \right).$$

Replace the value of $\mu(0)$ in (2.5),

$$\mu(\tau_k) = \mu(0) + \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma$$

and

$$\mu(0) = \mu(\tau_k) - \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k - s)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma. \tag{2.7}$$

We obtain m-point BC (1.4), by adding (2.6) and (2.7),

$$\mu(0) + \sum_{k=1}^{\infty} a_k \mu(\tau_k) = \mu_\circ.$$

□

It is obvious from assumption (i), the solution of the single-valued integral equation (2.2), where $\phi_1 \in S_{F_1}$, is a solution to the inclusion (1.1) with $\mu(0) + \sum_{k=1}^{m} a_k x(\tau_k) = \mu_\circ$.

Let's go on to the next step

$$\nu(\mathfrak{r}) = I^\sigma \phi_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))), \mathfrak{r} \in I. \tag{2.8}$$

The nonlinear functional integral equation (2.2) can thus be expressed as

$$\begin{aligned} \mu(\mathfrak{r}) = a &\left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma \right) \\ &+ \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma, \quad \mathfrak{r} \in I. \end{aligned} \tag{2.9}$$

As a result, the coupled system (2.8) and (2.9) and the functional integral equation (2.2) are equal.

Now, we investigate the existence of a continuous solution of the Eq. (2.2), that is a solution of inclusion (1.1) with nonlocal condition (1.4), by obtaining the continuous solution of the coupled system (2.8) and (2.9). Now for the existence of at least one solution $u = (\mu, \nu),\ \mu, \nu \in C(I)$ of the coupled system (2.8), (2.9) we have the following theorem.

Theorem 2.2. *Assume that assumptions* (i)-(iv) *hold. Then there exists at least one continuous solution* $u = (\mu, \nu), \mu, \nu \in C(I, \mathbb{R})$ *for the coupled system* (2.8), (2.9).

Proof. Let the set Q_r be defined as

$$Q_r = \{u = (\mu, \nu) \in R^2, \|u\| \leqslant r\},$$

where $r = r_1 + r_2 = \frac{a|\mu_\circ| + [k_1\|\mu\| + \mathcal{V}][a\sum_{k=1}^{m}|a_k|+1]\frac{\phi_1^*}{\Gamma(\eta+1)}}{1-[k_1\|\mu\|+\mathcal{V}][a\sum_{k=1}^{m}|a_k|+1]\frac{k}{\Gamma(\eta+1)}} + (1-\frac{b}{\Gamma(\sigma+1)})^{-1}\frac{M}{\Gamma(\sigma-\gamma+1)}$. It is clear that the set Q_r is nonempty, bounded, closed and convex.

Afterwards, let's indicate by A the operator defined on the space $C(I)$ by

$$\begin{aligned} Au(\mathfrak{r}) &= A(\mu,\nu)(\mathfrak{r}) = (A_1\nu(\mathfrak{r}), A_2\mu(\mathfrak{r})),\\ A_1\nu(\mathfrak{r}) &= a(\mu_\circ - \sum_{k=1}^{m} a_k\vartheta(\tau_k, \mu(\varphi_1(\tau_k)))\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma,\nu(\sigma))d\sigma)\\ &\quad + \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r}))\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma,\nu(\sigma)))d\sigma, \mathfrak{r}\in I, \end{aligned}$$

and

$$A_2\mu(\mathfrak{r}) = \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma\mathfrak{r}\in I.$$

Hence, according to $u = (\mu,\nu)\in Q_r$,

$$\begin{aligned} |A_1\nu(\mathfrak{r})| &= |a(\mu_\circ - \sum_{k=1}^{m} a_k\vartheta(\tau_k, \mu(\varphi_1(\tau_k)))\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma,\nu(\sigma))d\sigma)\\ &\quad + \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r}))\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma,\nu(\sigma)))d\sigma|\\ &\leqslant a|\mu_\circ| + a\sum_{k=1}^{m} a_k\vartheta(\tau_k, \mu(\varphi_1(\tau_k)))\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\sigma,\nu(\sigma))|d\sigma\\ &\quad + \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r}))\int_0^{\mathfrak{r}}\frac{(t-\sigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\sigma,\nu(\sigma)))|ds\\ &\leqslant a|\mu_\circ| + [a\sum_{k=1}^{m}|a_k|[k_1|\mu(\varphi_1(\tau_k)| + \mathcal{V}] + [k_1|\mu(\varphi_1(\mathfrak{r}))| + \mathcal{V}]\frac{(k|\nu| + \phi_1^*)}{\Gamma(\eta+1)}, \end{aligned}$$

then

$$\begin{aligned} \|A_1\nu\| &\leqslant a|\mu_\circ| + [k_1\|\mu\| + \mathcal{V}][a\sum_{k=1}^{m}|a_k| + 1]\frac{(k|\nu| + \phi_1^*)}{\Gamma(\eta+1)} = r_1,\\ r_1 &= \frac{a|\mu_\circ| + [k_1\|\mu\| + \mathcal{V}][a\sum_{k=1}^{m}|a_k| + 1]\frac{\phi_1^*}{\Gamma(\eta+1)}}{1-[k_1\|\mu\| + \mathcal{V}][a\sum_{k=1}^{m}|a_k| + 1]\frac{k}{\Gamma(\eta+1)}}. \end{aligned}$$

Also

$$\begin{aligned} |A_2\mu(\mathfrak{r})| &= |\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma|\\ &\leqslant \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\sigma,\mu(\varphi_2(\sigma)))|d\sigma \leqslant \int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)}[a(\sigma) + b|\mu(\varphi_2(\sigma))|]d\sigma. \end{aligned}$$

Taking supremum over $\mathfrak{r}\in I$,

$$\begin{aligned} \|A_2\mu\| &\leqslant \int_0^{\mathfrak{r}} a(\sigma)\frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)}d\sigma + \int_0^{\mathfrak{r}} b|\mu(\varphi(\sigma))|\frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)}d\sigma\\ &\leqslant I^{\sigma}a(\mathfrak{r}) + br_2\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)}d\sigma \end{aligned}$$

$$
\begin{aligned}
&\leqslant I^{\sigma-\gamma} I^{\gamma} a(\mathfrak{r}) + br_2 I^{\sigma}(\mathfrak{r}) \\
&\leqslant M \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\sigma-\gamma-1}}{\Gamma(\sigma-\gamma)} d\sigma + br_2 \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)} d\sigma \\
&\leqslant \frac{M\mathfrak{r}^{\sigma-\gamma}}{\Gamma(\sigma-\gamma+1)} + br_2 \frac{\mathfrak{r}^{\sigma}}{\Gamma(\sigma+1)} \\
&\leqslant \frac{M}{\Gamma(\sigma-\gamma+1)} + \frac{br_2}{\Gamma(\sigma+1)} = r_2, \quad r_2 = (1 - \frac{b}{\Gamma(\sigma+1)})^{-1} \frac{M}{\Gamma(\sigma-\gamma+1)}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\|Au\|_X &= \|A_1\nu\|_C + \|A_2\mu\|_C \leqslant r_1 + r_2 \\
&\leqslant \frac{a|\mu_\circ| + [k_1\|\mu\| + \mathcal{V}][a\sum_{k=1}^m |a_k| + 1]\frac{\phi_1^*}{\Gamma(\eta+1)}}{1 - [k_1\|\mu\| + \mathcal{V}][a\sum_{k=1}^m |a_k| + 1]\frac{k}{\Gamma(\eta+1)}} + (1 - \frac{b}{\Gamma(\sigma+1)})^{-1} \frac{M}{\Gamma(\sigma-\gamma+1)} = r.
\end{aligned}
$$

Hence the class $\{Au\}$, $u \in Q_r$ is uniformly bounded for $AQ_r \subset Q_r$. After that, for $x \in Q_r$, define the set

$$
\theta_{\vartheta}(\delta) = \sup\{|\vartheta(\mathfrak{r}_2, \mu) - \vartheta(\mathfrak{r}_1, \mu)| : \mathfrak{r}_1, \mathfrak{r}_2 \in I, \mathfrak{r}_1 < \mathfrak{r}_2, |\mathfrak{r}_2 - \mathfrak{r}_1| < \delta, |\mu| \leqslant r\}.
$$

Currently, for $u = (\mu, \nu) \in Q_r$, for all $\epsilon > 0$, $\delta > 0$, and for each $\mathfrak{r}_1, \mathfrak{r}_2 \in [0,1]$, $\mathfrak{r}_1 < \mathfrak{r}_2$ such that $|\mathfrak{r}_2 - \mathfrak{r}_1| < \delta$, we get

$$
\begin{aligned}
&|A_1\nu(\mathfrak{r}_2) - A_1\nu(\mathfrak{r}_1)| \\
&= |a(\mu_\circ - \sum_{k=1}^m a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma) \\
&\quad + \vartheta(\mathfrak{r}_2, \mu(\varphi_1(\mathfrak{r}_2)) \int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} f_1(\sigma, \nu(\sigma)) d\sigma \\
&\quad - a(\mu_\circ - \sum_{k=1}^m a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma) \\
&\quad - \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1)) \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma| \\
&= \Big|\vartheta(\mathfrak{r}_2, \mu(\varphi_1(\mathfrak{r}_2)) \int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma - \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1)) \int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma \\
&\quad + \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1)) \int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma - \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1)) \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma\Big| \\
&\leqslant \Big|\vartheta(\mathfrak{r}_2, \mu(\varphi_1(\mathfrak{r}_2)) - \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1))\Big| \int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\sigma, \nu(\sigma))| d\sigma \\
&\quad + \Big|\vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1)) \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma - \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1)) \int_0^{\mathfrak{r}_1} \frac{(\mathfrak{r}_1-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma\Big| \\
&\quad + \Big|\vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1))\Big| \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma)) d\sigma \\
&\leqslant \Big[\Big|\vartheta(\mathfrak{r}_2, \mu(\varphi_1(\mathfrak{r}_2)) - \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_2))| + |\vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_2)) - \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1))\Big|\Big] \\
&\quad \times \int_0^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\sigma, \nu(\sigma))| d\sigma + \Big|\vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1))\Big| \int_0^{\mathfrak{r}_1} \left[\frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} - \frac{(\mathfrak{r}_1-\sigma)^{\eta-1}}{\Gamma(\eta)}\right] |\phi_1(\sigma, \nu(\sigma))| d\sigma \\
&\quad + \vartheta(\mathfrak{r}_1, \mu(\varphi_1(\mathfrak{r}_1)) \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\sigma, \nu(\sigma))| d\sigma
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \left[\theta_\vartheta(\delta)+k_1|\mu(\varphi_1(\mathfrak{r}_2))-\mu(\varphi_1(\mathfrak{r}_1))|\right]\int_0^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)}(k|\nu|+\varphi_1^*)d\sigma\\
&\quad+[k_1|\mu(\varphi_1(\mathfrak{r}_1))|+\mathcal{V}]\int_0^{\mathfrak{r}_1}\left[\frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)}-\frac{(\mathfrak{r}_1-\sigma)^{\eta-1}}{\Gamma(\eta)}\right](k|\nu|+\varphi_1^*)d\sigma\\
&\quad+[k_1|\mu(\varphi_1(\mathfrak{r}_1))|+\mathcal{V}]\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\sigma)^{\eta-1}}{\Gamma(\eta)}(k|\nu|+\varphi_1^*)d\sigma\\
&\leqslant \left[\theta_\vartheta(\delta)+k_1|\mu(\varphi_1(\mathfrak{r}_2))-\mu(\varphi_1(\mathfrak{r}_1))|\right](k|\nu|+\varphi_1^*)\frac{(\mathfrak{r}_2)^\eta}{\Gamma(\eta+1)}+[k_1|\mu(\varphi_1(\mathfrak{r}_1))|+\mathcal{V}](k|\nu|+\varphi_1^*)\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^\eta}{\Gamma(\eta+1)}\\
&\quad+[k_1|\mu(\varphi_1(\mathfrak{r}_1))|+\mathcal{V}](k|\nu|+\varphi_1^*)\left(\frac{-(\mathfrak{r}_2-\mathfrak{r}_1)^\eta}{\Gamma(\eta+1)}+\frac{\mathfrak{r}_2^\sigma}{\Gamma(\eta+1)}-\frac{\mathfrak{r}_1^\eta}{\Gamma(\beta+1)}\right)\\
&\leqslant [k_1|\mu(\varphi_1(\mathfrak{r}_1))|+\mathcal{V}](k|\nu|+\varphi_1^*)\Big(\frac{\mathfrak{r}_2^\eta-\mathfrak{r}_1^\eta}{\Gamma(\eta+1)}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&|A_2\mu(\mathfrak{r}_2)-A_2\mu(\mathfrak{r}_1)|\\
&\quad\leqslant \Big|\int_0^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma-\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_1-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma\Big|\\
&\quad\leqslant \Big|\int_0^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma-\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma\Big|\\
&\qquad+\Big|\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma-\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_1-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma\Big|\\
&\quad\leqslant \Big|\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\alpha)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma\Big|+\Big|\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma\Big|\\
&\qquad-\Big|\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_1-\sigma)^{\sigma-1}}{\Gamma(\sigma)}\varphi_2(\sigma,\mu(\varphi_2(\sigma)))d\sigma\Big|\\
&\quad\leqslant \int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}|\varphi_2(\sigma,\mu(\varphi_2(\sigma)))|d\sigma+\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}-(\mathfrak{r}_1-\sigma)^{\sigma-1}}{\Gamma(\sigma)}|\varphi_2(\sigma,\mu(\varphi_2(\sigma)))|d\sigma\\
&\quad\leqslant \int_{\mathfrak{r}_1}^{\mathfrak{r}_2}[a+b|\mu(\varphi_2(\sigma))|]\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}d\sigma+\int_0^{\mathfrak{r}_1}[a+b|\mu(\varphi_2(s))|]\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}-(\mathfrak{r}_1-\sigma)^{\sigma-1}}{\Gamma(\sigma)}d\sigma\\
&\quad\leqslant (a+br_2)\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}}{\Gamma(\sigma)}ds+(a+br_2)\int_0^{\mathfrak{r}_1}\frac{(\mathfrak{r}_2-\sigma)^{\sigma-1}-(\mathfrak{r}_1-\sigma)^{\sigma-1}}{\Gamma(\sigma)}d\sigma\\
&\quad\leqslant (a+br_2)\frac{(\mathfrak{r}_2-\mathfrak{r}_1)^\sigma}{\Gamma(\sigma+1)}+(a+br_2)\left(\frac{-(\mathfrak{r}_2-\mathfrak{r}_1)^\sigma}{\Gamma(\sigma+1)}+\frac{\mathfrak{r}_2^\sigma}{\Gamma(\sigma+1)}-\frac{\mathfrak{r}_1^\sigma}{\Gamma(\sigma+1)}\right)\\
&\quad\leqslant (a+br_2)\frac{(\mathfrak{r}_2^\sigma-\mathfrak{r}_1^\sigma)}{\Gamma(\sigma+1)}.
\end{aligned}$$

For the operator A and $u\in Q_r$, we have

$$\begin{aligned}
Au(\mathfrak{r}_2)-Au(\mathfrak{r}_1)&=A(\mu,\nu)(\mathfrak{r}_2)-A(\mu,\nu)(\mathfrak{r}_1)\\
&=(A_2\mu(\mathfrak{r}_2),A_1\nu(\mathfrak{r}_2))-(A_2\mu(\mathfrak{r}_1),A_1\nu(\mathfrak{r}_1))=(A_2\mu(\mathfrak{r}_2)-A_2\mu(\mathfrak{r}_1),A_1\nu(\mathfrak{r}_2)-A_1\nu(\mathfrak{r}_1)),
\end{aligned}$$

then

$$\begin{aligned}
|Au(\mathfrak{r}_2)-Au(\mathfrak{r}_1)|_X&=|A(\mu,y)(\mathfrak{r}_2)-A(\mu,y)(\mathfrak{r}_1)|_X\\
&=|A_1\nu(\mathfrak{r}_2)-A_1\nu(\mathfrak{r}_2)|_C+|A_2\mu(\mathfrak{r}_2)-A_2\mu(\mathfrak{r}_1)|_C\\
&=[k_1|\mu(\varphi_1(\mathfrak{r}_1))|+\mathcal{V}](k|\nu|+\varphi_1^*)\Big(\frac{\mathfrak{r}_2^\eta-\mathfrak{r}_1^\eta}{\Gamma(\eta+1)}\Big)+(a+br_2)\frac{(\mathfrak{r}_2^\sigma-\mathfrak{r}_1^\sigma)}{\Gamma(\sigma+1)}.
\end{aligned}$$

As a result, the class of functions $\{Au\}$ is equi-continuous on Qr. The operator A is compact as a result of the Arzela-Ascoli Theorem [6]. The continuity of $A : Qr \to Qr$ still needs to be proven. Let $u_n = (\mu_n, \nu_n)$ be a sequence in Qr with $x_n \to \mu$, and $y_n \to y$ and since $\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))$ is continuous in $C(I, \mathbb{R})$, then $\phi_2(\mathfrak{r}, \mu_n(\mathfrak{r}))$ converges to $\phi_2(\mathfrak{r}, \mu(\mathfrak{r}))$, thus $\phi_2(\mathfrak{r}, \mu_n(\varphi_2(\mathfrak{r})))$ converges to $\phi_2(\mathfrak{r}, \mu(\varphi_2(\mathfrak{r})))$, using assumptions (iii)-(iv) and applying Lebesgue Dominated Convergence Theorem, we get

$$\lim_{n\to\infty} \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2(\sigma, \mu_n(\varphi(\sigma)))d\sigma = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2(\sigma, \mu(\varphi(\sigma)))d\sigma,$$

then

$$\begin{aligned}
\lim_{n\to\infty} A_2\mu_n(\mathfrak{r}) &= \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)} \lim_{n\to\infty} \phi_2(\sigma, \mu_n(\varphi(\sigma)))d\sigma = \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\sigma-1}}{\Gamma(\sigma)} \phi_2(\sigma, \mu(\varphi(\sigma)))ds = A_2\mu(\mathfrak{r}),\\
\lim_{n\to\infty} A_1\nu_n(\mathfrak{r}) &= a(\mu_\circ - \sum_{k=1}^{m} a_k \lim_{n\to\infty} \vartheta(\tau_k, \mu_n(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \lim_{n\to\infty} \phi_1(\sigma, \nu_n(\sigma))d\sigma)\\
&\quad + \lim_{n\to\infty} \vartheta(\mathfrak{r}, \mu_n(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \lim_{n\to\infty} \phi_1(\sigma, \nu_n(\sigma))d\sigma\\
&= a(x_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma)\\
&\quad + \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma = A_1\nu(\mathfrak{r}).
\end{aligned}$$

So,

$$\begin{aligned}
\lim_{n\to\infty} Au_n(\mathfrak{r}) &= \lim_{n\to\infty} (A_1\nu_n(\mathfrak{r}), A_2\mu_n(\mathfrak{r}))\\
&= (\lim_{n\to\infty} A_1\nu_n(\mathfrak{r}), \lim_{n\to\infty} A_2\mu_n(\mathfrak{r})) = (A_1\nu(\mathfrak{r}), A_2\mu(\mathfrak{r})) = Au(\mathfrak{r}).
\end{aligned}$$

Then $Au_n \to Au$ as $n \to 1$. The operator A is continuous as a result. While all criteria of the Schauder fixed-point theorem [7] are achieved, then A has a fixed point $u \in Q_r$, and then the system (2.9), (2.8) has at least one continuous solutions $u = (\mu, \nu) \in Q_r$, $\mu; \nu \in C(I, \mathbb{R})$.

Therefore, there is at least one solution $\mu \in C(I, \mathbb{R})$ to the functional integral equation (1.1).

Conversely, by differentiating (2.2), we get

$$\begin{aligned}
D^\eta \mu(\mathfrak{r}) &= D^\eta \Big\{ a\Big(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma\Big)\\
&\quad + \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma \Big\},\\
\nu(t) &= I^\sigma f_2(\mathfrak{r}, \mu(\varphi(\mathfrak{r}))).
\end{aligned}$$

Additionally, we derive from the integral equation (2.8)-(2.9),

$$\begin{aligned}
\mu(\tau_k) &= a\left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma\right)\\
&\quad + \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma,\\
\mu(0) &= a\left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, \nu(\sigma))d\sigma\right),\\
\nu(\mathfrak{r}) &= I^\sigma \phi_2(\mathfrak{r}, \nu(\varphi_2(\mathfrak{r}))),
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}\sum_{k=1}^{m} a_k\mu(\tau_k) &= a\sum_{k=1}^{m} a_k \left(\mu_\circ - \sum_{k=1}^{m} a_k\vartheta(\tau_k,\mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\beta)}\phi_1(\sigma,\nu(\sigma))d\sigma\right)\\ &\quad + \sum_{k=1}^{m} a_k\vartheta(\tau_k,\mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\beta)}\phi_1(\sigma,\nu(\sigma))d\sigma,\\ \nu(\mathfrak{r}) &= I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi_2(\mathfrak{r}))).\end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned}\mu(0)+\sum_{k=1}^{m} a_k\mu(\tau_k) &= a(1+\sum_{k=1}^{m} a_k) \left(\mu_\circ - \sum_{k=1}^{m} a_k\vartheta(\tau_k,\mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma,\nu(\sigma))d\sigma\right)\\ &\quad + \sum_{k=1}^{m} a_k\vartheta(\tau_k,\mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma,\nu(\sigma))d\sigma.\end{aligned}$$

Then

$$\mu(0)+\sum_{k=1}^{m} a_k\mu(\tau_k) = \mu_\circ.$$

Consequently, the nonlocal problem of functional differential inclusion (1.1)-(1.4) has at least one solution $\mu \in C(I,\mathbb{R})$. □

2.2. *Riemann-Stieltjes integral BCs* (1.2)

Let $\mu \in C(I,\mathbb{R})$ represent the solution to the non-local problem of (1.1)-(1.4). Let $a_k = h(\mathfrak{r}_k) - h(\mathfrak{r}_{k-1})$, the function h is nondecreasing, $\tau_k \in (\mathfrak{r}_{k-1},\mathfrak{r}_k)$, $0=\mathfrak{r}_0<\mathfrak{r}_1<\mathfrak{r}_2<\cdots<1$. The nonlocal condition (1.4) will then take the following form

$$\mu(0)+\sum_{k=1}^{m}\mu(\tau_k)(h(\mathfrak{r}_k)-h(\mathfrak{r}_{j-k})) = \mu_\circ.$$

We derive from [18] as $m\to\infty$ the continuation of the solution of the nonlocal problem (1.1)-(1.4).

$$\lim_{m\to\infty}\sum_{k=1}^{m}\mu(\tau_k)(h(\mathfrak{r}_j)-h(\mathfrak{r}_{j-1})) = \int_0^1 \mu(\sigma)dh(\sigma),$$

that is, the non-local conditions (1.4) is change to Riemann-Stieltjes integral condition as $m\to\infty$,

$$\mu(0)+\lim_{m\to\infty}\sum_{k=1}^{m}\mu(\tau_k)(h(\mathfrak{r}_k)-h(\mathfrak{r}_{k-1})) = \mu(0)+\int_0^1 \mu(\sigma)dh(\sigma) = \mu_\circ.$$

Theorem 2.3. *Assume that assumptions* (i)-(iv) *of Theorem* 2.2 *hold and* $h:[0,1]\to[0,1]$ *is an increasing function, then the Riemann-Stieltjes functional integral condition* (1.2) *and the non-local problem* (1.1) *have a solution* $\mu \in C(I,\mathbb{R})$.

2.3. *Infinite-point boundary condition* (1.3)

Take into account that $\mu \in C(I,\mathbb{R})$ be the solution to the non-local problem presented by (1.1) and (1.3).

Theorem 2.4. *Assume that assumptions* (i)-(iv) *of Theorem* 2.2 *hold and let* $B_m^{-1} = 1 + \sum_{k=1}^{m} a_k$ *be convergent sequence, then the non-local problem of* (1.1)-(1.3) *represented by the integral equation*

$$\begin{aligned}\mu(t) = B_m\mu_\circ - B_m \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu(\varphi(\sigma))))d\sigma \\ + \vartheta(t,\mu(\varphi_1(t)))\int_0^t \frac{(t-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu(\varphi(\sigma))))d\sigma,\end{aligned} \tag{2.12}$$

has at least one solution $\mu \in C(I, \mathbb{R})$.

Proof. Let $\mu \in C(I,\mathbb{R})$ be a solution of the infinite point BVP (1.1) and (1.3) provided by (2.2).

$$\begin{aligned}\mu_m(t) = \frac{1}{(1+\sum_{k=1}^m a_k)}\left(x_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\beta)}\phi_1(\sigma, I^\sigma f_2(\sigma,\mu(\varphi_2(\sigma))))d\sigma\right) \\ + \vartheta(t,\mu(\varphi_1(t)))\int_0^t \frac{(t-\sigma)^{\beta-1}}{\Gamma(\beta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu_m(\varphi_2(\sigma))))d\sigma.\end{aligned} \tag{2.13}$$

Consider the limit to (2.13), as $m \to \infty$, we obtain

$$\begin{aligned}\lim_{m\to\infty}\mu_m(t) &= \lim_{m\to\infty}\left[\frac{1}{(1+\sum_{k=1}^m a_k)}\left(\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu_m(\varphi_2(\sigma))))d\sigma\right)\right. \\ &\quad \left.+ \vartheta(t,\mu_m(\varphi_1(t)))\int_0^t \frac{(t-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu_m(\varphi_2(\sigma))))d\sigma\right] \\ &= \lim_{m\to\infty}\frac{1}{(1+\sum_{k=1}^m a_k)}\left[\mu_\circ - \sum_{k=1}^{m} a_k \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu_m(\varphi(\sigma))))ds\right] \\ &\quad + \lim_{m\to\infty}\vartheta(t,\mu_m(\varphi_1(t)))\int_0^t \frac{(t-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu_m(\varphi(\sigma))))d\sigma.\end{aligned} \tag{2.14}$$

Now $|a_k\mu(\tau_k)| \leqslant |a_k|\|\mu\|$, so using a comparison test, $\sum_{k=1}^m a_k\mu(\tau_k)$ is convergent. Also

$$\begin{aligned}&\left|\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu(\varphi(\sigma))))d\sigma\right| \\ &\leqslant \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}(k|I^\sigma\phi_2(\sigma,\mu(\varphi(\sigma)))| + \phi_1^*)ds \\ &\leqslant k\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}[I^\sigma a(\sigma) + b\int_0^\sigma \frac{(\sigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\theta))|d\theta]d\sigma + k\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1^* ds \\ &\leqslant k\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}[I^{\sigma-\gamma}I^\gamma a(\sigma) + br_2\int_0^\sigma \frac{(\sigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}d\theta]d\sigma + k\phi_1^*\frac{T^\beta}{\Gamma(\beta+1)} \\ &\leqslant kM\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\beta)}\int_0^\sigma \frac{(\sigma-\theta)^{\sigma-\gamma-1}}{\Gamma(\sigma-\gamma)}d\theta ds + kbr_2\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\beta)}\int_0^\sigma \frac{(\sigma-\theta)^{\sigma-1}}{\Gamma(\sigma)}d\theta d\sigma + \frac{k\phi_1^* T^\eta}{\Gamma(\eta+1)} \\ &\leqslant \frac{1}{\Gamma(\beta+1)}\left[\frac{kM}{\Gamma(\sigma-\gamma+1)} + \frac{kbr_2}{\Gamma(\sigma+1)} + k\phi_1^*\right] \leqslant N,\end{aligned}$$

then

$$|a_k||\vartheta(t,\mu(\varphi_1(t)))|\left|\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu(\varphi(\sigma))))d\sigma\right| \leqslant |a_k|[k_1 r + \mathcal{V}]N,$$

and by the comparison test, $\sum_{k=1}^m a_k \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\beta)}\phi_1(\sigma, I^\sigma\phi_2(\sigma,\mu(\varphi(\sigma))))d\sigma$ is convergent.

Using assumptions (i)-(iii) and applying Lebesgue dominated convergence theorem [15], from (2.14) we obtain (2.12). Furthermore, from (2.12), we have

$$(1+\sum_{k=1}^{m} a_k)\mu(\tau_k) = B_m^{-1}B_m x_\circ - B_m^{-1}B_m \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))))d\sigma$$
$$+(1+\sum_{k=1}^{m} a_k)\vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma,$$

then

$$\mu(\tau_k)+\sum_{k=1}^{m} a_k\mu(\tau_k) = x_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))))d\sigma$$
$$+\vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma$$
$$+\sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma,$$
$$\sum_{k=1}^{m} a_k\mu(\tau_k) = \mu_\circ - \mu(\tau_k) + \vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))d\sigma. \tag{2.15}$$

From (2.2), we obtain

$$\mu(0) = a\left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma))))ds\right)$$

and

$$\mu(\tau_k) = a\left(\mu_\circ \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\beta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))))d\sigma\right)$$
$$+\vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi_2(\sigma)))d\sigma,$$

So

$$\mu(0) = \mu(\tau_k) - \vartheta(\tau_k, \mu(\varphi_1(\tau_k))\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma, I^\sigma \phi_2(s, \mu(\varphi_2(\sigma))))d\sigma.$$

Going back to (2.15) we obtain infinite-point BC (1.3),

$$\mu(0)+\sum_{k=1}^{\infty} a_k\mu(\tau_k) = \mu_\circ.$$

Consequently, the nonlocal problem of functional differential inclusion (1.1)-(1.3) has at least one solution $x \in C(I, \mathbb{R})$. □

3. Existence of unique solutions

The necessary condition for the uniqueness result for non-local problem (1.1)-(1.4) is provided in this section. Assume the following assumption.

(iii)* Suppose that $\phi_2 : I \times \mathbb{R} \to \mathbb{R}$, is a continuous function that satisfies the Lipschitz condition, with $|\phi_2(t, \mu) - \phi_2(t, \nu)| \leqslant c|\mu - \nu|$.

Theorem 3.1. *Assume that assumptions of Theorem 2.2 hold with condition* (iii) *replaced by* (iii)*, *if*

$$\frac{(a\sum_{k=1}^{m} a_k k_1[a+b\|\mu_1\|]+c[k_1\|\mu_1\|+\mathcal{V}])2k}{\Gamma(\eta+1)\Gamma(\sigma+1)}<1,$$

then the non-local problem (1.1)-(1.4) *has a unique solution* $\mu\in C(I,\mathbb{R})$.

Proof. Let $\mu_1(\mathfrak{r})$ and $\mu_2(\mathfrak{r})$ be two solutions of the functional integral equation (2.2), then

$$\begin{aligned}
&\mu_1(\mathfrak{r})-\mu_2(\mathfrak{r})\\
&\quad=a\left(\mu_\circ-\sum_{k=1}^{m}a_k\vartheta(\tau_k,\mu_1(\varphi_1(\tau_k))\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_1(\varphi_2(\sigma))))d\sigma\right)\\
&\qquad+\vartheta(\mathfrak{r},\mu_1(\varphi_1(\mathfrak{r}))\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_1(\varphi_2(\sigma))))d\sigma\\
&\qquad-a\left(\mu_\circ-\sum_{k=1}^{m}a_k\vartheta(\tau_k,\mu_2(\varphi_1(\tau_k))\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_2(\varphi_2(\sigma))))d\sigma\right)\\
&\qquad-\vartheta(\mathfrak{r},\mu_2(\varphi_1(\mathfrak{r}))\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))))d\sigma\\
&\quad\leqslant a\sum_{k=1}^{m}a_k|\vartheta(\tau_k,\mu_1(\varphi_1(\tau_k))-\vartheta(\tau_k,\mu_2(\varphi_1(\tau_k))|\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))))\\
&\qquad+a\sum_{k=1}^{m}a_k|\vartheta(\tau_k,\mu_1(\varphi_1(\tau_k))|(\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))))-\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_1(\varphi(\sigma))))|d\sigma)\\
&\qquad+|\vartheta(\mathfrak{r},\mu_1(\varphi_1(\mathfrak{r}))-\vartheta(\mathfrak{r},\mu_2(\varphi_1(\mathfrak{r}))|\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_2(\varphi_2(\sigma))))|d\sigma\\
&\qquad+\vartheta(\mathfrak{r},\mu_1(\varphi_1(\mathfrak{r}))\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\beta-1}}{\Gamma(\beta)}|\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_1(\varphi(\sigma))))-\phi_1(\sigma,I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))))|d\sigma.
\end{aligned}$$

Lipschitz condition for ϕ_1 allows us to obtain

$$\begin{aligned}
&\leqslant a\sum_{k=1}^{m}a_k k_1|\mu_1(\varphi_1(\tau_k))-\mu_2(\varphi_1(\tau_k))|\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}[k|I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))|+\phi_1^*]\\
&\quad+a\sum_{k=1}^{m}a_k[k_1|\mu(\varphi_1(\tau_k)|+\mathcal{V}](\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}k|I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))))-I^\sigma\phi_2(\sigma,\mu_1(\varphi(\sigma))))|d\sigma)\\
&\quad+k_1|\mu_1(\varphi_1(\mathfrak{r}))-\mu_2(\varphi_1(\mathfrak{r}))|\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}[k|I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))|+\phi_1^*]d\sigma\\
&\quad+[k_1|\mu(\varphi_1(\mathfrak{r}))|+\mathcal{V}]\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\beta-1}}{\Gamma(\beta)}|I^\sigma\phi_2(\sigma,\mu_1(\varphi(\sigma))))-I^\sigma\phi_2(\sigma,\mu_2(\varphi(\sigma))))|d\sigma\\
&\leqslant a\sum_{k=1}^{m}a_k k_1|\mu_1(\varphi_1(\tau_k))-\mu_2(\varphi_1(\tau_k))|\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}k[\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\tau,\mu_1(\varphi(\tau)))|d\tau+\phi_1^*]d\sigma\\
&\quad+a\sum_{k=1}^{m}a_k[k_1|\mu(\varphi_1(\tau_k)|+\mathcal{V}](\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}k\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu_1(\varphi(\tau)))-\phi_2(\tau,\mu_2(\varphi(\tau)))|d\tau d\sigma\\
&\quad+k_1|\mu_1(\varphi_1(\mathfrak{r}))-\mu_2(\varphi_1(\mathfrak{r}))|\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}k[\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}\phi_2(\tau,\mu_1(\varphi(\tau)))|d\tau+\phi_1^*]d\sigma\\
&\quad+[k_1|\mu(\varphi_1(\mathfrak{r}))|+\mathcal{V}]\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\beta-1}}{\Gamma(\beta)}k\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu_1(\varphi(\tau)))-\phi_2(\tau,\mu_2(\varphi(\tau)))|d\tau d\sigma.
\end{aligned}$$

Lipschitz condition for ϕ_2 allows us to obtain

$$\begin{aligned}
&|\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r})| \\
&\leqslant a\sum_{k=1}^{m} a_k k_1 k|\mu_1(\varphi_1(\tau_k)) - \mu_2(\varphi_1(\tau_k))| \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}[\int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}[a+b|\mu_1(\varphi(\tau))|]d\tau + \phi_1^*]d\sigma \\
&\quad + a\sum_{k=1}^{m} a_k kc[k_1|\mu(\varphi_1(\tau_k)|+\mathcal{V}](\int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)} k\int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu_1(\varphi(\tau))) - \mu_2(\varphi(\tau))|d\tau d\sigma \\
&\quad + k_1|\mu_1(\varphi_1(\mathfrak{r})) - \mu_2(\varphi_1(\mathfrak{r}))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}[\int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}[a+b|\mu_1(\varphi(\tau))|]d\tau + \phi_1^*]d\sigma \\
&\quad + kc[k_1|\mu(\varphi_1(\mathfrak{r}))|+\mathcal{V}]\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\beta-1}}{\Gamma(\beta)} k\int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu_1(\varphi(\tau)) - \mu_2(\varphi(\tau))|d\tau d\sigma \\
&\leqslant a\sum_{k=1}^{m} a_k k_1 k\|\mu_1-\mu_2\| \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}[\int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}[a+b\|\mu_1\|]d\tau + \phi_1^*]d\sigma \\
&\quad + kc[k_1\|\mu_1\|+\mathcal{V}]\|\mu_1-\mu_2\| \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} d\tau d\sigma \\
&\quad + a\sum_{k=1}^{m} a_k kc\|\mu_1-\mu_2\| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}[\int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}[a+b\|\mu_1\|]d\tau + \phi_1^*]d\sigma \\
&\quad + kc[k_1\|\mu_1\|+\mathcal{V}]\|\mu_1-\mu_2\| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\sigma} \frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)} d\tau d\sigma,
\end{aligned}$$

then

$$\|\mu_1-\mu_2\| \leqslant \frac{(a\sum_{k=1}^{m} a_k k_1[a+b\|\mu_1\|] + c[k_1\|\mu_1\|+\mathcal{V}])2k}{\Gamma(\eta+1)\Gamma(\sigma+1)}\|\mu_1-\mu_2\|.$$

Hence

$$\left(1 - \frac{(a\sum_{k=1}^{m} a_k k_1[a+b\|\mu_1\|] + c[k_1\|\mu_1\|+\mathcal{V}])2k}{\Gamma(\eta+1)\Gamma(\sigma+1)}\right)\|\mu_1-\mu_2\| \leqslant 0.$$

Since $\frac{(a\sum_{k=1}^{m} a_k k_1[a+b\|\mu_1\|]+c[k_1\|\mu_1\|+\mathcal{V}])2k}{\Gamma(\eta+1)\Gamma(\sigma+1)} < 1$, then $\mu_1(\mathfrak{r}) = \mu_2(\mathfrak{r})$ and the solution of the integral equation (2.2) is unique, and as consequence the integral equation (2.2) has a unique solution, and as a result, this establishes the existence of unique solutions to the non-local problem (1.1)-(1.4). □

3.1. Continuous dependence of solutions

Theorem 3.2. *Assume that assumptions of Theorem 3.1 hold. Then the solution of the non-local problem (1.1)-(1.4) is continuously dependent on the S_{Φ_1}, the set of all Lipschitzian selections of Φ_1.*

Proof. Let $\phi_1(\mathfrak{r},\mu(\mathfrak{r}))$ and $\phi_1^*(\mathfrak{r},\mu(\mathfrak{r}))$ be two separate Lipschitzian selections of $\Phi_1(\mathfrak{r},\mu(\mathfrak{r}))$, so that

$$|\phi_1(\mathfrak{r},\mu(\mathfrak{r})) - \phi_1^*(\mathfrak{r},\mu(\mathfrak{r})| < \epsilon,\ \epsilon > 0,\ \mathfrak{r}\in I,$$

then we have the following for the two related solutions $\mu_{\phi_1}(\mathfrak{r})$ and $\mu_{\phi_1^*}(\mathfrak{r})$ of (2.2).

$$\begin{aligned}
&|\mu_{\phi_1}(\mathfrak{r}) - \mu^*_{\phi_1}(\mathfrak{r})| \\
&= |a\sum_{k=1}^{m} a_k\vartheta(\tau_k,\mu(\varphi_1(\tau_k)) \int_0^{\tau_k} \frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\left[\phi_1(\sigma, I^{\sigma}\phi_2(\sigma,\mu(\varphi(\sigma))) - \phi_1^*(\mathfrak{r}, I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))\right]ds
\end{aligned}$$

$$
\begin{aligned}
&+\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r}))\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\beta)}\left[\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu(\varphi(\sigma)))-\phi_1^*(\mathfrak{r},I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))\right]ds|\\
&\leqslant a\sum_{k=1}^m a_k|\vartheta(\tau_k,\mu(\varphi_1(\tau_k))|\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\beta)}|\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu(\varphi(\sigma)))-\phi_1^*(\mathfrak{r},I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|d\sigma\\
&\quad+|\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r}))|\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))-\phi_1^*(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|d\sigma\\
&\leqslant a\sum_{k=1}^m a_k|\vartheta(\tau_k,\mu(\varphi_1(\tau_k))|\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu(\varphi(\sigma)))-\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|d\sigma\\
&\quad+a\sum_{k=1}^m a_k|\vartheta(\tau_k,\mu(\varphi_1(\tau_k))|\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^{\sigma}\phi_1(\sigma,\mu^*(\varphi(\sigma)))-\phi_1^*(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|d\sigma\\
&\quad+|\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r}))|\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu(\varphi(\sigma)))-\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|ds\\
&\quad+|\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r}))|\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}|\phi_1(\sigma,I^{\sigma}\phi_1(\sigma,\mu^*(\varphi(\sigma)))-\phi_1^*(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|d\sigma\\
&\leqslant a\sum_{k=1}^m a_k|\vartheta(\tau_k,\mu(\varphi_1(\tau_k))|\\
&\quad\times\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}(|\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu(\varphi(\sigma)))-\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|+\delta)d\sigma\\
&\quad+|\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r}))|\Big[\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\beta-1}}{\Gamma(\eta)}[|\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu(\varphi(\sigma)))-\phi_1(\sigma,I^{\sigma}\phi_2(\sigma,\mu^*(\varphi(\sigma)))|+\delta]d\sigma\Big]\\
&\leqslant a\sum_{k=1}^m a_k|\vartheta(\tau_k,\mu(\varphi_1(\tau_k))|k\Big(\int_0^{\mathfrak{r}_k}|I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r}))-I^{\sigma}\phi_2(\mathfrak{r},\mu^*(\varphi(\mathfrak{r}))|ds+\frac{\delta}{\Gamma(\eta+1)}\Big)\\
&\quad+k|\vartheta(\mathfrak{r},\mu(\varphi_1(\mathfrak{r}))|\Big[\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\beta-1}}{\Gamma(\beta)}|I^{\sigma}\phi_2(\mathfrak{r},\mu(\varphi(\mathfrak{r}))-I^{\sigma}\phi_2(\mathfrak{r},\mu^*(\varphi(\mathfrak{r}))|ds+\frac{\delta}{\Gamma(\beta+1)}\Big]\\
&\leqslant a\sum_{k=1}^m a_k k[k_1|\mu(\varphi_1(\tau_k))|+\mathcal{V}]\\
&\quad\times\Big(\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\beta-1}}{\Gamma(\beta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu(\varphi(\tau)))-\phi_2(\tau,\mu^*(\varphi(\tau)))|d\tau d\sigma+\frac{\delta T^{\beta}}{\Gamma(\eta+1)}\Big)\\
&\quad+k[k_1|\mu(\varphi_1(\mathfrak{r}))|+\mathcal{V}]\Big[\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\beta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\phi_2(\tau,\mu(\varphi(\tau)))-\phi_2(\tau,\mu^*(\varphi(\tau)))|d\tau d\sigma+\frac{\delta}{\Gamma(\eta+1)}\Big]\\
&\leqslant a\sum_{k=1}^m a_k kc[k_1|\mu(\varphi_1(\tau_k))|+\mathcal{V}]\Big(\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\sigma))-\mu^*(\varphi(\sigma))|d\tau d\sigma+\frac{\delta}{\Gamma(\eta+1)}\Big)\\
&\quad+kc[k_1|\mu(\varphi_1(\mathfrak{r}))|+\mathcal{V}]\Big[\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\beta-1}}{\Gamma(\eta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\sigma))-\mu^*(\varphi(\sigma))|d\tau d\sigma+\frac{\delta}{\Gamma(\beta+1)}\Big]\\
&\leqslant\|\mu_{\phi_1}-\mu_{\phi_1^*}\|[k_1\|\mu\|+\mathcal{V}]\Big(a\sum_{k=1}^m a_k kc\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}d\tau d\sigma\\
&\quad+kc\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\sigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\sigma}\frac{(s-\tau)^{\sigma-1}}{\Gamma(\sigma)}d\tau d\sigma\Big)+[k_1\|\mu\|+\mathcal{V}]\Big(a\sum_{k=1}^m a_k kc+1\Big)\frac{\delta}{\Gamma(\eta+1)},
\end{aligned}
$$

$$
\|\mu_{\phi_1}-\mu_{\phi_1^*}\|\leqslant\frac{(a\sum_{k=1}^m a_k+1)kc[k_1\|\mu\|+\mathcal{V}]}{\Gamma(\eta+1)\Gamma(\sigma+1)}\|\mu_{\phi_1}-\mu_{\phi_1^*}\|+\Big(a\sum_{k=1}^m a_k kc+1\Big)\frac{\delta[k_1\|\mu\|+\mathcal{V}]}{\Gamma(\eta+1)}
$$

$$\|\mu_{\Phi_1} - \mu_{\Phi_1^*}\| \leqslant \left(1 - \frac{(a\sum_{k=1}^{m} a_k + 1)kc[k_1\|\mu\| + \mathcal{V}]}{\Gamma(\eta+1)\Gamma(\sigma+1)}\right)^{-1} (a\sum_{k=1}^{m} a_k kc + 1)\frac{\delta[k_1\|\mu\| + \mathcal{V}]}{\Gamma(\eta+1)} = \epsilon.$$

Hence,

$$\|\mu_{\Phi_1} - \mu_{\Phi_1^*}\| \leqslant \epsilon.$$

It demonstrates the solution on the set S_{Φ_1} of all Lipschitzian selection of Φ_1 is continuous dependence. □

Theorem 3.3. *Let the assumptions of Theorem 3.1 be satisfied. Then the solution of the non-local problem (1.1)-(1.4) depends continuously on initial data $\mu_\circ$.*

Proof. The solution of the integral inclusion (2.2) depends continuously on initial data $\mu_\circ$, if

$$\forall \epsilon > 0, \ \exists\, \delta(\epsilon) \text{ s.t. } |\mu_\circ - \mu_\circ^*| < \delta \Rightarrow \|\mu - \mu^*\| < \epsilon,$$

then for the two corresponding solutions $\mu(\mathfrak{r})$ and $\mu^*(\mathfrak{r})$ of the integral equation (2.2), we obtain

$$\begin{aligned}
&|\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})|\\
&= |a\left(\mu_\circ - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k))) \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu_2(\varphi(\sigma)))d\sigma\right)\\
&\quad + \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r}))) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \sigma)^{\eta-1}}{\Gamma(\beta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))))d\sigma\\
&\quad - a\left(\mu_\circ^* - \sum_{k=1}^{m} a_k \vartheta(\tau_k, \mu(\varphi_1(\tau_k))) \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\beta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu^*(\varphi(\sigma)))d\sigma\right)\\
&\quad - \vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r}))) \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \sigma)^{\eta-1}}{\Gamma(\eta)} \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu^*(\varphi(\sigma))))d\sigma|\\
&\leqslant a|\mu_\circ - \mu_\circ^*|\\
&\quad + a\sum_{k=1}^{m} a_k |\vartheta(\tau_k, \mu(\varphi_1(\tau_k)))| \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))) - \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu^*(\varphi(\sigma)))|d\sigma\\
&\quad + |\vartheta(\mathfrak{r}, \mu(\varphi_1(\mathfrak{r})))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \sigma)^{\eta-1}}{\Gamma(\eta)} |\phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))) - \phi_1(\sigma, I^\sigma \phi_2(\sigma, \mu^*(\varphi(\sigma)))|d\sigma\\
&\leqslant a\delta + a\sum_{k=1}^{m} a_k [k_1|\mu(\varphi_1(\mathfrak{r}))| + \mathcal{V}]k \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} |I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))) - I^\sigma \phi_2(\sigma, \mu^*(\varphi(\sigma)))|d\sigma\\
&\quad + k[k_1|\mu(\varphi_1(\mathfrak{r}))| + \mathcal{V}] \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \sigma)^{\beta-1}}{\Gamma(\eta)} |I^\sigma \phi_2(\sigma, \mu(\varphi(\sigma))) - I^\sigma \phi_2(\sigma, \mu^*(\varphi(\sigma)))|d\sigma\\
&\leqslant a\delta + a\sum_{k=1}^{m} a_k [k_1|\mu(\varphi_1(\mathfrak{r}))| + \mathcal{V}]k\\
&\quad \times \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\sigma} \frac{(\sigma - \tau)^{\sigma-1}}{\Gamma(\sigma)} |\phi_2(\tau, \mu(\varphi(\tau))) - \phi_2(\tau, \mu^*(\varphi(\tau)))|d\tau d\sigma\\
&\quad + k[k_1|\mu(\varphi_1(\mathfrak{r}))| + \mathcal{V}] \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \sigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\sigma} \frac{(\sigma - \tau)^{\sigma-1}}{\Gamma(\sigma)} |\phi_2(\tau, \mu(\varphi(\tau))) - \phi_2(\tau, \mu^*(\varphi(\tau)))|d\tau d\sigma\\
&\leqslant a\delta + a\sum_{k=1}^{m} a_k kc[k_1\|\mu\| + \mathcal{V}] \int_0^{\tau_k} \frac{(\tau_k - \sigma)^{\eta-1}}{\Gamma(\eta)} \int_0^{\sigma} \frac{(\sigma - \tau)^{\sigma-1}}{\Gamma(\sigma)} |\mu(\varphi(\tau)) - \mu^*(\varphi(\tau))|d\tau d\sigma
\end{aligned}$$

$$+ kc[k_1\|\mu\| + \mathcal{V}]\int_0^{\tau}\frac{(\tau-\sigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}|\mu(\varphi(\tau)) - \mu^*(\varphi(\tau))|d\tau d\sigma$$
$$\leqslant a\delta + a\sum_{k=1}^{m} a_k kc[k_1\|\mu\| + \mathcal{V}]\|\mu-\mu^*\|\int_0^{\tau_k}\frac{(\tau_k-\sigma)^{\eta-1}}{\Gamma(\eta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}d\tau d\sigma$$
$$+ kc[k_1\|\mu\| + \mathcal{V}]\|\mu-\mu^*\|\int_0^{\tau}\frac{(\tau-\sigma)^{\beta-1}}{\Gamma(\eta)}\int_0^{\sigma}\frac{(\sigma-\tau)^{\sigma-1}}{\Gamma(\sigma)}d\tau d\sigma,$$

taking supermum over $\tau \in I$,

$$\|\mu-\mu^*\| \leqslant a\delta + \frac{[a\sum_{k=1}^{m} a_k + 1]kc[k_1\|\mu\| + \mathcal{V}]}{\Gamma(\sigma+1)\Gamma(\eta+1)}\|\mu-\mu^*\|,$$
$$\|\mu-\mu^*\| \leqslant (1 - \frac{[a\sum_{k=1}^{m} a_k + 1]kc[k_1\|\mu\| + \mathcal{V}]}{\Gamma(\sigma+1)\Gamma(\eta+1)})^{-1}a\delta = \epsilon.$$

Hence,

$$\|\mu-\mu^*\| \leqslant \epsilon.$$

Thus, the integral equation (2.2) has a continuous dependence on $\mu_\circ$. So the solution of the non-local problem (1.1)-(1.4) depends continuously on initial data $\mu_\circ$. □

4. Illustrative example

We provide an example in this section to demonstrate our findings.

Example 4.1. Consider the following nonlinear integro-differential inclusion:

$$^{c}D^{\eta}\left(\frac{\mu(\tau)-\mu(0)}{\vartheta(\tau,\mu(\varphi_1(\tau)))}\right) \in \Phi_1(\tau, I^{\sigma}\phi_2(\tau,\mu(\varphi(\tau))),\ \tau\in[0,1],\ \eta,\sigma\in(0,1) \tag{4.1}$$

with infinite point boundary condition

$$\mu(0) + \sum_{k=1}^{\infty}\frac{1}{k^2}\mu(\frac{k-1}{k}) = \mu_\circ. \tag{4.2}$$

We choose $\Phi_1 : [0,1]\times\mathbb{R} \to 2^{\mathbb{R}^+}$ in (4.1) as

$$\Phi_1(\tau, I^{\frac{1}{4}}\phi_2(\tau,\mu(\tau)) = \left[0, \tau^2 + 1 + \int_0^{\tau}\frac{(\tau-\sigma)^{-\frac{3}{4}}}{2\Gamma(\frac{3}{4})}\big(\sin(\mu(\sigma)+1) + \frac{\mu(\sigma)}{e^{\sigma}}\big)ds\right],$$

set

$$\phi_2(\tau,\mu(\tau)) = \frac{1}{2}\big(\sin(\mu(\sigma)+1) + \frac{\mu(\sigma)}{e^{\sigma}}\big).$$

Define the continuous map $\phi_1 : [0,1]\times\mathbb{R}\to\mathbb{R}$, notice that for $\phi_1 \in S_{\Phi_1}$, then we have

$$|\phi_1(\tau, I^{\frac{1}{4}}\phi_2(\tau,\mu(\varphi(\tau)))) - \phi_1(\tau, I^{\frac{1}{4}}\phi_2(\tau,\nu(\varphi(\tau))))| \leqslant \frac{1+e}{2e\Gamma(\frac{1}{4})}|\mu-\nu|,$$

and

$$\big|\phi_2(\tau,\mu(\tau))\big| \leqslant \frac{1}{2}|\sin(\mu(\tau)+1)| + \frac{|\mu(\tau)|}{2e},\quad \vartheta(\tau,\mu(\varphi_1(\tau))) = \frac{e^{-\pi t}}{1+t} + \frac{e^{-\ln(t+1)}x(t)}{1+x(t)},$$

and

$$|\vartheta(\tau,\mu_1(\varphi_1(\tau))) - \vartheta(\tau,\mu_2(\varphi_1(\tau)))| \leqslant e^{-\ln(t+1)}|\mu_1-\mu_2|.$$

As a result, conditions of Theorem 2.2 are hold with $k = \frac{1+e}{2e\Gamma(\frac{1}{4})} \approx 0.1889 < 1$, $a(\tau) = \frac{1}{2}\sin(\mu(t)+1) \in L^1[0,1]$, $b = \frac{1}{2e}$ and the series $\sum_{k=1}^{\infty}\frac{1}{k^4}$ is convergent. Also, $[k_1\|\mu\| + \mathcal{V}][a(|\mu_\circ| + \sum_{k=1}^{m}|a_k|) + 1]k \approx 0.6136 < 1$ and $\frac{b}{\Gamma(\sigma+1)} \approx 0.2029 < 1$. We deduce from Theorem 2.2 that the nonlocal problem (4.1)-(4.2) has at least one continuous solution.

5. Conclusion

In our recent studies, for a class of a hybrid differential inclusion with integrated boundary conditions BCs or with infinite-point BCs we considered a continuous solution. We have demonstrated that if the continuous solutions for BVPs can be obtained with m-point BCs, then solutions for this problem can also be obtained with integrated BCs or with infinite-point BCs. It is possible to use the suggested method to more fractional hybrid differential inclusion. Additionally, various types of inclusion can be solved using this method. In future work, we will investigate positive solution and its stability for the following boundary value problem of the fractional hybrid differential inclusion:

$$ {}^{H}D^{\eta}\left(\frac{D^{\alpha}\mu(\mathfrak{r})-\mu(0)}{\vartheta(\mathfrak{r},\mu(\mathfrak{r}))}\right) \in \Phi_1(\mathfrak{r}, D^{\alpha}\mu(\mathfrak{r}), I^{\sigma}\varphi_2(\mathfrak{r}, D^{\alpha}\mu(\mathfrak{r}))),\ \alpha \in (0,1),\ \mathfrak{r} \in (0,1], $$

equipped with Riemann-Stieltjes integro boundary conditions (1.2) or the infinite-point boundary conditions (1.3), where ${}^{H}D^{\eta}$ is the standard Hilfer fractional derivative.

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