J. Math. Computer Sci., 32 (2024), 43–53

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

New results on soft generalized topological spaces

Jawaher Al-Mufarrij[a], S. Saleh[b,c,*]

[a]Department of Mathematics, Women section, King Saud University, Riyadh 12372, KSA.

[b]Department of Computer Science, Cihan University-Erbil, Erbil, Iraq.

[c]Department of Mathematics, Hodeidah University, Hodeidah, Yemen.

Abstract

This work aims to introduce and discuss two new classes of separation properties namely, soft generalized R_0 and R_1 in a soft generalized topological space defined on an initial universe set, by using the notions of soft g-open sets and soft g-closure operator. We investigate some of their properties and characterizations. We further, investigate the relationships between different generalized structures of soft topology, providing some illustrative examples and results. Additionally, we present connections between these separation properties and those in some generated topologies. Furthermore, we show that being SGR_i, $i = 0, 1$ are soft generalized topological properties.

Keywords: Soft sets, soft g-open sets, soft generalized topology, Sg-closure, SG-kernel, soft generalized R_0 and R_1 spaces.

2020 MSC: 54A05, 54A40, 54D15.

1. Introduction and preliminaries

Molodtsov [21] introduced the concept of soft sets(or S-sets), in 1999 as a tool for dealing with uncertain problems. Since then, many works have been published on S-set theory and its applications in various fields, as in [1, 3, 4, 7–9, 13, 14, 16, 19]. Shabir-Naz [25] introduced the topological structure of S-sets and studied various related concepts, leading to the development of generalized structures of soft topology, including supra soft topology [12], infra soft topology [5], and soft generalized topology [27]. While many results from soft topology hold true in these generalized structures, some become invalid. On the other hand, Csaszar [10] introduced the concept of generalized topology as a generalization of general topology. Al-Omari-Noiri [2] proposed a unified theory of contra-(μ, λ)-continuous functions on generalized topological spaces. Jyothis-Sunil [27, 28] defined the concept of soft generalized topology on S-sets and studied some related notions.

Soft separability properties have been studied in many articles as in [6, 15, 23–26]. Jyothis-Sunil [29] defined and studied some soft generalized separation axioms. In this work, we continue to study soft generalized separation axioms and generalize some soft separability properties by defining the properties SGR_i, $i = 0, 1$. We discuss some results, characterizations, and relationships with supporting examples.

*Corresponding author

Email addresses: jmufarij@ksu.edu.sa (Jawaher Al-Mufarrij), s_wosabi@yahoo.com (S. Saleh)

doi: 10.22436/jmcs.032.01.04

Received: 2023-01-30 *Revised:* 2023-03-02 *Accepted:* 2023-05-10

This work is organized as follows. in the introduction, we review some known definitions and results in the soft setting that will be used in the subsequent sections. In Section 2, we define a soft generalized topology on a universe set U and discuss various examples, concepts, and properties, as well as the relationships between the generalized structures of soft topology. In Section 3, we present the definitions of the soft generalized separation properties SGR_i , $i = 0, 1$ and investigate the basic properties, characterizations, and related theorems for them. In Section 4, we present more properties, results, relationships with some necessary examples. We show that the SGR_i, $i = 0, 1$ are soft generalized topological property.

In all the paper, U refers to an initial universe set, T is the set of all parameters for U, 2^U is the power set of U, and SG-refers to soft generalized. Next, we give some concepts and results about S-set theory, for more details see [11, 17, 18, 20, 21, 30].

An S-set $H_T = (H, T)$ on U is a mapping $H : T \longrightarrow 2^U$ that is, H_T can be written as a set of ordered pairs $H_T = \{(t, h(t)) : t \in T,\ H(t) \in 2^U\}$. The class of all S-sets on U is symbolized by $SS(U)$.

For $H_T, K_T \in SS(U)$ and $x \in U$, we have following.

(i) If $H(t) = \emptyset$ (resp. $H(t) = U$) for any $t \in T$, then H_T is called a null (resp. universal) S-set and symbolized by $\widetilde{\emptyset}$ (resp. $\tilde{U}$).

(ii) The relative complement H^c_T of H_T, where $H^c : T \longrightarrow 2^U$ is a mapping given by $H^c(t) = U - H(t)$ for every $t \in T$. Clearly $(H^c_T)^c = H_T$.

(iii) H_T is an S-subset of K_T is symbolized by $H_T \widetilde{\subseteq} K_T$ if $H(t) \subseteq K(t)$ for all $t \in T$.

(iv) The S-union (resp. S-intersection) of H_T and K_T is an S-set G_T (resp. L_T) given by $G(t) = H(t) \cup K(t)$ (resp. $L(t) = H(t) \cap K(t)$) for all $t \in T$ and is symbolized by $H_T \widetilde{\cup} G_T$ (resp. $H_T \widetilde{\cap} G_T$).

For $H_T \in SS(U)$, $Y \subseteq U$, and $x \in U$, we have following.

(i) If $H(t) = \{x\}$ and $H(t') = \emptyset$ for every $t' \in T - \{t\}$, then H_T is called an S-point on U symbolized by x_t. We write $x_t \widetilde{\in} H_T$ if for the element $t \in T$, $x \in H(t)$. The class of all S-points in $\tilde{U}$ is denoted by $SP(U)$.

(ii) $x \in H_T$ if $x \in H(t)$ for all $t \in T$, and $x \notin H_T$ if $x \notin H(t)$ for some $t \in T$.

(iii) If $H(t) = \{x\}$ for all $t \in T$, then H_T is called an S-singleton point denoted by x_T. We write $x_T \widetilde{\in} H_T \iff x \in H_T \iff x_t \widetilde{\in} H_T$ for all $t \in T$.

(iv) $\tilde{Y} = (Y, T)$ refers to the S-set on U for which $Y(t) = Y$ for all $t \in T$, is called stable. We write $x_t \neq y_t$ if $x \neq y$.

Definition 1.1. Let $SS(U)$ and $SS(V)$ be the two families of all S-sets on U, V respectively and let $u : U \longrightarrow V$ and $p : T \longrightarrow E$ be two maps, then the map $f_{up} : SS(U) \longrightarrow SS(V)$ is said to be a soft map (briefly, S-map) and we have:

(i) for $H_T \in SS(U)$, the image $f_{up}(H_T)$ of H_T is the S set on V given by $f_{up}(H_T)(e) = \cup\{u(H(t)) : t \in p^{-1}(e)\}$ if $p^{-1}(e) \neq \emptyset$ and $f_{up}(H_T)(e) = \widetilde{\emptyset}$ otherwise for any $e \in E$;

(ii) for $G_E \in SS(V)$, the preimage $f^{-1}_{up}(G_E)$ of G_E is the S-set on U given by $f^{-1}_{up}(G_E)(t) = u^{-1}(G(p(t)))$ for any $t \in T$.

The S-map f_{up} is called one-one (resp. onto and bijective), if u and p are one-one (resp. onto and bijective). For more details about the properties of S-maps see [17].

Definition 1.2 ([10])**.** A generalized topology (or GT) on U is a collection σ of subsets of U, which is closed under arbitrary unions and satisfies $\emptyset \in \sigma$. Any set in (U, σ) is called an g-open set.

Definition 1.3 ([22])**.** An GTS (U, τ) is said to be:

(i) GR_0 if for any $x \neq y \in U$ with $cl(x) \neq cl(y)$ implies $cl(x) \cap cl(y) = \emptyset$;

(ii) GR_1 if for any $x \neq y \in U$ with $cl(x) \neq cl(y)$, there are disjoint g-open subsets F, G of U such that $x \in F$ and $y \in G$.

Definition 1.4 ([25])**.** A family $\tau \subseteq SS(U)$ under a fixed set of parameters T is called a soft topology on U if τ is closed under arbitrary S-unions, finite S-intersections and $\widetilde{U}, \widetilde{\phi} \in \tau$. The triple (U, τ, T) is called a soft topological space (or STS). Any element in τ is called an S-open set, and the complement of any S-open set is called an S-closed set.

The S-closure $cl(F_T)$ of F_T in (U, τ, T) is the S-intersection of all S-closed super sets of F_T, and the S-interior $int(F_T)$ of F_T is the S-union of all S-open sets contained in F_T.

Definition 1.5 ([23])**.** An STS (U, τ, T) is said to be:

(i) SR_0 if for any $x_t \neq y_t \in SP(U)$ with $x_t \widetilde{\in} cl(y_t)$ implies $y_t \widetilde{\in} cl(x_t)$;
(ii) SR_1 if for any $x_t \neq y_t \in SP(U)$ with $cl(x_t) \neq cl(y_t)$, there are disjoint S-open subsets F_T, G_T of U such that $x_t \in F_T$ and $y_t \in G_T$.

2. On soft generalized topological spaces

Jyothis-Sunil [27] gave the definition of soft generalized topology on a soft set. In this section, we give the definition of soft generalized topology on an initial universe set U as one of the generalized structures of soft topology. More examples, concepts, and properties are presented. In addition, the connections with other generalized structures of soft topology are examined.

First, we recall the definitions of some generalized structures of soft topology such as supra soft topology [12] and infra soft topology [5] as follows.

Definition 2.1. A family $\sigma \subseteq SS(U)$ with a fixed set of parameters T is said to be:

(i) a supra soft topology (briefly, SST) on U if the S-union of any number of S-sets in σ belongs to σ and $\widetilde{\emptyset}$, $\tilde{U} \in \sigma$;
(ii) an infra soft topology (briefly, IST) on U if it is closed under finite S-intersections and $\widetilde{\emptyset} \in \sigma$.

Definition 2.2. A collection g of S-sets on U with a fixed set of parameters T is said to be a soft generalized topology on U if $\widetilde{\emptyset} \in g$ and it is closed under arbitrary S-unions of members in g. The triple (U, g, T) is called a soft generalized topological space (briefly, SGTS), any element in g is called a soft g-open set (briefly, Sg-open), and its relative complement is called an Sg-closed set. The set of all Sg-closed sets in U is denoted by g^c.

In the next, we give some examples of soft generalized topologies on U.

Example 2.3. The following classes are soft generalized topologies on U.

(1) $g_1 = \{H_{iT} \in SS(U) : H_{1T} \subseteq H_{2T} \subseteq \cdots \subseteq H_{iT},\ i \in J\}$.
(2) $g_2 = \left\{H_T \in SS(U) : x_T \widetilde{\in} H_T\right\} \cup \{\widetilde{\emptyset}\}$.
(3) $g_3 = \{\widetilde{\emptyset}, \widetilde{U},\ F_T,\ F_T^c\}$ for any $F_T \in SS(U)$.
(4) $g_4 = \{\widetilde{\emptyset}, H_T\}$ for any $H_T \in SS(U)$.

Remark 2.4. Let (U, g, T) be an SGTS and (U, τ, T) be STS, then we have:

(1) if H_T and F_T are two Sg-open sets, then $H_T \widetilde{\cap} F_T$ need not be Sg-open set;
(2) if H_T is an Sg-open set and F_T is S-open set, then $H_T \widetilde{\cap} F_T$ need not be Sg-open set, but if $g = SS(U)$, $H_T \in g$, and $F_T \in \tau$, then $H_T \widetilde{\cap} F_T$ is Sg-open set;
(3) if g_i are SGTs on U for all $i \in J$, then $\cap g_i$ is SGT on U.

Now by the next results and examples, we can describe the relationships between the generalized structures of ST such as SST, IST, and SGT as follows.

Result 2.5. Clearly, every ST on $\mathcal{U}$ is an SST, SGT, and IST on $\mathcal{U}$, but the converse is not true in general. The next examples show it.

Example 2.6. Let $\mathcal{U} = \{a, b, c, d, e\}$, $T = \{t_1, t_2\}$, and $\sigma = \{\widetilde{\emptyset}, \tilde{\mathcal{U}}, F_T, G_T, H_T\}$, where, $F_T = \{(t_1,\{a, d\}), (t_2, \{a, c\})\}$, $G_T = \{(t_1,\{b, d\}), (t_2,\{b, c\})\}$, and $H_T = \{(t_1,\{a, b, d\}), (t_2,\{a, b, c\})\}$. One can verify that σ is an SST and SGT on $\mathcal{U}$, but not ST.

Example 2.7. Let $\mathcal{U} = \{a, b, c\}$, $T = \{t_1, t_2\}$, and $\sigma = \{\widetilde{\emptyset}, \tilde{\mathcal{U}}, F_T, H_T\}$, where $F_T = \{(t_1,\{a\}), (t_2, \emptyset)\}$, $H_T = \{(t_1,\{b, c\}), (t_2,\{b\})\}$. One can check that σ is an IST on $\mathcal{U}$, but not (ST, SST, SGT) on $\mathcal{U}$.

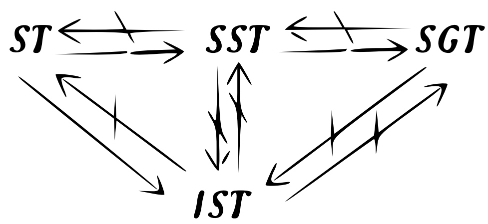
Result 2. Clearly, every SST is an SGT, but the converse is not true in general. The next example shows it.

Example 2.8. From the Example 2.6, consider the class $g = \{\widetilde{\emptyset}, G_T, H_T\}$, it is clear that g is an SGT on $\mathcal{U}$, but not SST.

Result 3. An IST is independent of SST and SGT. The next example shows it.

Example 2.9. Let $\mathcal{U} = \{a, b, c\}$, $T = \{t_1, t_2\}$, and $\sigma = \{\widetilde{\emptyset}, \tilde{\mathcal{U}}, F_T, H_T\}$, where, $F_T = \{(t_1,\{a, c\}), (t_2, \mathcal{U})\}$, $H_T = \{(t_1,\{b\}), (t_2,\{b\})\}$, then σ is an GST and SST on $\mathcal{U}$, but not IST. On other hand, the collection σ in Example 2.7 is an IST on $\mathcal{U}$, but is neither SST and nor SGT on $\mathcal{U}$.

The relationships among ST, SST, IST, and SGT can be summarized as follows:



Definition 2.10. Let $(\mathcal{U}, \sigma)$ be a generalized topological space and T be a fixed set of parameters. The family $g_\sigma = \{F_T : F(t) = A \text{ for all } t \in T \text{ and } A \in \sigma\}$ defines an SGT, called stable SGT on $\mathcal{U}$ generated by σ. In general, an SGTS $(\mathcal{U}, g, T)$ is called stable if any Sg-open set in $(\mathcal{U}, g, T)$ is stable.

Definition 2.11. An SGTS $(\mathcal{U}, g, T)$ is called a strong stable soft generalized topological space (briefly, strong stable SGTS) if $g = \{H_T : H(t) = B \text{ for all } t \in T \text{ and } B \subset \mathcal{U}\}$. In this case any S-singleton point x_T in $\mathcal{U}_T$ is an Sg-open set.

Definition 2.12. Let $(\mathcal{U}, g, T)$ be an SGTS, the collection $g_t = \{H(t) : H_T \in g\}$ for each $t \in T$ defines a generalized topology on $\mathcal{U}$, called a parametric GT.

Remark 2.13.

(1) If $(\mathcal{U}, g, T)$ is a strong stable SGTS, we have:
  (i) any element in $(\mathcal{U}, g, T)$ is both Sg-open and Sg-closed set;
  (ii) $(\mathcal{U}, g_t)$ is a discrete space for all $t \in T$;
  (iii) every $(\mathcal{U}, g_\sigma, T)$ is a subspace of a strong stable SGTS $(\mathcal{U}, g, T)$;
  (iv) every strong stable SGTS $(\mathcal{U}, g, T)$ is a subspace of soft discrete space $(\mathcal{U}, \tau, T)$.

(2) Let $(\mathcal{U}, \sigma)$ be a discrete TS, we have the SGT g_σ, which is defined in Definition 2.10, is a strong stable SGT on $\mathcal{U}$.

Definition 2.14. For SGT $(\mathcal{U}, g, T)$, if $H^c \in g$ for every $H \in g$, then $(\mathcal{U}, g, T)$ is called a complemental SGTS.

Example 2.15.

(1) Let $U = \{a, b, c\}$, $T = \{t_1, t_2\}$, and the class $\sigma = \{\emptyset, U, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}\}$. Then σ is a GT on U and from Definition 2.10, we have the class $g_\sigma = \{\widetilde{\emptyset}, \tilde{U}, \widetilde{\{c\}}, \widetilde{\{a, b\}}, \widetilde{\{b, c\}}, \widetilde{\{a, c\}}\}$ is an SGT on U.

(2) Let $U = \{x, y, z\}$, $T = \{t_1, t_2\}$ and $\tau = SS(U)$ be a soft discrete topology on U. The class $g = \{\widetilde{\emptyset}, \tilde{U}, x_T, y_T, z_T, \widetilde{\{x, y\}}, \widetilde{\{x, z\}}, \widetilde{\{y, z\}}\}$ is a strong stable SGT on U and any element in g is both Sg-open and Sg-closed set. Moreover, g is a complemental SGT and it is a subspace of τ. On other hand, $g_{t_1} = g_{t_2} = \{\emptyset, U, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}\}$ is a discrete topology on U.

Definition 2.16. An S-set H_T in (U, g, T) is said to be an Sg-neighborhood (briefly, Sg-nbd) of x_t if there is an $F_T \in g$ such that $x_t \widetilde{\in} F_T \widetilde{\subseteq} H_T$.

Notation. O_{x_t} refers to an Sg-open set containing x_t, and is called an Sg-open nbd of x_t.

Definition 2.17. Let H_T be an S-set in SGTS (U, g, T). Then the Sg-closure $cl_g(H_T)$ of H_T is the S-intersection of all Sg-closed super sets of H_T, and the Sg-interior $int_g(H_T)$ of H_T is the S-union of all Sg-open sets contained in H_T.

Proposition 2.18. *Let* (U, g, T) *be an* SGTS *and* $H_T,\ K_T \in SS(U)$, *we have:*

(i) $H_T \in g^c$ *if and only if* $cl_g(H_T) = H_T$;
(ii) $H_T \widetilde{\subseteq} F_T$ *implies* $cl_g(H_T) \widetilde{\subseteq} cl_g(F_T)$;
(iii) $x_t \widetilde{\in} cl_g(H_T)$ *if and only if* $O_{x_t} \widetilde{\cap} H_T \neq \widetilde{\emptyset}$ *for all* $O_{x_t} \in g$.

Proof. The proofs of (i) and (ii) are obvious.

(iii) Let $x_t \widetilde{\in} cl_g(H_T)$, then $x_t \widetilde{\in} F_T$ for all $F_T \in g_c$ such that $H_T \widetilde{\subseteq} F_T$. Suppose that there is an Sg-open set O_{x_t} containing x_t with $O_{x_t} \widetilde{\cap} H_T = \widetilde{\emptyset}$, then $H_T \widetilde{\subseteq} O^c_{x_t}$. This is a contradiction. Hence the result holds. Conversely, suppose that $x_t \widetilde{\notin} cl_g(H_T)$, then $x_t \widetilde{\in} (cl_g\,(H_T))^c = O_{x_t}$, i.e., there is an Sg-open set containing x_t such that $H_T \widetilde{\cap} (cl_g\,(H_T))^c = \widetilde{\emptyset}$, and the result holds. □

Definition 2.19. Let (U, g, T) be an SGTS and $Y \subseteq U$. The family $g_Y = \{\tilde{Y} \widetilde{\cap} H_T : H_T \in g\}$ is an SGT on Y, and (Y, g_Y, T) is called an SGT-subspace of (U, g, T). For the SGT-subspace (Y, g_Y, T) of (U, g, T) and $H_E \in SS(Y)$ we have H_T is an Sg-open set in Y if and only if $H_T = \tilde{Y} \widetilde{\cap} G_T$ for some $G_T \in g$.

Definition 2.20. Let (U, g, T) be an SGTS, $H_T \in SS\,(U)$, and $x_t \in SP(U)$, then the soft generalized kernel of H_T, denoted as $SGK\,(H_T)$ is the S-set given as $SGK\,(H_T) = \widetilde{\cap}\{F_T \in g : H_T \widetilde{\subseteq} F_T\}$. In particular, the soft generalized kernel of $x_t \in SP(U)$ is given by $SGK\,(x_t) = \widetilde{\cap}\{F_T \in g : x_t \widetilde{\in} F_T\}$.

Lemma 2.21. *Let* (U, g, T) *be an* SGTS *and* $H_T \in SS(U)$. *Then* $SGK\,(H_T) = \widetilde{\cup}\{x_t \in SP(U) : cl_g(x_t) \widetilde{\cap} H_T \neq \emptyset_T\}$.

Proof. Let $x_t \widetilde{\in} SGK\,(H_T)$. Suppose that $cl_g\,(x_t) \widetilde{\cap} H_T = \widetilde{\emptyset}$, then $H_T \widetilde{\subseteq} (cl_g\,(x_t))^c$ and $x_t \widetilde{\notin} (cl_g\,(x_t))^c$, which is an Sg-open set containing H_T. This contradicts with $x_t \widetilde{\in} SGK\,(H_T)$. So, $cl_g\,(x_t) \widetilde{\cap} H_T \neq \widetilde{\emptyset}$ and $SGK\,(H_T) \widetilde{\subseteq} \widetilde{\cup}\{x_t \in SP\,(U) : cl_g\,(x_t) \widetilde{\cap} H_T \neq \widetilde{\emptyset}\}$.

Conversely, let $cl_g\,(x_t) \widetilde{\cap} H_T \neq \widetilde{\emptyset}$. Suppose $x_t \widetilde{\notin} GSK\,(H_T)$, then there is an $K_T \in g$ such that $H_T \widetilde{\subseteq} K_T$ and $x_t \widetilde{\notin} K_T$. Now let $y_t \widetilde{\in} cl_g\,(x_t) \widetilde{\cap} H_T$, we have $y_t \widetilde{\in} cl_g\,(x_t)$ and since K_T is an Sg-open set containing y_t this implies $x_t \widetilde{\in} K_T$, a contradiction. So $x_t \widetilde{\in} SGK\,(H_T)$. □

Lemma 2.22. *Let* (U, g, T) *be an* SGTS *and* $x_t \in SP(U)$, *then* $y_t \widetilde{\in} SGK\,(x_t)$ *if and only if* $x_t \widetilde{\in} cl_g(y_t)$.

Proof. It is obvious. □

Lemma 2.23. *Let* (U, g, T) *be an* SGTS *and* $x_t,\ y_t \in SP(U)$, *then* $SGK(x_t) \neq SGK(y_t)$ *if and only if* $cl_g\,(x_t) \neq cl_g\,(y_t)$.

Proof. Let $SGK(x_t) \neq SGK(y_t)$, there is $z_t \in SP(U)$ with $z_t \widetilde{\in} SGK(x_t)$ and $z_t \widetilde{\notin} SGK(y_t)$. If $z_t \widetilde{\in} SGK(x_t)$, from Lemma 2.21, we get $x_t \widetilde{\cap} cl_g(z_t) \neq \widetilde{\emptyset}$ implies $x_t \widetilde{\in} cl_g(z_t)$, that is $cl_g(x_t) \widetilde{\subseteq} cl_g(z_t)$. Similarly, if $z_t \widetilde{\notin} SGK(y_t)$ we get $y_t \widetilde{\notin} cl_g(z_t)$. Since $cl_g(x_t) \widetilde{\subset} cl_g(z_t)$ and $y_t \widetilde{\notin} cl_g(x_t)$, we have $y_e \widetilde{\notin} cl_g(x_t)$. Hence $cl_g(x_t) \neq cl_g(y_t)$.

Conversely, let $cl_g(x_t) \neq cl_g(y_t)$, there is $z_t \in SP(U)$ with $z_t \widetilde{\in} cl_g(x_t)$ and $z_e \widetilde{\notin} cl_g(y_t)$. Thus, there is an Sg-open set containing z_t and so x_t but not y_t. Hence $y_t \widetilde{\notin} SGK(x_t)$ and the proof is complete. □

Now, let us give the next definition which is obtained by replacing τ and S-open sets in [29] with g and Sg-open sets, respectively.

Definition 2.24. An SGTS (U, g, T) is said to be:

(i) soft generalized T_0 (briefly, SGT_0) iff for any x_t, $y_t (x \neq y)$ there are Sg-open sets H_T and F_T such that $x_t \widetilde{\in} F_T$ and $y_t \widetilde{\notin} H_T$ or $y_t \widetilde{\in} H_T$ and $x_t \widetilde{\notin} H_T$;

(ii) soft generalized T_1 (briefly, SGT_1) iff for any x_t, $y_t (x \neq y)$ there are Sg-open sets H_T and F_T such that $x_t \widetilde{\in} H_T$, $y_t \widetilde{\notin} H_T$ and $y_t \widetilde{\in} F_T$, $x_t \widetilde{\notin} F_T$;

(iii) soft generalized T_2 (briefly, SGT_2) iff for any x_t, $y_t (x \neq y)$ there are Sg-open sets H_T and F_T such that $x_t \widetilde{\in} H_T$, $y_t \widetilde{\in} F_T$ and $H_T \widetilde{\cap} F_{ET} = \widetilde{\emptyset}$.

Remark 2.25. Clearly, $SGT_2 \Longrightarrow SGT_1 \Longrightarrow SGT_0$

3. On soft generalized R_0 and R_1 spaces

In the following, we introduce and study two new classes of soft generalized separation properties, called SGR_i, $i = 0, 1$ and investigate some characterizations for them.

Definition 3.1. An SGTS (U, g, T) is called soft Generalized R_0 (briefly, SGR_0) iff for any $x_t \neq y_t \in SP(U)$ with $x_t \widetilde{\in} cl_g(y_t)$ implies $y_t \widetilde{\in} cl_g(x_t)$.

Theorem 3.2. *An SGTS (U, g, T) is SGR_0 if and only if $cl_g(x_t) \widetilde{\subseteq} H_T$ for all $H_T \in g$, $x_t \widetilde{\in} H_T$.*

Proof. Let (U, g, T) be SGR_0. Suppose $cl_g(x_t) \widetilde{\nsubseteq} H_T$ for some $H_T \in g$ and $x_t \widetilde{\in} H_T$, there is an S-point y_t such that $y_t \widetilde{\in} cl_g(x_t)$, $y_t \widetilde{\notin} H_T$. So that $y_t \widetilde{\cap} H_T = \widetilde{\emptyset}$ for some $H_T \in g$, $x_t \widetilde{\in} H_T$ and x_t, $y_t \in SP(U)$ with $x \neq y$. Thus $x_t \widetilde{\notin} cl_g(y_t)$. This is a contradiction. Thus, the necessary part holds.

Conversely, let $x_t \widetilde{\notin} cl_g(y_t)$, there is an Sg-open set K_T containing x_t such that $y_t \widetilde{\cap} K_T = \widetilde{\emptyset}$ this implies that $y_t \widetilde{\notin} K_T$. By hypothesis $cl_g(x_t) \widetilde{\subseteq} K_T$, we get $y_t \widetilde{\notin} cl_g(x_t)$. Therefore (U, g, T) is SGR_0. □

Theorem 3.3. *For SGTS (U, g, T) and $x_t \in SP(U)$, the next items are equivalent:*

(1) *(U, g, T) is SGR_0;*

(2) *for any $H_T \in g^c$ with $x_t \widetilde{\notin} H_T$, we have $cl_g(x_t) \widetilde{\cap} H_T = \widetilde{\emptyset}$;*

(3) *for any x_t, $y_t \in SP(U)$ $(x \neq y)$, either $cl_g(x_t) = cl_g(y_t)$ or $cl_g(x_t) \widetilde{\cap} cl_g(y_t) = \widetilde{\emptyset}$.*

Proof.

(1) $\Longrightarrow$ (2) It follows from that of the above theorem.

(2) $\Longrightarrow$ (3) Let $x_t \neq y_t \in SP(U)$ with $cl_g(x_t) \neq cl_g(y_t)$, there is $z_t \widetilde{\in} cl_g(x_t)$ and $z_t \widetilde{\notin} cl_g(y_t)$(or, $z_t \widetilde{\in} cl_g(y_t)$ and $z_t \widetilde{\notin} cl_g(x_t)$). Thus there is $H_T \in g$ such that $y_t \widetilde{\notin} H_T, z_t \widetilde{\in} H_T$ and so, $x_t \widetilde{\in} H_T$. Therefore, $x_t \widetilde{\notin} cl_g(y_t)$. From (2) we get, $cl_g(x_t) \widetilde{\cap} cl_g(y_t) = \widetilde{\emptyset}$. The proof of the rest case is similar.

(3) $\Longrightarrow$ (1) Let $x_t \neq y_t \in SP(U)$ with $x_t \widetilde{\notin} cl_g(y_t)$, we get $cl_g(x_t) \neq cl_g(y_t)$. From (3), we have $cl_g(x_t) \widetilde{\cap} cl_g(y_t) = \widetilde{\emptyset}$ which implies $y_t \widetilde{\in} cl_g(y_t) \widetilde{\subset} (cl_g(x_t))^c$ and so, $y_t \widetilde{\notin} cl_g(x_t)$. □

Corollary 3.4. *An* SGTS $(\mathcal{U}, g, T)$ *is* SGR_0 *if and only if for any* $x_t \neq y_t \in SP(\mathcal{U})$ *with* $cl_g(x_t) \neq cl_g(y_t)$ *implies* $cl_g(x_t) \widetilde{\cap} cl_g(y_t) = \widetilde{\emptyset}$.

Proof. It follows from that of the above theorems. □

Theorem 3.5. *For* SGTS $(\mathcal{U}, g, T)$ *the next statements are equivalent:*

(1) $(\mathcal{U}, g, T)$ *is* SGR_0;
(2) $H_T \in g^c \Longrightarrow H_T = SGK(H_T)$;
(3) $H_T \in g^c$ *and* $x_t \widetilde{\in} H_T \Longrightarrow SGK(x_t) \widetilde{\subseteq} H_T$;
(4) $x_t \in SP(\mathcal{U}) \Longrightarrow SGK(x_t) \widetilde{\subseteq} cl_g(x_t)$.

Proof.

(1) $\Longrightarrow$ (2) Let $H_T \in g^c$. Suppose that $x_t \widetilde{\notin} H_T$, we have $x_t \widetilde{\in} H_T^c$ which is an Sg-open set containing x_t. Since $(\mathcal{U}, g, T)$ is SGR_0, we get $cl_g(x_t) \widetilde{\subseteq} H_T^c$ implies $cl_g(x_t) \widetilde{\cap} H_T = \widetilde{\emptyset}$. From Lemma 2.21, we get $x_t \widetilde{\notin} SGK(H_T)$. So, $H_T = SGK(H_T)$.

(2) $\Longrightarrow$ (3) It follows from that $F_T \widetilde{\subseteq} G_T$ implies $SGK(F_T) \widetilde{\subseteq} SGK(G_T)$.

(3) $\Longrightarrow$ (4) Obvious.

(4) $\Longrightarrow$ (1) Let $x_t \neq y_t \in SP(\mathcal{U})$ with $x_t \widetilde{\in} cl_g(y_t)$. From Lemma 2.22, we get $y_t \widetilde{\in} SGK(x_t)$. Since $x_t \widetilde{\in} cl_g(x_t)$, which is an Sg-closed set and from (4), we have $y_t \widetilde{\in} SGK(x_t) \widetilde{\subseteq} cl_g(x_t)$, that is $y_t \widetilde{\in} cl_g(x_t)$ and this completes the proof. □

Proposition 3.6. *An* SGTS $(\mathcal{U}, g, T)$ *is* SGR_0 *if and only if* $cl_g(x_t) \widetilde{\subseteq} SGK(x_t)$ *for all* $x_t \in SP(\mathcal{U})$.

Proof. It follows from Lemma 2.22 and Theorem 3.2. □

From Lemma 2.22 and the above proposition, one can verify the next corollary.

Corollary 3.7. *An* SGTS $(\mathcal{U}, g, T)$ *is* SGR_0 *if for any* $x_t \in SP(\mathcal{U})$, $SGK(x_t) = cl_g(x_t)$.

Theorem 3.8. *An* SGTS $(\mathcal{U}, g, T)$ *is* SGR_0 *if and only if for any* $x_t \neq y_t \in SP(\mathcal{U})$ *with* $SGK(x_t) \neq SGK(y_t)$ *implies* $SGK(x_t) \widetilde{\cap} SGK(y_t) = \widetilde{\emptyset}$.

Proof.

$\Longrightarrow$ Let $(\mathcal{U}, g, T)$ be an SGR_0 and $x_t \neq y_t \in SP(\mathcal{U})$ with $GSK(x_t) \neq GSK(y_t)$. By Lemma 2.23, we get $cl_g(x_t) \neq cl_g(y_t)$. Suppose $SGK(x_t) \widetilde{\cap} SGK(y_t) \neq \widetilde{\emptyset}$, there is $z_t \widetilde{\in} SGK(x_t) \widetilde{\cap} SGK(y_t)$. Since $z_t \widetilde{\in} SGK(x_t)$, from Lemma 2.22 we have, $x_t \widetilde{\in} cl_g(z_t)$ implies $cl_g(x_t) \widetilde{\subset} cl_g(z_t)$. Since $x_t \widetilde{\in} cl_g(x_t)$ and from Corollary 3.4, we get $cl_g(x_t) = cl_g(z_t)$. Similarly, since $z_t \widetilde{\in} SGK(y_t)$, we have $cl_g(y_t) = cl_g(z_t) = cl_g(x_t)$. This is a contradiction. Therefore, $SGK(x_e) \widetilde{\cap} SGK(y_e) = \widetilde{\emptyset}$.

$\Longleftarrow$ Let $x_t \neq y_t \in SP(\mathcal{U})$ with $cl_g(x_t) \neq cl_g(y_t)$. From Lemma 2.23, we have $SGK(x_t) \neq GSK(y_t)$. By hypothesis, we get $SGK(x_e) \widetilde{\cap} SGK(y_e) = \widetilde{\emptyset}$. Suppose that $cl_g(x_t) \widetilde{\cap} cl_g(y_t) \neq \widetilde{\emptyset}$, there is $z_t \in SP(\mathcal{U})$ such that $z_t \widetilde{\in} cl_g(x_t)$ and $z_t \widetilde{\in} cl_g(y_t)$. Form Lemma 2.22, we have $x_t \widetilde{\in} GSK(z_t)$ and $y_t \widetilde{\in} SGK(z_t)$ and by Lemma 2.21, we obtain, $SGK(x_t) \widetilde{\cap} SGK(z_t) \neq \widetilde{\emptyset}$ and $SGK(y_t) \widetilde{\cap} SGK(z_t) \neq \widetilde{\emptyset}$. By hypothesis we get, $SGK(x_t) = SGK(z_t)$ and $SGK(y_t) = SGK(z_t) = SGK(x_t)$. So, $SGK(x_t) \widetilde{\cap} SGK(y_t) \neq \widetilde{\emptyset}$. This is a contradiction. Thus $cl_g(x_t) \widetilde{\cap} cl_g(y_t) = \widetilde{\emptyset}$. Hence by Corollary 3.4, we obtain the result. □

Definition 3.9. An SGTS $(\mathcal{U}, g, T)$ is called soft generalized R_1 (briefly, SGR_1) iff for any $x_t \neq y_t \in SP(\mathcal{U})$, with $cl_g(x_t) \neq cl_g(y_t)$, there are Sg-open sets H_T, K_T such that $x_t \widetilde{\in} H_T$ and $y_t \widetilde{\in} K_T$ with $H_T \widetilde{\cap} K_T = \widetilde{\phi}$.

Proposition 3.10. *Every* SGR_1 *space is* SGR_0.

Proof. Let $x_t \neq y_t \in SP(U)$ with $x_t \widetilde{\notin} cl_g(y_t)$, then $cl_g(x_t) \neq cl_g(y_t)$. Since (U, g, T) is SGR_1, there is $H_T \in g$ such that $y_t \widetilde{\in} H_T$ and $x_t \widetilde{\notin} H_T$. So $y_t \widetilde{\notin} cl_g(x_t)$, and this completes the proof. □

The converse of the above theorem is not necessary true, the next example shows it.

Example 3.11. Let U be an infinite set. The class $g = \{\widetilde{\emptyset}\} \cup \{H_T : (H(t))^c \text{ is a finite subset of } U \text{ for all } t \in T\}$ is SGT on U and (U, g, T) is called an SG cofinite space. Now one can verify g is SGR_0. But it is not SGR_1. Indeed, suppose that (U, g, T) is SGR_1 and $x_t \neq y_t \in SP(U)$ with $cl_g(x_t) \neq cl_g(y_t)$, there are $F_T, G_T \in g$ such that $x_t \widetilde{\in} F_T, y_t \widetilde{\in} G_T$ and $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$ implies $(F(t))^c \cup (G(t))^c = U$. Since $(F(t))^c$, $(G(t))^c$ are finite subsets of U, this means that U is finite. This is a contradiction. Thus (U, g, T) is not SGR_1.

Theorem 3.12. *Every strong stable* SGTS (U, g, T) *is* $SGR_i, i = 0, 1$.

Proof. For the case $i=1$, let (U, g, T) be a strong SGTS and $x_t, y_t \in SP(U)(x \neq y)$ such that $cl_g(x_t) \neq cl_g(y_t)$, there are Sg-open sets x_T, y_T such that $x_t \widetilde{\in} x_T$ and $y_t \widetilde{\in} y_T$ with $x_T \widetilde{\cap} y_T = \widetilde{\emptyset}$. Hence (U, g, T) is SGR_1. The proof of other case is obvious. □

Corollary 3.13. *Every stable* SGTS (U, g, T) *is* SGR_i, $i = 0, 1$.

Theorem 3.14. *An* SGTS (U, g, T) *is* SGR_1 *if and only if for any* $x_t \neq y_t \in SP(U)$ *with* $SGK(x_t) \neq SGK(y_t)$, *there are* $H_T, K_T \in g$ *such that* $cl_g(x_t) \widetilde{\subseteq} H_T$, $cl_g(y_t) \widetilde{\subseteq} K_T$ *and* $H_T \widetilde{\cap} K_T = \widetilde{\emptyset}$.

Proof. It follows by using Lemma 2.22. □

Proposition 3.15. *For* SGTS (U, g, T), *the next statements are equivalent.*

(1) (U, g, T) *is* SGR_1.
(2) *For any* $x_t \neq y_t \in SP(U)$ *with* $x_t \widetilde{\notin} cl_g(y_t)$, *there are* $F_T, G_T \in g$ *such that* $x_t \widetilde{\in} F_T$, $y_t \widetilde{\in} G_T$, *and* $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$.
(3) *For any* $x_t \neq y_t \in SP(U)$ *with* $cl_g(x_t) \neq cl_g(y_t)$, *there are* $F_T, G_T \in g$ *such that* $cl_g(x_t) \widetilde{\subseteq} F_T$ *and* $cl_g(y_t) \widetilde{\subseteq} G_T$ *with* $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$.

Proof. It follows from the above theorem and Lemma 2.23. □

Theorem 3.16. *Every complemental* SGTS (U, g, T) *is* SGR_i, $i = 0, 1$.

Proof. We will prove only the case $i = 1$. The proof of other case is similar. Let $x_t \neq y_t \in SP(U)$ and $x_t \widetilde{\notin} cl_g(y_t)$, then $x_t \widetilde{\in} (cl_g(y_t))^c = H_T \in g$. Since (U, g, T) is a complemental SGTS, we have $y_t \widetilde{\in} cl_g(y_t) = G_T \in g$. Clearly, $H_T \widetilde{\cap} G_T = \widetilde{\emptyset}$ and so, from Proposition 3.15 (2), the result holds. □

Corollary 3.17. *Every* SR_i *space is* SGR_i, *for* $i = 0, 1$.

4. More properties and relations

Theorem 4.1. *Every* SGT *subspace* (Y, g_Y, T) *of* SGR_i (U, g, T) *is* SGR_i, $i = 0, 1$.

Proof. We will show the case $i = 1$. The proof of the rest case is similar. Let $x_t \neq y_t \in SP(Y)$ with $cl_g(x_t) \neq cl_g(y_t)$, then x_t, y_t are different S-points in U with $cl_g(x_t) \neq cl_g(y_t)$. Since (U, g, T) is SGR_1, there are $F_T, G_T \in g$ such that $x_t \widetilde{\in} F_T$ and $y_t \widetilde{\in} G_T$ with $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$. So there are Sg-open sets $H_T^Y = Y_T \widetilde{\cap} F_T \in g_Y$ and $V_T^Y = Y_T \widetilde{\cap} G_T \in g_Y$ containing x_t, y_t, respectively, with $U_E^Y \widetilde{\cap} V_E^Y = \widetilde{\emptyset}$. Therefore (Y, g_Y, T) is SGR_1. □

The next example shows a SGTS with SGR_i and another GTS which does not have GR_i for $i = 0, 1$.

Example 4.2. Let $U = \{a, b, c\}$ and $T = \{t_1, t_2\}$. The class $g = \{\widetilde{\emptyset}, \tilde{U}, F_{1T}, F_{2T}\}$, where $F_{1T} = \{(t_1, U)\}$ and $F_{2T} = \{(t_2, U)\}$ is a SGT on U. One can verify that (U, g, T) is SGR_0 and SGR_1. On the other hand, the class $\tau = \{\emptyset, \{a\}, \{a, b\}\}$ is a GT on U which is not GR_0. Indeed, for the different points $a, b \in U$ with $cl(a) = U \neq cl(b) = \{b, c\}$, we have $cl(a) \cap cl(b) = \{b, c\} \neq \emptyset$.

Theorem 4.3. *If* (U, g, T) *is* SGR_i*, then* (U, g_t) *is* GR_i *for all* $t \in T$, $i = 0, 1$.

Proof. We will prove the case $i = 1$. The proof of the case $i = 0$ is similar. Let $x, y \in U$ and $x \neq y$ with $cl(x) \neq cl(y)$, then either $x \notin cl(y)$ or $y \notin cl(x)$. Thus, $x_t \notin cl_g(y_t)$ or $y_t \notin cl_g(x_t)$ this implies $cl_g(x_t) \neq cl_g(y_t)$. Since (U, g, T) is SGR_1, there are $H_T, K_T \widetilde{\in} g$ such that $x_t \widetilde{\in} H_T$ and $y_t \widetilde{\in} K_T$ with $H_T \widetilde{\cap} K_T = \widetilde{\emptyset}$ and so, there are $H(t)$ and $K(t) \in g_t$ such that $x \in H(t)$ and $y \in K(t)$ with $H(t) \cap K(t) = \emptyset$ for all $t \in T$. Therefore (U, g_t) is GR_1 for all $t \in T$. □

The next example shows that the converse of the above theorem may not be true.

Example 4.4. Let $U = \{a, b\}$ and $T = \{t_1, t_2\}$. Consider the class $g = \{\widetilde{\emptyset}, \widetilde{U}, H_{1T}, H_{2T}, H_{3T}, H_{4T}\}$, where $H_{1T} = \{(t_1, \{a\})\}$, $H_{2T} = \{(t_1, \{a\}), (t_2, \{a\})\}$, $H_{3T} = \{(t_1, \{a\}), (t_2, \{b\})\}$, and $H_{4T} = \{(t_1, \{a\}), (t_2, U)\}$, which is a SGT on U and the class $g_{t_2} = \{\emptyset, U, \{a\}, \{b\}\}$ is a GT on U. It is clear that (U, g_{t_2}) is GR_1 and GR_0. But (U, g, T) is not SGR_0. Indeed, for $a_{t_1}, b_{t_1} \in SP(U)(a \neq b)$, we have, $\tilde{U} = cl_g(a_{t_1}) \neq cl_g(b_{t_1}) = b_{t_1}$ but $cl_g(a_{t_1}) \widetilde{\cap} cl_g(b_{t_1}) \neq \widetilde{\emptyset}$. Hence (U, g, T) is not SGR_1.

Proposition 4.5. *Let* (U, g, T) *be a strong stable* SGTS*, then* (U, g, T) *is* SGR_i *if and only if* (U, g_t) *is* GR_i *for all* $t \in T$ *and* $i = 0, 1$.

Proof. We will give the proof for $i = 1$. The proof for the case $i = 0$ is similar.

$\Longrightarrow$ The proof follows from that of Theorem 4.3.

$\Longleftarrow$ Let $x_t \neq y_t \in SP(U)$ with $cl_g(x_t) \neq cl_g(y_t)$, then $x \neq y$ with $cl(x) \neq cl(y)$. Since (U, g_t) is GR_1, there are g-open subsets F, K of U such that $x \in F$ and $y \in K$ with $F \cap K = \emptyset$ imply there are $H_T, V_T \widetilde{\in} g$ such that $F = H(t)$ and $K = V(t)$ for all $t \in T$ with $x_t \widetilde{\in} H_T$ and $y_t \widetilde{\in} V_T$ with $H_T \widetilde{\cap} V_T = \widetilde{\emptyset}$. Therefore, (U, g, T) is SGR_1. □

Theorem 4.6. *A* GTS (U, σ) *is* GR_i *if and only if* (U, g_σ, T) *is* SGR_i, $i = 0, 1$.

Proof. We will give the proof for $i = 1$. The proof for the case $i = 0$ is similar.

$\Longrightarrow$ The proof is similar to that of the converse part in the above proposition.

$\Longleftarrow$ Let $x \neq y \in U$ with $cl(x) \neq cl(y)$, we have either $x \notin cl(y)$ or $y \notin cl(x)$ and this implies that $x_t \notin cl_g(y_t)$ or $y_t \notin cl_g(x_t)$, then $cl_g(x_t) \neq cl_g(y_t)$. Since (U, g_σ, T) is SGR_1, there are $F_T, G_T \widetilde{\in} g_\sigma$ such that $x_t \widetilde{\in} F_T, y_t \widetilde{\in} G_T$ and $F_T \widetilde{\cap} G_T = \widetilde{\emptyset}$. Thus, there are disjoint g-open sets $A, B \in \sigma$ such that $x \in F(t) = A$ and $y \in G(t) = B$ for all $t \in T$. Hence (U, σ) is GR_1. □

Theorem 4.7. *If* (U, g, T) *is* SGT_i*, then it is* SGR_{i-1}*, for* $i = 1, 2$.

Proof. We will prove the case $i = 1$. The proof for the case $i = 2$ is obvious. Let (U, g, T) be SGT_1 and H_T be an Sg-open set containing x_t. We need to prove that $cl_g(x_t) \widetilde{\subseteq} H_T$. So let $y_t \widetilde{\notin} H_T$, then $x_t \widetilde{\notin} cl_g(y_t)$ and x_t, y_t are different S-points. Since (U, g, T) is SGT_1, there is $K_T \widetilde{\in} g$ such that $y_t \widetilde{\in} K_T$ and $x_t \widetilde{\notin} K_T$, then $y_t \widetilde{\notin} cl_g(x_t)$. Therefore $cl_g(x_t) \widetilde{\subseteq} H_T$. This completes the proof. □

The converse of the above theorem may not be true. The next example shows it.

Example 4.8. Let $U = \{a, b\}$ and $T = \{t_1, t_2\}$. The class $g = \{\widetilde{\emptyset}, \tilde{U}, F_{1T}, F_{2T}\}$, where, $F_{1T} = \{(t_1, U)\}$ and $F_{2T} = \{(t_2, U)\}$ is an SGT on U. One can verify (U, g, T) is SGR_0 and SGR_1 but not SGT_1. Indeed, for two S-points a_{t_1}, b_{t_1}, the Sg-open sets which are containing a_{t_1} are $\tilde{U}$ and F_{1T} but also, they are containing b_{t_1}. Thus (U, g, T) is not SGT_1. Moreover, one can check that (U, g, T) is not SGT_2.

Theorem 4.9. *For* SGTS (U, g, T), *we have:*

(1) (U, g, T) *is* $SGT_2 \iff$ *it is both* SGR_1 *and* SGT_0;
(2) (U, g, T) *is* $SGT_1 \iff$ *it is both* SGR_0 *and* SGT_0.

Proof. We will show the case (1). The proof of the other case is similar.

The necessity part follows from Theorem 4.7 and Remark 2.25.

Conversely, let $x_t \neq y_t \in SP(U)$ with $x_t \widetilde{\notin} cl_g(y_t)$. Since (U, g, T) is SGR_0, then $y_t \widetilde{\notin} cl_g(x_t)$ and so, $cl_g(x_t) \neq cl_g(y_t)$. Again, (U, g, T) is SGR_1, so there are disjoint Sg-open sets F_T, H_T containing x_t, y_t, respectively. Hence (U, g, T) is SGT_2. □

Corollary 4.10. (U, g, T) *is* $SGT_2 \iff$ *it is both* SGR_1 *and* SGT_1.

Definition 4.11. An S-map $f_{up} : (U, g, T) \longrightarrow (V, \sigma, E)$ is called:

(i) Sg-continuous if $f_{up}^{-1}(F_E) \in g$ for any Sg-open set $F_E \in \sigma$ ([27]);
(ii) Sg-open if $f_{up}(G_T) \in \sigma$ for any Sg-open set $G_T \in g$ ([27]);
(iii) Sg-homeomorphism if it is bijective, Sg-continuous, and Sg-open.

Definition 4.12. A property is called a soft generalized-topological property if the property is preserved by Sg-homeomorphism.

Theorem 4.13. *For a bijective* Sg-*continuous map* $f_{up} : (U, g, T) \longrightarrow (V, \sigma, E)$, *if* (V, σ, E) *is* SGR_i, *then* (U, g, T) *is also* SGR_i, $i = 0.1$.

Proof. We will prove only the case $i = 1$. The proof of the rest case is similar. To show that (U, g, T) is SGR_1, let $x_t, y_t \in SP(U)(x \neq y)$. Since f_{up} is one-one, there are two distinct S-points a_e, b_e in V such that $f_{up}(x_t) = a_e$ and $f_{up}(y_t) = b_e$. Since (V, σ, E) is SGR_1, there are two Sg-open sets $H_{1E}, H_{2E} \in \sigma$ such that $a_e \widetilde{\in} H_{1E}$ and $b_e \widetilde{\in} H_{2E}$ and so, $x_t \widetilde{\in} f_{up}^{-1}(H_{1E})$ and $y_t \widetilde{\in} f_{up}^{-1}(H_{2E})$. Since f_{up} is Sg-continuous, we have $f_{up}^{-1}(H_{1E}), f_{up}^{-1}(H_{2E})$ are Sg-open sets in (U, g, T) with $f_{up}^{-1}(H_{1E}) \widetilde{\cap} f_{up}^{-1}(H_2E) = \widetilde{\emptyset}$. Thus (U, g, T) is SGR_1. □

Theorem 4.14. *For a bijective* Sg-*open map* $f_{up} : (U, g, T) \longrightarrow (V, \sigma, E)$, *if* (U, g, T) *is* SGR_i, *then* (V, σ, E) *is also* SGR_i, $i = 0.1$.

Proof. We will prove only the case $i = 1$. The proof of the rest case is similar. To show that (V, σ, E) is SGR_1. Let $a_e \neq b_e \in SP(V)$. Since f_{up} is onto, there are two distinct S-points x_t, y_t in U such that $f_{up}(x_t) = a_e$ and $f_{up}(y_t) = b_e$. By hypothesis, there are two Sg-open sets $H_{1T}, H_{2T} \in g$ such that $x_t \widetilde{\in} H_{1E}, y_t \widetilde{\in} H_{2E}$ and so, $a_e \widetilde{\in} f_{up}(H_{1T})$ and $b_e \widetilde{\in} f_{up}(H_{2T})$. Since f_{up} is Sg-open, we have $f_{up}(H_{1T})$, $f_{up}(H_{2T})$ are Sg-open sets in (V, σ, E) with $f_{up}(H_{1E}) \widetilde{\cap} f_{up}(H_{2T}) = \widetilde{\emptyset}$. Hence (V, σ, E) is SGR_1. □

From the above two theorems, we have the next theorem.

Theorem 4.15. *Let* $f_{up} : (U, G, T) \longrightarrow (V, \sigma, E)$ *be an* Sg-*homeomorphism map, then* (U, g, T) *is* SGR_i *if and only if* (V, σ, E) *is* SGR_i, $i = 0.1$.

Corollary 4.16. *The soft generalized properties* SGR_i *are* SG-*topological property, for* $i = 0, 1$.

Corollary 4.17. *From Remark 2.25, Proposition 3.10, Corollary 3.17, and Theorems 4.7 and 4.9, the following implications hold and describe the relationships between* SGR_i *and other soft separation properties.*

$$\begin{array}{ccccc} SGT_2 & \Longrightarrow & SGT_1 & \Longrightarrow & SGT_0 \\ \Downarrow & & \Downarrow & & \\ SGR_1 & \Longrightarrow & SGR_0 & & \\ \Uparrow & & \Uparrow & & \\ SR_1 & \Longrightarrow & SR_0 & & \end{array}$$

5. Conclusion

In this work, we defined and studied a new class of soft generalized properties called soft generalized R_0 and R_1 axioms in soft generalized topological spaces, and have obtained some characterizations of these properties. We also, investigated the relationships between various generalized topological structures of soft topology and presented several results with supported examples. In the future work, we will study the notions of R_0 and R_1 properties in supra soft topological spaces and investigate some soft generalized notions such as compactness and connectedness in this new setting. It is stated that the results obtained in the paper may be useful for further research on soft set theory and its applications.

Acknowledgement

The authors are grateful to the anonymous referees for their valuable comments. This study was supported by King Saud University. Deanship of Scientific Research. College of Science Research Center.

References

[1] H. Aktaş, N. Çağman, *Soft sets and soft groups*, Inform. Sci., **177** (2007), 2726–2735. 1
[2] A. Al-Omari, T. Noiri, *A unified theory of contra-(μ, λ)-continuous functions in generalized topological spaces*, Acta Math. Hungar., **135** (2012), 31–41. 1
[3] A. Al-Omari, *Soft topology in ideal topological spaces*, Hacet. J. Math. Stat., **48** (2019), 1277–1285 1
[4] H. S. Al-Saadi, H. Aygün, A. Al-Omari, *Some notes on soft hyperconnected spaces*, J. Anal., **28** (2020), 351–362. 1
[5] T. M. Al-Shami, *New Structure: Infra Soft Topological Spaces*, Math. Probl. Eng., **2021** (2021), 12 Pages. 1, 2
[6] T. M. Al-Shami, *Partial belong relation on soft separation axioms, two birds with one stone*, Soft Comput., **24** (2020), 5377–5387. 1
[7] N. Çağman, S. Enginoğlu, *Soft set theory and uni-int decision making*, European J. Oper. Res., **207** (2010), 848–855. 1
[8] N. Çağman, S. Karataş, S. Enginoglu, *Soft topology*, Comput. Math. Appl., **62** (2011), 351–358.
[9] B. Chen, *Soft semi-open sets and related properties in soft topological spaces*, Appl. Math. Inf. Sci., **7** (2013), 287–294. 1
[10] Á. Császár, *Generalized topology, generalized continuity*, Acta Math. Hungar., **96** (2002), 351–357. 1, 1.2
[11] S. Das, S. K. Samanta, *Soft Matric*, Annals Fuzzy Math. Info., **6** (2013), 77–94. 1
[12] S. A. El-Sheikh, A. M. Abd El-Latif, *Decompositions of some types of supra soft sets and soft continuity*, Int. J. Math. Trends Tech., **9** (2014), 37–56. 1, 2
[13] F. Feng, Y. B. Jun, X. Zhao, *Soft semirings*, Comput. Math. Appl., **56** (2008), 2621–2628. 1
[14] D. N. Georgiou, A. C. Megaritis, *Soft set theory and topology*, Appl. Gen. Topol., **15** (2014), 93–109. 1
[15] O. Göçür, A. Kopuzlu, *Some new properties on soft separation axioms*, Ann. Fuzzy Math. Inform., **9** (2015), 421–429. 1
[16] Y. B. Jun, C. H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Inform. Sci., **178** (2008), 2466–2475. 1
[17] A. Kharal, B. Ahmad, *Mappings on soft classes*, New Math. Nat. Comput., **7** (2011), 471–481. 1, 1
[18] P. K. Maji, R. Biswas, R. Roy, *Soft set theory*, Comput. Math. Appl., **45** (2003), 555–562. 1
[19] P. K. Maji, A. R. Roy, R. Biswas, *An application of soft sets in a decision making problem*, Comput. Math. Appl., **44** (2002), 1077–1083. 1
[20] W. K. Min, *A note on soft topological spaces*, Comput. Math. Appl., **62** (2011), 3524–3528. 1
[21] D. Molodtsov, *Soft set theory—first results*, Comput. Math. Appl., **37** (1999), 19–31. 1
[22] B. Roy, *On generalized R_0 and R_1 spaces*, Acta Math. Hungar., **127** (2010), 291–300. 1.3
[23] S. Saleh, *On some new properties in soft topological spaces*, Ann. Fuzzy Math. Inform., **17** (2019), 303–312. 1, 1.5
[24] S. Saleh, K. Hur, *On some lower soft separation axioms*, Ann. Fuzzy Math. Inform., **19** (2020), 61–72.
[25] M. Shabir, M. Naz, *On soft topological spaces*, Comput. Math. Appl. **61** (2011), 1786–1799. 1, 1.4
[26] O. Tantawy, S. A. El-Sheikh, S. Hamde, *Separation axioms on soft topological spaces*, Ann. Fuzzy Math. Inform., **11** (2016), 511–525. 1
[27] J. Thomas, S. J. John, *On soft generalized topological spaces*, J. New results Sci., **3** (2014), 1–15. 1, 2, i, ii
[28] J. Thomas, J. John, *On soft μ-compact soft generalized topological spaces*, J. Uncertain. Math. Sci., **2014** (2014), 1–9. 1
[29] J. Thomas, J. John, *Soft generalized separation axioms in soft generalized topological spaces*, Int. J. Sci. Eng. Res., **6** (2015), 969–974. 1, 2
[30] İ. Zorlutuna, M. Akdag, W. K. Min, S. Atmaca, *Remarks on soft topological spaces*, Ann. Fuzzy Math. Inform., **3** (2012), 171–185. 1