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Maximal, equilibrium, and coincidence points for majorized type correspondences

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Abstract

We present an equilibrium result for abstract economies for majorized condensing type correspondences on Hausdorff topological vector spaces. In addition we obtain new maximal element and coincidence point results for collectively multi-valued maps.

Keywords: Maximal and coincidence point theory, abstract economies.

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1. Introduction

The main aim of this paper is to present an existence theory for maximal element type and coincidence elements for multi-valued maps of condensing or compact type in the topological vector space setting. First we present an equilibrium theory for abstract economies for maps (constraints, preferences) majorized by upper semicontinuous maps with closed convex values and this abstract economy result will motivate the general maximal element type result.

Now we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q-dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$, where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geqslant 1$, and $H_0(X) \approx K$.

Let X, Y, and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

(ii) p is a perfect map; i.e.; p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

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Let $\phi : X \to Y$ be a multi-valued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map;
(ii) $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [6]. An upper semicontinuous map $\phi : X \to Y$ with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y.

Later we will use $C(Y)$ which will denote the family of nonempty convex closed subsets of Y.

We also discuss the following classes of maps in this paper. Let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and G a multifunction. We say $G \in DKT(Z, W)$ [3] if W is convex and there exists a map $S : Z \to W$ with $co(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w) = \{z \in Z : w \in S(z)\}$ is open (in Z) for each $w \in W$. We say $G \in HLPY(Z, W)$ [7, 9] if W is convex and there exists a map $S : Z \to W$ with $co(S(x)) \subseteq G(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\text{int}S^{-1}(w) : \ w \in W\}$.

Now we consider a general class of maps, namely the PK maps of Park. Let X and Y be Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \ \text{Fix}\, F \neq \emptyset \ \text{ for all } \ F \in \mathcal{X}(Z, Z)\},$$

where FixF denotes the set of fixed points of F. The class $\mathcal{U}$ of maps is defined by the following properties:

(i) $\mathcal{U}$ contains the class **C** of single valued continuous functions;
(ii) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued; and
(iii) $B^n \in \mathcal{F}(\mathcal{U}_c)$ for all $n \in \{1, 2, \ldots\}$; here $B^n = \{x \in \mathbf{R}^n : \ \|x\| \leqslant 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathcal{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$. Recall PK is closed under compositions.

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α-close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α-close to f_0.

Let V be a subset of a Hausdorff topological vector space E. Then we say V is Schauder admissible if for every compact subset K of V and every covering $\alpha \in Cov_V(K)$ there exists a continuous function $\pi_\alpha : K \to V$ such that

(i) π_α and $i : K \to V$ are α-close;
(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in AES(\text{compact})$.

Theorem 1.1 ([1, 11])**.** *Let X be a Schauder admissible subset of a Hausdorff topological vector space and $\Psi \in PK(X, X)$ a compact upper semicontinuous map with closed values. Then there exists an $x \in X$ with $x \in \Psi(x)$.*

Remark 1.2. Other variations of Theorem 1.1 can be found in [12] (see also [8] for another result).

We now list two well known results from the literature [16, 17].

Theorem 1.3. *Let* X *and* Y *be two topological spaces and* A *an open subset of* X. *Suppose* $F_1 : X \to 2^Y$, $F_2 : A \to 2^Y$ *are upper semicontinuous such that* $F_2(x) \subset F_1(x)$ *for all* $x \in A$. *Then the map* $F : X \to 2^Y$ *defined by*

$$F(x) = \begin{cases} F_1(x), & x \notin A, \\ F_2(x), & x \in A, \end{cases}$$

is upper semicontinuous.

Theorem 1.4. *Let* X *be a topological vector space and* Y *a normal space. If* $F, G : X \to Y$ *have closed values and are upper semicontinuous at* $x \in X$, *then* $F \cap G$ *is also upper semicontinuous at* x.

2. Abstract economies and maximal elements

Let I be the set of agents and we describe the abstract economy as $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$, where $A_i, B_i : X \equiv \prod_{i \in I} X_i \to E_i$ are constraint correspondences, $P_i : X \to E_i$ is a preference correspondence and X_i is a choice (or strategy) set which is a subset of a Hausdorff topological vector space E_i. We are interested in finding an equilibrium point for Γ, i.e., a point $x \in X$ with $x_i \in \overline{B_i}(x)$ and $A_i(x) \cap P_i(x) = \emptyset$; here x_i denotes the projection of x on X_i.

We begin by discussing maximal type maps motivated from the literature (see [9, 10, 14–17]). Let Z and W be sets in a Hausdorff topological vector space E with Z paracompact and W a closed convex normal subset of E. Suppose $H : Z \to W$ and for each $x \in Z$ assume there exists a map $A_x : Z \to W$ and an open set O_x containing x with $H(z) \subseteq A_x(z)$ for every $z \in O_x$, $A_x : O_x \to W$ is upper semicontinuous with closed convex values. We claim there exists a map $\Psi : Z \to W$ with $H(z) \subseteq \Psi(z)$ for $z \in Z$ and $\Psi : Z \to W$ is upper semicontinuous with closed convex values. To see this note $\{O_x\}_{x \in Z}$ is an open covering of Z and since Z is paracompact there exists [4, 5] a locally finite open covering $\{V_x\}_{x \in Z}$ of Z with $x \in V_x$ and $V_x \subseteq O_x$ for $x \in Z$, and for each $x \in Z$ let

$$Q_x(z) = \begin{cases} A_x(z), & z \in V_x, \\ W, & z \in Z \backslash V_x. \end{cases}$$

Now Theorem 1.3 guarantees that $Q_x : Z \to W$ is upper semicontinuous with closed convex values. Next note $H(z) \subseteq Q_x(z)$ for every $z \in Z$ since if $z \in V_x$, then since $V_x \subseteq U_x$ and $H(w) \subseteq A_x(w)$ for $w \in O_x$ we have $H(z) \subseteq Q_x(z)$ whereas if $z \in Z \backslash V_x$, then it is immediate since $Q_x(z) = W$. Now define $\Psi : Z \to W$ by

$$\Psi(z) = \bigcap_{x \in Z} Q_x(z) \quad \text{for} \quad z \in Z.$$

Note $\Psi : Z \to W$ has closed convex values with $H(w) \subseteq \Psi(w)$ for $w \in Z$ since $H(z) \subseteq Q_x(z)$ for every $z \in Z$ (for each $x \in X$). It remains to show $\Psi : Z \to W$ is upper semicontinuous. Let $u \in Z$. There exists an open neighbourhood N_u of u such that $\{x \in Z : N_u \cap V_x \neq \emptyset\} = \{x_1, \dots, x_{n_u}\}$ (a finite set). Note if $x \notin \{x_1, \dots, x_{n_u}\}$, then $\emptyset = V_x \cap N_u$ so $Q_x(z) = W$ for $z \in N_u$ and so we have

$$\Psi(z) = \bigcap_{x \in Z} Q_x(z) = \bigcap_{j=1}^{n_u} Q_{x_j}(z) \quad \text{for} \quad z \in N_u.$$

Now for $j \in \{1, \dots, n_u\}$ note $Q_{x_j} : Z \to W$ is upper semicontinuous (so $Q^{\star}_{x_j} : N_u \to W$, the restriction of Q_{x_j} to N_u, is upper semicontinuous) so Theorem 1.4 guarantees that $\Psi : N_u \to W$ is upper semicontinuous (at u). Since N_u is open we have that $\Psi : Z \to W$ is upper semicontinuous (at u).

We begin with our abstract economy result when the maps B_i are of compact or condensing type.

Theorem 2.1. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy with $\{X_i\}_{i\in I}$ a family of nonempty convex closed normal sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$ assume the following conditions hold:*

$$clB_i(\equiv \overline{B_i}) : X \equiv \prod_{i\in I} X_i \to C(X_i) \ \textit{ is upper semicontinuous}$$

and

$$U_i = \{x \in X : \ A_i(x) \cap P_i(x) \neq \emptyset\} \ \textit{ is open in } \ X \tag{2.1}$$

(recall $C(X_i)$ denotes the family of nonempty convex closed subsets of X_i). Also for each $i \in I$ suppose there exists a map $\Psi_i : U_i \to X_i$ with $(A_i \cap P_i)(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i : U_i \to X_i$ is upper semicontinuous with closed convex values. In addition assume either

$$y_i \notin \Psi_i(y) \ \textit{ for all } \ y \in U_i, \ \forall i \in I \tag{2.2}$$

or

$$\textit{there exists a } j_0 \in I \ \textit{ with } \ y_{j_0} \notin \Psi_{j_0}(y) \ \textit{ for } \ y \in U_{j_0} \tag{2.3}$$

occurs (here y_i denotes the projection of y on X_i). Suppose there is a compact subset K of X with $B(K) \subseteq K$ (here $B(x) = \prod_{i\in I} \overline{B_i}(x)$ for $x \in X$) and assume

$$K \textit{ is a Schauder admissible subset of } E \equiv \prod_{i\in I} E_i.$$

Then there exists an $x \in X$ with $x_i \in \overline{B_i}(x)$ for each $i \in I$ and if (2.2) holds we have $A_i(x) \cap P_i(x) = \emptyset$ for each $i \in I$ whereas if (2.3) holds we have $A_{j_0}(x) \cap P_{j_0}(x) = \emptyset$.

Proof. If $U_i = \emptyset$ for all $i \in I$, then from Theorem 1.1 (applied to $B = \prod_{i\in I} \overline{B_i} : K \to CK(K)$) there exists a $y \in K$ with $y \in B(y)$, i.e., $y_i \in \overline{B_i}(y)$ for each $i \in I$. Now since $U_i = \emptyset$ for all $i \in I$, then by definition we have $A_i(x) \cap P_i(x) = \emptyset$ for all $i \in I$.

As a result we assume for the remainder of the proof that there exists an $i_0 \in I$ with $U_{i_0} \neq \emptyset$. We will assume that $U_i \neq \emptyset$ for each $i \in I$ (we will also remark on the situation that $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I\backslash J$ at each step below). Let $i \in I$. Note $\overline{B_i}|_{U_i} : U_i \to C(X_i)$ is upper semicontinuous so from Theorem 1.4 (note X_i is a normal subset of E_i) we have that $\Psi_i \cap \overline{B_i} : U_i \to X$ is upper semicontinuous. Let $F_i : X \to X_i$ be given by

$$F_i(x) = \begin{cases} \overline{B_i}(x), & x \in U_i, \\ (\Psi_i \cap \overline{B_i})(x), & x \in X\backslash U_i, \end{cases}$$

so Theorem 1.3 guarantees that $F_i : X \to X_i$ is upper semicontinuous with nonempty convex and closed values (note for $x \in U_i$ that $(\Psi_i \cap \overline{B_i})(x) \subseteq \overline{B_i}(x)$). Note we also remark that if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I\backslash J$, then choose F_i as above if $i \in J$ whereas choose $F_i = \overline{B_i}$ if $i \in I\backslash J$.

Let $F : X \to C(X)$ be given by

$$F(x) = \prod_{i\in I} F_i(x) \quad \text{for} \quad x \in X,$$

and note $F : K \to CK(K)$ so $F \in Kak(K, K)$. Now Theorem 1.1 guarantees a $y \in X$ with $y \in F(y)$, i.e., $y_i \in F_i(y)$ for each $i \in I$. Thus $y_i \in \overline{B_i}(y)$ for each $i \in I$ since if $y \notin U_i$ we have $F_i(y) = \overline{B_i}(y)$ whereas if $y \in U_i$ we have $F_i(y) = (\Psi_i \cap \overline{B_i})(y) \subseteq \overline{B_i}(y)$ (we have a similar result if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I\backslash J$).

First suppose (2.2) occurs. Fix $i \in I$. We claim $y \notin U_i$. If not, then $y \in U_i$ so $y_i \in (\Psi_i \cap \overline{B_i})(y) \subseteq \Psi_i(y)$, i.e., $y_i \in \Psi_i(y)$, a contradiction. Thus $y \notin U_i$. We can do this argument for all $i \in I$ so the result in the statement of Theorem 2.1 holds (note if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I\backslash J$, then note if $i \in I\backslash J$ we have $U_i = \emptyset$ so $y \notin U_i$ whereas if $i \in J$, then the argument above gives $y \notin U_i$, so in both cases we have $y \notin U_i$).

Next suppose (2.3) occurs. Suppose $y \in U_{j_0}$. Then $y_{j_0} \in (\Psi_{j_0} \cap \overline{B_{j_0}})(y) \subseteq \Psi_{j_0}(y)$, i.e., $y_{j_0} \in \Psi_{j_0}(y)$, a contradiction. Thus $y \notin U_{j_0}$ so the result in the statement of Theorem 2.1 holds (note if $U_i \neq \emptyset$ for $i \in J \subseteq I$ and $U_i = \emptyset$ for $i \in I\backslash J$, then note if $j_0 \in I\backslash J$ we have $U_{j_0} = \emptyset$ so $y \notin U_{j_0}$ whereas if $j_0 \in J$, then the argument above gives $y \notin U_{j_0}$, so in both cases we have $y \notin U_{j_0}$). □

Remark 2.2. For each $i \in I$ suppose (2.1) is changed to U_i is paracompact and open in X. Also for each $i \in I$ suppose $A_i \cap P_i : U_i \to X_i$ and for each $x \in U_i$ assume there exists a map $A_{i,x} : U_i \to X_i$ and an open set $O_{i,x}$ (in U_i) containing x with $(A_i \cap P_i)(z) \subseteq A_{i,x}(z)$ for every $z \in O_{i,x}$ and $A_{i,x} : O_{i,x} \to X_i$ is upper semicontinuous with closed convex values. Also suppose either

$$z_i \notin A_{i,x}(z) \quad \text{for all } z \in O_{i,x},\ \forall x \in U_i,\ \forall i \in I, \tag{2.4}$$

or

$$\text{there exists } j_0 \in I \text{ with } z_{j_0} \notin A_{j_0,x}(z) \text{ for all } z \in O_{j_0,x},\ \forall x \in U_{j_0} \tag{2.5}$$

occurs.

Then the discussion before Theorem 2.1 (with $Z = U_i$, $W = X_i$, $H = A_i \cap P_i$ and $A_x = A_{i,x}$) guarantees that there exists a map $\Psi_i : U_i \to X_i$ as described in the statement of Theorem 2.1, i.e., $(A_i \cap P_i)(z) \subseteq \Psi_i(z)$ for $z \in U_i$ and $\Psi_i : U_i \to X_i$ is upper semicontinuous with closed convex values: here $\{O_{i,x}\}_{x \in U_i}$ is an open covering of U_i so there exists a locally finite open covering $\{V_{i,x}\}_{x \in U_i}$ of U_i (recall U_i is assumed paracompact) with $x \in V_{i,x}$ and $V_{i,x} \subseteq O_{i,x}$ for $x \in U_i$, and for each $x \in U_i$,

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in V_{i,x}, \\ X_i, & z \in U_i \backslash V_{i,x}, \end{cases}$$

and $\Psi_i : U_i \to X_i$ is

$$\Psi_i(z) = \bigcap_{x \in U_i} Q_{i,x}(z) \quad \text{for } z \in U_i.$$

Also note (2.4) (respectively (2.5)) implies (2.2) (respectively (2.3)). To see this let $i \in I$ and $y \in U_i$. Now $y \in V_{i,x}$ for some $x \in U_i$ since $\{V_{i,x}\}_{x \in U_i}$ is a locally finite open covering of U_i with $x \in V_{i,x}$ and $V_{i,x} \subseteq O_{i,x}$ for $x \in U_i$. Now note $Q_{i,x}(y) = A_{i,x}(y)$ so

$$\Psi_i(y) = \bigcap_{z \in U_i} Q_{i,z}(y) \subseteq Q_{i,x}(y) = A_{i,x}(y).$$

Next note the following maximal element type result.

Theorem 2.3. *Let $\{X_i\}_{i \in I}$ be a family of nonempty sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$ suppose $F_i : X \equiv \prod_{i \in I} X_i \to X_i$ and assume there exists a map $\Psi_i : X \to X_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to X_i$ is upper semicontinuous with closed convex values. Assume there is a compact subset K of X with $\Psi(K) \subseteq K$ (here $\Psi(x) = \prod_{i \in I} \Psi_i(x)$ for $x \in X$) and suppose*

$$K \ \textit{is a Schauder admissible subset of } E \equiv \prod_{i \in I} E_i. \tag{2.6}$$

Also suppose for each $x \in X$ there is a $j \in I$ with $x_j \notin \Psi_j(x)$. Then there exists a $y \in X$ and an $i_0 \in I$ with $F_{i_0}(y) = \emptyset$.

Proof. Suppose the conclusion is false. Then for each $x \in X$ we have $F_i(x) \neq \emptyset$ for all $i \in I$ and since $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$, then $\Psi_i(x) \neq \emptyset$ for all $i \in I$. As a result $\Psi \in K\alpha k(K, K)$ so Theorem 1.1 guarantees a $y \in K$ with $y \in \Psi(y)$. Thus $y_i \in \Psi_i(y)$ for $i \in I$, a contradiction. □

Remark 2.4.

(i). Note that Theorem 2.3 improves Theorems 2.6 and 2.9 in [13].

(ii). We can rewrite Theorem 2.3 with $\Psi_i = F_i$. Let $\{X_i\}_{i\in I}$ be a family of nonempty sets each in a Hausdorff topological vector space E_i (here I is an index set). For each $i \in I$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to X_i$ is upper semicontinuous with nonempty closed convex values. Assume there is a compact subset K of X with $F(K) \subseteq K$ (here $F(x) = \prod_{i\in I} F_i(x)$ for $x \in X$) and suppose (2.6) holds. Then there exists a $y \in X$ with $y_i \in F_i(y)$ for $i \in I$.

The proof is immediate since $F \in K\alpha k(K, K)$ so Theorem 1.1 gives the result.

(iii). Let X be paracompact and $\{X_i\}_{i\in I}$ a family of nonempty convex closed normal sets each in a Hausdorff topological vector space E_i. For each $i \in I$ suppose $F_i : X \to X_i$ and for each $x \in X$ assume there is a map $A_{i,x} : X \to X_i$ and an open set $U_{i,x}$ containing x with $F_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$ and $A_{i,x} : U_{i,x} \to X_i$ is upper semicontinuous with closed convex values. Also assume

$$\text{there exists } j_0 \in I \text{ with } z_{j_0} \notin A_{j_0,x}(z) \text{ for all } z \in X, \forall x \in X. \tag{2.7}$$

Now from the discussion before Theorem 2.1 (with $Z = X$, $W = X_i$, $H = F_i$ and $A_x = A_{i,x}$) there exists a map $\Psi_i : X \to X_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to X_i$ is upper semicontinuous with closed convex values: here $\{U_{i,x}\}_{x\in X}$ is an open covering of X so there exists a locally finite open covering $\{V_{i,x}\}_{x\in X}$ of X (recall X is assumed paracompact) with $x \in V_{i,x}$ and $V_{i,x} \subseteq U_{i,x}$ for $x \in X$, and for each $x \in X$,

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in V_{i,x}, \\ X_i, & z \in X \backslash V_{i,x}, \end{cases}$$

and $\Psi_i : X \to X_i$ is

$$\Psi_i(z) = \bigcap_{x\in X} Q_{i,x}(z) \quad \text{for} \quad z \in X.$$

Now (2.7) implies that for each $x \in X$ there is a $j \in I$ with $x_j \notin \Psi_j(x)$. To see this fix a $y \in X$. From (2.7) for all $x \in X$ there exists a $j_0 \in I$ with $y_{j_0} \notin A_{j_0,x}(y)$. Now since $\{V_{j_0,x}\}_{x\in X}$ is a locally finite open covering of X there exists an $x^\star \in X$ with $y \in V_{j_0,x^\star}$ so

$$\Psi_{j_0}(y) = \bigcap_{x\in X} Q_{j_0,x}(y) \subseteq Q_{j_0,x^\star}(y) = A_{j_0,x^\star}(y),$$

and so $y_{j_0} \notin \Psi_{j_0}(y)$.

Next we consider some collectively coincidence type results.

Theorem 2.5. *Let $\{X_i\}_{i\in I}$, $\{Y_i\}_{i\in J}$ be families of nonempty sets each in a Hausdorff topological vector space E_i (here I and J are index sets). For each $i \in J$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to Y_i$ and assume there exists a map $\Psi_i : X \to Y_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to Y_i$ is upper semicontinuous with closed convex values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and $G_j \in K\alpha k(Y, X_j)$. Also assume there is a subset Ω of X and a compact subset K of Y with $\Psi(\Omega) \subseteq K$ and $G(K) \subseteq \Omega$ and suppose*

$$\Omega \text{ is a Schauder admissible subset of } E \equiv \prod_{i\in I} E_i; \tag{2.8}$$

here $\Psi(x) = \prod_{i\in J} \Psi_i(x)$ for $x \in X$ and $G(y) = \prod_{i\in I} G_i(y)$ for $y \in Y$. Finally suppose either for each $(x, y) \in X \times Y$ with $x_i \in G_i(y)$ for all $i \in I$ there exists a $j_0 \in J$ with $y_{j_0} \notin \Psi_{j_0}(y)$ or for each $(x, y) \in X \times Y$ with $y_j \in \Psi_j(x)$ for all $j \in J$ there exists an $i_0 \in I$ with $x_{i_0} \notin G_{i_0}(y)$ occurs. Then there exists a $y \in X$ and an $i_0 \in J$ with $F_{i_0}(y) = \emptyset$.

Proof. Suppose the conclusion is false. Then for each $x \in X$ we have $F_i(x) \neq \emptyset$ for all $i \in J$ and since $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$, then $\Psi_i(x) \neq \emptyset$ for all $i \in J$. Now note $\Psi \in Kak(\Omega, K)$ and $G \in Kak(K, \Omega)$ so $G\,\Psi \in Ad(\Omega, \Omega)$ is a compact map (recall K is compact). Theorem 1.1 guarantees an $x \in X$ (in fact $x \in \Omega$) with $x \in G(\Psi(x))$. Let $y \in \Psi(x)$ with $x \in G(y)$. Then $y_j \in \Psi_j(x)$ for all $j \in J$ and $x_i \in G_i(y)$ for all $i \in I$, a contradiction. □

Remark 2.6.

(i). Note one could also consider the map $\Psi\,G$ instead of $G\,\Psi$ in the proof of Theorem 2.5 if one rephrases the statement of Theorem 2.5.

(ii). To get a contradiction in the proof of Theorem 2.5 one only needs the statement "there exists an $x \in X$ (in fact $x \in \Omega$) with $x \in G(\Psi(x))$" to be false, so one could list other conditions to guarantee the contradiction.

(iii). We could rewrite Theorem 2.5 with $\Psi_i = F_i$. Let $\{X_i\}_{i\in I}$, $\{Y_i\}_{i\in J}$ be families of nonempty sets each in a Hausdorff topological vector space E_i (here I and J are index sets). For each $i \in J$ suppose $F_i : X \equiv \prod_{i\in I} X_i \to Y_i$ is upper semicontinuous with nonempty closed convex values. For each $j \in I$ suppose $G_j : Y \equiv \prod_{i\in J} Y_i \to X_j$ and $G_j \in Kak(Y, X_j)$. Also assume there is a subset Ω of X and a compact subset K of Y with $F(\Omega) \subseteq K$ and $G(K) \subseteq \Omega$ and suppose (2.8) holds; here $F(x) = \prod_{i\in J} F_i(x)$ for $x \in X$ and $G(y) = \prod_{i\in I} G_i(y)$ for $y \in Y$. Then there exists an $x \in X$ and a $y \in Y$ with $y_j \in F_j(x)$ for all $j \in J$ and $x_i \in G_i(y)$ for all $i \in I$.

The proof is immediate since $F \in Kak(\Omega, K)$, $G \in Kak(K, \Omega)$ so $G\,F \in Ad(\Omega, \Omega)$ and apply Theorem 1.1.

(iv). Let X be paracompact and $\{Y_i\}_{i\in J}$ a family of nonempty convex closed normal sets each in E_i. For each $i \in J$ suppose $F_i : X \to Y_i$ and for each $x \in X$ assume there is a map $A_{i,x} : X \to Y_i$ and an open set $U_{i,x}$ containing x with $F_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$ and $A_{i,x} : U_{i,x} \to Y_i$ is upper semicontinuous with closed convex values. Also assume

$$\begin{cases} \text{for each } (w, y) \in X \times Y \text{ with } w_i \in G_i(y) \text{ for all } i \in I, \\ \text{there exists a } j_0 \in J \text{ with } y_{j_0} \notin A_{j_0,x}(w) \text{ for all } x \in X. \end{cases}$$

Now from the discussion before Theorem 2.1 (with $Z = X$, $W = Y_i$, $H = F_i$ and $A_x = A_{i,x}$) there exists a map $\Psi_i : X \to Y_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to Y_i$ is upper semicontinuous with closed convex values: here $\{U_{i,x}\}_{x\in X}$ is an open covering of X so there exists a locally finite open covering $\{V_{i,x}\}_{x\in X}$ of X (recall X is assumed paracompact) with $x \in V_{i,x}$ and $V_{i,x} \subseteq U_{i,x}$ for $x \in X$, and for each $x \in X$,

$$Q_{i,x}(z) = \begin{cases} A_{i,x}(z), & z \in V_{i,x}, \\ X_i, & z \in X \backslash V_{i,x}, \end{cases}$$

and $\Psi_i : X \to Y_i$ is

$$\Psi_i(z) = \bigcap_{x\in X} Q_{i,x}(z) \quad \text{for} \quad z \in X.$$

We now claim for each $(w, y) \in X \times Y$ with $w_i \in G_i(y)$ for all $i \in I$ there exists a $j_0 \in J$ with $y_{j_0} \notin \Psi_{j_0}(w)$. To see this note for each $x \in X$ there is a $j_0 \in J$ with $y_{j_0} \notin A_{j_0,x}(w)$. Now since $\{V_{j_0,x}\}_{x\in X}$ is a locally finite open covering of X there exists an $x^\star \in X$ with $w \in V_{j_0,x^\star}$ so

$$\Psi_{j_0}(w) = \bigcap_{x\in X} Q_{j_0,x}(w) \subseteq Q_{j_0,x^\star}(w) = A_{j_0,x^\star}(y),$$

and so $y_{j_0} \notin \Psi_{j_0}(w)$.

Next in Theorem 2.5 we will replace $G_j \in Kak(Y, X_j)$ with $G_j \in Ad(Y, X_j)$. To do this recall the finite product of admissible with respect to Gorniewicz maps is admissible with respect to Gorniewicz.

Theorem 2.7. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of nonempty sets each in a Hausdorff topological vector space E_i. For each $i \in \{1,\dots,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and assume there exists a map $\Psi_i : X \to Y_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to Y_i$ is upper semicontinuous with closed convex values. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in Ad(Y, X_j)$. Also assume there is a subset Ω of X and a compact subset K of Y with $\Psi(\Omega) \subseteq K$ and $G(K) \subseteq \Omega$ and suppose*

$$\Omega \text{ is a Schauder admissible subset of } E \equiv \prod_{i=1}^{N} E_i;$$

here $\Psi(x) = \prod_{i=1}^{N_0} \Psi_i(x)$ for $x \in X$ and $G(y) = \prod_{i=1}^{N} G_i(y)$ for $y \in Y$. Finally suppose either for each $(x,y) \in X \times Y$ with $x_i \in G_i(y)$ for all $i \in \{1,\dots,N\}$ there exists a $j_0 \in \{1,\dots,N_0\}$ with $y_{j_0} \notin \Psi_{j_0}(y)$ or for each $(x,y) \in X \times Y$ with $y_j \in \Psi_j(x)$ for all $j \in \{1,\dots,N_0\}$ there exists an $i_0 \in \{1,\dots,N\}$ with $x_{i_0} \notin G_{i_0}(y)$ occurs. Then there exists a $y \in X$ and an $i_0 \in \{1,\dots,N_0\}$ with $F_{i_0}(y) = \emptyset$.

Proof. Suppose the conclusion is false. Then note $\Psi \in Kak(\Omega, K)$ and $G \in Ad(K, \Omega)$ so $G\,\Psi \in Ad(\Omega,\Omega)$ is a compact map. Theorem 1.1 guarantees an $x \in X$ (in fact $x \in \Omega$) with $x \in G(\Psi(x))$, a contradiction. □

Remark 2.8.

(i) Note $\{1,\dots,N_0\}$ could be replaced by J (an index set) in Theorem 2.7.
(ii) There is an analogue of Remark 2.6 (i)-(iv) in this situation also.

Theorem 2.9. *Let $\{X_i\}_{i=1}^{N}$, $\{Y_i\}_{i=1}^{N_0}$ be families of nonempty sets each in a Hausdorff topological vector space E_i and in addition $\{X_i\}_{i=1}^{N}$ is a family of convex sets. For each $i \in \{1,\dots,N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^{N} X_i \to Y_i$ and assume there exists a map $\Psi_i : X \to Y_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to Y_i$ is upper semicontinuous with closed convex values. For each $j \in \{1,\dots,N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \to X_j$ and $G_j \in DKT(Y, X_j)$. Also assume there is a convex subset Ω_i of X_i for each $i \in \{1,\dots,N\}$ and a compact subset K_j of Y_j for each $j \in \{1,\dots,N_0\}$ with $\Psi_i(\Omega) \subseteq K_i$ for each $i \in \{1,\dots,N_0\}$ and $G_j(K) \subseteq \Omega_j$ for each $j \in \{1,\dots,N\}$; here $\Omega = \prod_{i=1}^{N} \Omega_i$ and $K = \prod_{i=1}^{N_0} K_i$. Finally suppose either for each $(x,y) \in X \times Y$ with $x_i \in G_i(y)$ for all $i \in \{1,\dots,N\}$ there exists a $j_0 \in \{1,\dots,N_0\}$ with $y_{j_0} \notin \Psi_{j_0}(y)$ or for each $(x,y) \in X \times Y$ with $y_j \in \Psi_j(x)$ for all $j \in \{1,\dots,N_0\}$ there exists an $i_0 \in \{1,\dots,N\}$ with $x_{i_0} \notin G_{i_0}(y)$ occurs. Then there exists a $y \in X$ and an $i_0 \in \{1,\dots,N_0\}$ with $F_{i_0}(y) = \emptyset$.*

Proof. Suppose the conclusion is false. Fix $j \in \{1,\dots,N_0\}$. Now let $S_j : Y \to X_j$ with $S_j(y) \neq \emptyset$ for $y \in Y$, $co\,(S_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $S_j^{-1}(w)$ is open (in Y) for each $w \in X_j$. Let $G_j^\star$ (respectively, $S_j^\star$) denote the restriction of G_i (respectively, S_j) to K. Note $G_j^\star \in DKT(K, \Omega_j)$ since for $x \in \Omega_j$ we have

$$(S_j^\star)^{-1}(x) = \{z \in K : x \in S_j^\star(z)\} = \{z \in K : x \in S_j(z)\} = K \cap \{z \in Y : x \in S_j(z)\} = K \cap S_j^{-1}(x),$$

which is open in $K \cap Y = K$. Now for each $j \in \{1,\dots,N\}$ from [2, 3] there exists a continuous (single valued) selection $g_j : K \to \Omega_j$ of G_j with $g_j(y) \in G_j(y)$ for $y \in K$ and there exists a finite set R_j of Ω_j with $g_j(K) \subseteq co\,(R_j) \equiv Q_j$. Let $Q = \prod_{i=1}^{N} Q_i$ ($\subseteq \Omega$; note for each $j \in \{1,\dots,N\}$ that $Q_j = co\,(R_j) \subseteq \Omega_j$ since Ω_j is convex) and note Q is compact. Let

$$g(y) = \prod_{i=1}^{N} g_i(y) \quad \text{for} \quad y \in K,$$

and note $g : K \to Q$ is continuous. Now let

$$\Psi(x) = \prod_{i=1}^{N_0} \Psi_i(x),$$

and note $\Psi \in Kak(Q, K)$. Then $g\,\Psi \in Ad(Q,Q)$ and Q is a compact convex set in a finite dimensional subspace of $E = \prod_{i=1}^{N} E_i$, so Theorem 1.1 guarantees an $x \in Q$ with $x \in g\,(\Psi(x))$. Let $y \in \Psi(x)$ with $x = g(y)$ so $y_i \in \Psi_i(y)$ for $i \in \{1,\dots,N_0\}$ and $x_i = g_i(y) \in G_i(y)$ for $i \in \{1,\dots,N\}$, a contradiction. □

Remark 2.10. In Theorem 2.9 note $G_j(Y, X_j)$ could be replaced by $G_j \in HLPY(Y, X_j)$ since one has $G_j^\star \in HLPY(K, \Omega_j)$ and, then, from [7, 9] one can deduce the existence of a continuous selection $g_j : K \to \Omega_j$ of G_j. To see that $G_j^\star \in HLPY(K, \Omega_j)$ let $S_j : Y \to X_j$ with $S_j(y) \neq \emptyset$ for $y \in Y$, $co\,(S_j(y)) \subseteq G_j(y)$ for $y \in Y$ and $Y = \{\text{int}\, S_j^{-1}(w) : \; w \in X_j\}$. Also let $S_j^\star$ denotes the restriction of S_j to K. To show $G_j^\star \in HLPY(K, \Omega_j)$ we need to show $K = \bigcup\{\text{int}_K\, S_j^{-1}(w) : \; w \in \Omega_j\}$. Note

$$K = K \cap Y = K \cap \left(\bigcup\{\text{int} S_j^{-1}(w) : \; w \in X_j\}\right) = \bigcup\{K \cap \, \text{int}\, S_j^{-1}(w) : \; w \in X_j\},$$

so $K \subseteq \bigcup\{\text{int}_K\, S_j^{-1}(w) : \; w \in X_j\}$ since for each $w \in X_j$ we have that $K \cap \text{int}\, S_j^{-1}(w)$ is open in K. On the other hand clearly $\bigcup\{\text{int}_K\, S_j^{-1}(w) : \; w \in X_j\} \subseteq K$ so as a result

$$K = \bigcup\{\text{int}_K\, S_j^{-1}(w) : \; w \in X_j\}.$$

Now for any $y \in K$ from above there exists a $w \in X_j$ with $y \in \text{int}_K\, S_j^{-1}(w) \subseteq S_j^{-1}(w)$ so $w \in S_j(y) \subseteq \Omega_j$ since $co\,(S_j^\star(y)) \subseteq G_j^\star(y)$ and $G_j(K) \subseteq \Omega_j$, i.e., for any $y \in K$ there exists a $w \in \Omega_j$ with $y \in \text{int}_K\, S_j^{-1}(w)$. Thus

$$K = \bigcup\{\text{int}_K\, S_j^{-1}(w) : \; w \in \Omega_j\},$$

so $G_j^\star \in HLPY(K, \Omega_j)$.

Remark 2.11.

(i) Note one could also consider the map $\Psi\, g$ instead of $g\, \Psi$ in the proof of Theorem 2.9 if one rephrases the statement of Theorem 2.9.

(ii) To get a contradiction in the proof of Theorem 2.9 one only needs the statement "there exists an $x \in X$ (in fact $x \in Q$) with $x \in g(\Psi(x))$" to be false, so one could list other conditions to guarantee the contradiction.

(iii) We can replace $\{1, \ldots, N\}$ and $\{1, \ldots, N_0\}$ with I and J index sets in Theorem 2.9 provided we rephrase Theorem 2.9 so that Theorem 1.1 can be applied.

Remark 2.12.

(i). Let X be paracompact and $\{Y_i\}_{i=1}^{N_0}$ a family of nonempty convex closed normal sets each in E_i. For each $i \in \{1, \ldots, N_0\}$ suppose $F_i : X \to Y_i$ and for each $x \in X$ assume there is a map $A_{i,x} : X \to Y_i$ and an open set $U_{i,x}$ containing x with $F_i(z) \subseteq A_{i,x}(z)$ for every $z \in U_{i,x}$ and $A_{i,x} : U_{i,x} \to Y_i$ is upper semicontinuous with closed convex values. Also assume

$$\begin{cases} \text{for each } \; (w, y) \in X \times Y \; \text{ with } \; w_i \in G_i(y) \; \text{ for all } \; i \in \{1, \ldots, N\}, \\ \text{there exists a } \; j_0 \in \{1, \ldots, N_0\} \; \text{ with } \; y_{j_0} \notin A_{j_0,x}(w) \; \text{ for all } \; x \in X. \end{cases}$$

Then as in Remark 2.6 (ii) there exists a map $\Psi_i : X \to Y_i$ with $F_i(z) \subseteq \Psi_i(z)$ for $z \in X$ and $\Psi_i : X \to Y_i$ is upper semicontinuous with closed convex values. Also for each $(w, y) \in X \times Y$ with $w_i \in G_i(y)$ for all $i \in \{1, \ldots, N\}$, there exists a $j_0 \in \{1, \ldots, N_0\}$ with $y_{j_0} \notin \Psi_{j_0}(w)$.

(ii). Note Theorem 2.9 improves Theorem 2.17 in [13], where part of an assumption there was inadvertently omitted (but in fact it is a condition mentioned in Remark 2.11 (ii)).

(iiii). In Theorem 2.9 if $\Psi_i = F_i$ for $i \in \{1, \ldots, N_0\}$, then Theorem 2.9 improves Theorem 2.15 in [13], where there part of an assumption there was inadvertently omitted (but in fact it is a condition mentioned in Remark 2.11 (ii)). In fact one can rephrase Theorem 2.9, when $\Psi_i = F_i$ as follows. Let $\{X_i\}_{i=1}^N$, $\{Y_i\}_{i=1}^{N_0}$ be families of nonempty sets each in a Hausdorff topological vector space E_i and in addition $\{X_i\}_{i=1}^N$ is a family of convex sets. For each $i \in \{1, \ldots, N_0\}$ suppose $F_i : X \equiv \prod_{i=1}^N X_i \; \to Y_i$ is upper semicontinuous with nonempty closed convex values. For each $j \in \{1, \ldots, N\}$ suppose $G_j : Y \equiv \prod_{i=1}^{N_0} Y_i \; \to X_j$ and $G_j \in DKT(Y, X_j)$. Also assume there is a subset Ω_i of X_i for each $i \in \{1, \ldots, N\}$ and a compact subset K_j of Y_j for each $j \in \{1, \ldots, N_0\}$ with $\Psi_i(\Omega) \subseteq K_i$ for each $i \in \{1, \ldots, N_0\}$ and $G_j(K) \subseteq \Omega_j$ for each $j \in \{1, \ldots, N\}$; here $\Omega = \prod_{i=1}^N \Omega_i$ and $K = \prod_{i=1}^{N_0} K_i$. Then there exists an $x \in X$ and a $y \in Y$ with $y_i \in F_i(x)$ for $i \in \{1, \ldots, N_0\}$ and $x_i \in G_i(y)$ for $i \in \{1, \ldots, N\}$.

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