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Multiple solutions for a class of perturbed damped vibration problems

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Abstract

The existence of three distinct weak solutions for a class of perturbed damped vibration problems with nonlinear terms depending on two real parameters is investigated. Our approach is based on variational methods.

Keywords: Multiple solutions, perturbed damped vibration problem, critical point theory, variational methods.
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1. Introduction

Consider the following perturbed damped vibration problem

$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \lambda \nabla F(t, u(t)) + \mu \nabla G(t, u(t)) & a.e. \;\; t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $q \in L^1(0, T; \mathbb{R})$, $Q(t) = \int_0^t q(s)ds$ for all $t \in [0, T]$, $Q(T) = 0$, $A : [0, T] \to \mathbb{R}^{N\times N}$ is a continuous map from the interval $[0, T]$ to the set of N-order symmetric matrices, $\lambda > 0$, $\mu \geq 0$ and $F, G : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ are measurable with respect to t, for all $u \in \mathbb{R}^N$, continuously differentiable in u, for almost every $t \in [0, T]$, satisfies the following standard summability condition

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$$\sup_{|\xi|\leq a} \max\{|F(\cdot,\xi)|,\ |G(\cdot,\xi)|, |\nabla F(\cdot,\xi)|,\ |\nabla G(\cdot,\xi)|\} \in L^1([0,T]) \tag{1.2}$$

for any $a > 0$, and $F(t,0,\dots,0) = G(t,0,\dots,0) = 0$ for all $t \in [0,T]$.

Assume that $\nabla F, \nabla G$ are continuous in $[0,T] \times \mathbb{R}^N$, then the condition (1.2) is satisfied.

As a special case of dynamical systems, Hamiltonian systems are very important in the study of fluid mechanics, gas dynamics, nuclear physics and relativistic mechanics. Inspired by the monographs [21–23], the existence and multiplicity of periodic solutions for Hamiltonian systems have been investigated in many papers (see [2, 5–7, 13, 14, 16, 17, 25–27] and the references therein). For example, in [27] the authors obtained existence theorems for periodic solutions of a class of unbounded non-autonomous non-convex subquadratic second order Hamiltonian systems by using the minimax methods in critical point theory. In [13] Cordaro established a multiplicity result to an eigenvalue problem related to second-order Hamiltonian systems, and proved the existence of an open interval of positive eigenvalues in which the problem admits three distinct periodic solutions. In [16] Faraci studied the multiplicity of solutions of a second order non-autonomous system.

Very recently, some researchers have paid attention to the existence and multiplicity of solutions for damped vibration problems, for instance, see [9, 10, 12, 28–31] and references therein. For example, Chen in [9, 10] studied a class of non-periodic damped vibration systems with subquadratic terms and with asymptotically quadratic terms, respectively, and obtained infinitely many nontrivial homoclinic orbits by a variant fountain theorem developed recently by Zou [33]. Wu and Chen in [30] based on variational principle presented three existence theorems for periodic solutions of a class of damped vibration problems. In particular, the authors in [29] based on variational methods and critical point theory studied the existence of one solution and multiple solutions for damped vibration problems. In [31], the authors using critical point theory and variational methods investigated the solutions of a Dirichlet boundary value problem for damped nonlinear impulsive differential equations.

In [11, 18, 19, 24] using variational methods and critical point theory the existence of multiple solutions for a class of perturbed second-order impulsive Hamiltonian systems was discussed.

We also cite the paper [20] in which employing a critical point theorem (local minimum result) for differentiable functionals, the existence of at least one non-trivial weak solution for a class of impulsive damped vibration systems under an asymptotical behaviour of the nonlinear datum at zero was proved.

In the present paper, motivated by [29], using two kinds of three critical points theorems obtained in [3, 8] which we recall in the next section (Theorems 2.1 and 2.2), we ensure the existence of at least three solutions for the problem (1.1); see Theorems 3.1 and 3.2.

We also refer the reader to the papers [1, 4, 15] in which the existence of multiple solutions for boundary value problems is ensured.

The present paper is arranged as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple solutions for the problem (1.1).

2. Preliminaries

Our main tools are three critical points theorems that we recall here in a convenient form. In the first one, the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed. The first has been obtained in [8], and it is a more precise version of Theorem 3.2 of [3]. The second has been established in [3].

Theorem 2.1 ([8, Theorem 3.6])**.** *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be a coercive continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose*

Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$.
Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that

(a_1) $\dfrac{\sup_{x\in\Phi^{-1}(-\infty,r]} \Psi(x)}{r} < \dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(a_2) *for each* $\lambda \in \Lambda_r := \left(\dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{r}{\sup_{x\in\Phi^{-1}(-\infty,r]} \Psi(x)}\right)$ *the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

Theorem 2.2 ([3, Corollary 3.1], [4, Theorem 2.2])**.** *Let X be a reflexive real Banach space, $\Phi : X \longrightarrow \mathbb{R}$ be convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \longrightarrow \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that*

1. $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$;
2. *for each $\lambda > 0$ and for every $x_1, x_2 \in X$ which are local minima for the functional $\Phi - \lambda\Psi$ and such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, one has*

$$\inf_{s\in[0,1]} \Psi(sx_1 + (1-s)x_2) \geq 0.$$

Assume that there are two positive constants r_1, r_2 and $\overline{x} \in X$, with $2r_1 < \Phi(\overline{x}) < \frac{r_2}{2}$, such that

(b_1) $\dfrac{\sup_{x\in\Phi^{-1}(-\infty,r_1)} \Psi(x)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(b_2) $\dfrac{\sup_{x\in\Phi^{-1}(-\infty,r_2)} \Psi(x)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$.

Then, for each $\lambda \in \left(\dfrac{3}{2}\dfrac{\Phi(\overline{x})}{\Psi(\overline{x})},\ \min\left\{\dfrac{r_1}{\sup_{u\in\Phi^{-1}(-\infty,r_1)} \Psi(x)}, \dfrac{\frac{r_2}{2}}{\sup_{x\in\Phi^{-1}(-\infty,r_2)} \Psi(x)}\right\}\right)$, *the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(-\infty, r_2)$.*

We assume that the matrix A satisfies the following conditions:

(A1) $A(t) = (a_{kl}(t))$, $k = 1, \ldots, N$, $l = 1, \ldots, N$, is a symmetric matrix with $a_{kl} \in L^\infty[0,T]$ for any $t \in [0,T]$;

(A2) there exists $\delta > 0$ such that $(A(t)\xi, \xi) \geq \delta|\xi|^2$ for any $\xi \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$.

Let us recall some basic concepts. Denote

$$E = \{u : [0,T] \to \mathbb{R}^N |\ u \text{ is absolutely continuous},\ \ u(0) = u(T),\ \ \dot{u} \in L^2([0,T], \mathbb{R}^N)\},$$

with the inner product

$$\prec u, v \succ_E = \int_0^T [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))]dt.$$

The corresponding norm is defined by

$$\|u\|_E = \Big(\int_0^T (|\dot{u}(t)|^2 + |u(t)|^2)dt\Big)^{\frac{1}{2}} \ \forall\ u \in E.$$

For every $u, v \in E$, we define

$$\prec u, v \succ = \int_0^T [e^{Q(t)}(\dot{u}(t), \dot{v}(t)) + e^{Q(t)}(A(t)u(t), v(t))]dt,$$

and we observe that, by the assumptions (A1) and (A2), it defines an inner product in E. Then E is a separable and reflexive Banach space with the norm

$$\|u\| = \prec u, u \succ^{\frac{1}{2}} \quad , \quad \forall u \in E.$$

Obviously, E is a uniformly convex Banach space. Clearly, the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_E$ (see [17]).

Since $(E, \|\cdot\|)$ is compactly embedded in $C([0,T], \mathbb{R}^N)$ (see [21]), there exists a positive constant c such that

$$\|u\|_\infty \leq c \parallel u \parallel, \tag{2.1}$$

where $\|u\|_\infty = \max_{t\in[0,T]} \mid u(t) \mid$.

We use the following notations:

$$G^\theta := \int_0^T e^{Q(t)} \sup_{|x|\leq\theta} G(t,x)dt, \quad t \in [0,T], \quad \forall\ \theta > 0,$$

and

$$G_{x_0} := \int_0^T e^{Q(t)} G(t, x_0)dt, \quad \forall\ x_0 \in \mathbb{R}^N.$$

We mean by a (weak) solution of the problem (1.1), any function $u \in E$ such that

$$\int_0^T e^{Q(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{Q(t)}(A(t)u(t), v(t))dt - \lambda \int_0^T e^{Q(t)}(\nabla F(t, u(t)), v(t))dt$$
$$- \mu \int_0^T e^{Q(t)}(\nabla G(t, u(t)), v(t))dt = 0$$

for every $v \in E$.

3. Main results

In order to introduce our first result, fix $\theta > 0$ and nonzero vector $x_0 \in \mathbb{R}^N$ such that

$$\frac{|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt} < \frac{\theta^2}{c^2 \displaystyle\int_0^T e^{Q(t)} \sup_{|x|\leq\theta} F(t,x)dt},$$

and pick

$$\lambda \in \Lambda := \left] \frac{|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}{2\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt}, \frac{\theta^2}{2c^2 \displaystyle\int_0^T e^{Q(t)} \sup_{|x|\leq\theta} F(t,x)dt} \right[,$$

and let

$$\delta_{\lambda,G} := \min\left\{ \frac{\theta^2 - 2c^2\lambda \int_0^T e^{Q(t)} \sup_{|x|\le\theta} F(t,x)dt}{2c^2 G^\theta}, \right.$$

$$\left. \frac{|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt - 2\lambda \int_0^T e^{Q(t)} F(t,x_0)dt}{2G_{x_0}} \right\},$$

and

$$\overline{\delta}_{\lambda,G} := \min\left\{ \delta_{\lambda,g}, \ \frac{1}{\max\left\{0, 2c^2 \limsup_{|x|\to\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^2}\right\}} \right\}, \tag{3.1}$$

where we read $\rho/0 = +\infty$, so that, for instance, $\overline{\delta}_{\lambda,G} = +\infty$ when

$$\limsup_{|x|\to\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^2} \le 0,$$

and $G_{x_0} = G^\theta = 0$. Now, we formulate our main result.

Theorem 3.1. *Suppose that the assumptions* (A1) *and* (A2) *hold. Assume that there exist a positive constant θ and and a non-zero vector $x_0 \in \mathbb{R}^N$ with $\theta < c(\delta \int_0^T e^{Q(t)}dt)^{\frac{1}{2}}|x_0|$ such that*

(B_1) $$\frac{\int_0^T e^{Q(t)} \sup_{|x|\le\theta} F(t,x)dt}{\theta^2} < \frac{1}{c^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt} \frac{\int_0^T e^{Q(t)} F(t,x_0)dt}{|x_0|^2};$$

(B_2) *there exist functions $h_1, h_2 \in L^1(0,T,\mathbb{R})$ and two numbers $\alpha \in [0,1)$, $M > 0$ such that $\frac{(\nabla F(t,x),x)}{|x|} \le h_1(t)|x|^\alpha + h_2(t)$ for all $x \in \mathbb{R}^N$ with $|x| \ge M$ and a.e $t \in [0,T]$.*

Then, for each $\lambda \in \Lambda$ and for every function $G : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ which is measurable with respect to t, for all $u \in \mathbb{R}^N$, continuously differentiable in u, for almost every $t \in [0,T]$, satisfying (1.2) *and the condition*

$$\limsup_{|x|\to\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^2} < +\infty,$$

there exists $\overline{\delta}_{\lambda,G} > 0$ given by (3.1) *such that, for each $\mu \in [0, \overline{\delta}_{\lambda,G}[$, the problem* (1.1) *admits at least three distinct weak solutions.*

Proof. Fix λ as in the conclusion. Take $X = E$ and define the functionals $\Phi,\ \Psi : X \to \mathbb{R}$ as follows

$$\Phi(u) = \frac{1}{2}\|u\|^2,$$

and

$$\Psi(u) = \int_0^T e^{Q(t)}(F(t,u(t)) + \frac{\mu}{\lambda} G(t,u(t))dt$$

for every $u \in X$. It is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_0^T e^{Q(t)}(\nabla F(t,u(t)), v(t))dt + \frac{\mu}{\lambda}\int_0^T e^{Q(t)}(\nabla G(t,u(t)), v(t))dt$$

for every $v \in X$, and $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^T e^{Q(t)}(\dot{u}(t), \dot{v}(t))dt + \int_0^T e^{Q(t)}(A(t)u(t), v(t))dt$$

for every $v \in X$. Since Φ' is uniformly monotone on X, coercive and hemicontinuous in X, applying [32, Theorem 26. A] it admits a continuous inverse on X^*. Furthermore, Φ is sequentially weakly lower semicontinuous. Put $r = \frac{1}{2}(\frac{\theta}{c})^2$ and $w(t) := x_0$ for all $t \in [0,T]$. It is easy to see that $w \in X$ and, in particular, one has

$$\frac{1}{2}|x_0|^2\delta\int_0^T e^{Q(t)}dt \le \Phi(w) \le \frac{1}{2}|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt. \tag{3.2}$$

This together with the condition $\theta < c(\delta\int_0^T e^{Q(t)}dt)^{\frac{1}{2}}|x_0|$ ensures

$$0 < r < \Phi(w).$$

Bearing (2.1) in mind, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r]) &= \{u \in X;\ \Phi(u) \le r\}\\ &= \left\{u \in X;\ \frac{||u||^2}{2} \le r\right\}\\ &\subseteq \{u \in X;\ |u(t)| \le \theta \text{ for each } t \in [0,T]\},\end{aligned}$$

and it follows that

$$\begin{aligned}\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r])}\int_0^T e^{Q(t)}[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt\\ &\le \int_0^T e^{Q(t)}\sup_{|x|\le\theta}F(t,x)dt + \frac{\mu}{\lambda}G^\theta.\end{aligned}$$

So,

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} &= \frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\int_0^T e^{Q(t)}[F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt}{r}\\ &\le 2c^2\frac{\int_0^T e^{Q(t)}\sup_{|x|\le\theta}F(t,x)dt + \frac{\mu}{\lambda}G^\theta}{\theta^2},\end{aligned} \tag{3.3}$$

and

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt + \frac{\mu}{\lambda}\int_0^T e^{Q(t)}G(t,x_0)dt}{\frac{1}{2}|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}. \tag{3.4}$$

Since $\mu < \delta_{\lambda,g}$, one has

$$\mu < \frac{\theta^2 - 2c^2\lambda \displaystyle\int_0^T e^{Q(t)} \sup_{|x|\leq\theta} F(t,x)dt}{2c^2 G^\theta},$$

that is,

$$\frac{\displaystyle\int_0^T e^{Q(t)} \sup_{|x|\leq\theta} F(t,x)dt + \frac{\mu}{\lambda}G^\theta}{\frac{1}{2}(\frac{\theta}{c})^2} < \frac{1}{\lambda}.$$

Furthermore,

$$\mu < \frac{|x_0|^2(\sum_{i,j=1}^N \|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt - 2\lambda \displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt}{2G_{x_0}},$$

that is,

$$\frac{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt + \frac{\mu}{\lambda}G_{x_0}}{\frac{1}{2}|x_0|^2(\displaystyle\sum_{i,j=1}^N \|a_{ij}\|_\infty)\displaystyle\int_0^T e^{Q(t)}dt} > \frac{1}{\lambda}.$$

Then,

$$\frac{\displaystyle\int_0^T e^{Q(t)} \sup_{|x|\leq\theta} F(t,x)dt + \frac{\mu}{\lambda}G^\theta}{\frac{1}{2}(\frac{\theta}{c})^2} < \frac{1}{\lambda} < \frac{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt + \frac{\mu}{\lambda}G_{x_0}}{\frac{1}{2}|x_0|^2(\displaystyle\sum_{i,j=1}^N \|a_{ij}\|_\infty)\displaystyle\int_0^T e^{Q(t)}dt}. \tag{3.5}$$

Hence from (3.3)–(3.5), we see that the condition (a_1) of Theorem 2.1 is fulfilled. Finally, since $\mu < \overline{\delta}_{\lambda,G}$, we can fix $l > 0$ such that

$$\limsup_{|x|\to\infty} \frac{\sup_{t\in[0,T]} G(t,x)}{|x|^2} < l,$$

and $\mu l < \frac{1}{2c^2}$. Therefore, there exists a function $\rho \in L^1([0,T])$ such that

$$G(t,x) \leq l|x|^2 + \rho(t)$$

for every $t \in [0,T]$ and $x \in \mathbb{R}^N$.

Let

$$\eta(t,x) = \frac{(\nabla F(t,x),x)}{|x|} - h_1(t)|x|^\alpha - h_2(t) \text{ for all } t \in [0,T] \text{ and } x \in \mathbb{R}^N.$$

Let $\beta(t) = \sup_{|x|<M} \eta(t,x)$. Then, by ($B_2$),

$$\eta(t,x) \le \begin{cases} 0 & \text{if } |x| \ge M \\ \beta(t) & \text{if } |x| < M. \end{cases}$$

Thus

$$\eta(t,sx) \le \begin{cases} 0 & \text{if } s \ge \dfrac{M}{|x|} \\ \beta(t) & \text{if } 0 < s < \dfrac{M}{|x|}. \end{cases}$$

Therefore,

$$\begin{aligned} F(t,x) - F(t,0) &= \int_0^1 \nabla F(t,sx)\cdot x ds \\ &= \int_0^1 |x|[\eta(t,sx) + h_1(t)s^\alpha |x|^\alpha + h_2(t)]ds \\ &\le M\beta(t) + \frac{1}{\alpha+1}|x|^{\alpha+1}h_1(t) + |x|h_2(t). \end{aligned}$$

Consequently, for $\lambda > 0$ we have

$$\begin{aligned} \Phi(u) - \lambda\Psi(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_0^T e^{Q(t)}(F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))dt \\ &\ge \frac{1}{2}\|u\|^2 - \lambda \int_0^T e^{Q(t)}\Big[M\beta(t) + \frac{1}{\alpha+1}|u(t)|^{\alpha+1}h_1(t) + |u(t)|h_2(t)\Big]dt \\ &\quad - \lambda \int_0^T e^{Q(t)}F(t,0)dt - \mu l \int_0^T |u(t)|^2 dt - \mu\|\rho\|_1 \\ &\ge (\frac{1}{2} - \mu l c^2)\|u\|^2 - c_1\|u\|^{\alpha+1} - c_2\|u\| + c_3 \end{aligned}$$

for some constants c_1, c_2 and c_3. Since $\alpha \in [0,1)$, this follows $\lim_{\|u\|\to\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty, \ \ \forall\lambda > 0$, which means the functional $\Phi - \lambda\Psi$ is coercive, and the condition (a_2) of Theorem 2.1 is satisfied.

From (3.3) and (3.5) one also has

$$\lambda \in \left]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[.$$

Finally, since the weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$ (see [29, Theorem 2.2]), Theorem 2.1 (with $\overline{x} = w$) concludes the result. □

Now, we present a variant of Theorem 3.1. Here no asymptotic condition on the nonlinear term G is requested; on the other hand, the functions F and G are supposed to be nonnegative. Fix $\theta_1, \theta_2 > 0$ and nonzero vector $x_0 \in \mathbb{R}^N$ such that

$$\frac{3}{2}\frac{|x_0|^2(\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt} < \frac{1}{c^2}\min\left\{\frac{\theta_1^2}{\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_1}F(t,x)dt}, \frac{\theta_2^2}{2\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_2}F(t,x)dt}\right\},$$

and picking

$$\lambda\in\Lambda' := \left]\frac{3}{4}\frac{|x_0|^2(\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt}, \frac{1}{2c^2}\right.$$
$$\left.\min\left\{\frac{\theta_1^2}{\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_1}F(t,x)dt}, \frac{\theta_2^2}{2\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_2}F(t,x)dt}\right\}\right[,$$

put

$$\delta^*_{\lambda,G} := \min\left\{\frac{\theta_1^2 - 2\lambda c^2\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_1}F(t,x)dt}{2c^2G^{\theta_1}}, \frac{\theta_2^2 - 4\lambda c^2\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_2}F(t,x)dt}{4c^2G^{\theta_2}}\right\}. \tag{3.6}$$

Theorem 3.2. *Suppose that the assumptions* (A1) *and* (A2) *hold. Let* $F : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ *be a nonnegative function satisfies the assumption* (1.2)*. Assume that there exist a non-zero vector* $x_0 \in \mathbb{R}^N$ *and two positive constants* θ_1 *and* θ_2 *with*

$$\frac{2(\frac{\theta_1}{c})^2}{\delta\int_0^T e^{Q(t)}dt} < |x_0|^2 < \frac{\frac{1}{2}(\frac{\theta_2}{c})^2}{(\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt},$$

such that

$$(C_1)\qquad \frac{\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_1}F(t,x)dt}{\theta_1^2} < \frac{2}{3}\frac{1}{c^2(\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}\frac{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt}{|x_0|^2};$$

$$(C_2)\qquad \frac{\displaystyle\int_0^T e^{Q(t)}\sup_{|x|\le\theta_2}F(t,x)dt}{\theta_2^2} < \frac{1}{3}\frac{1}{c^2(\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}\frac{\displaystyle\int_0^T e^{Q(t)}F(t,x_0)dt}{|x_0|^2}.$$

Then, for every $\lambda\in\Lambda'$ *and for every nonnegative function* $G : [0,T]\times\mathbb{R}^N\to\mathbb{R}$ *satisfying the assumption* (1.2)*, there exists* $\delta^*_{\lambda,G} > 0$ *given by* (3.6) *such that for each* $\mu\in[0,\delta^*_{\lambda,G})$*, the problem* (1.1) *has at least three weak solutions* v^j*;* $j = 1,2,3$ *such that* $\|v^j\|_\infty < \theta_2$*,* $\forall t\in[0,T]$*,* $j = 1,2,3$*.*

Proof. Fix λ and μ as in the conclusion and let Φ and Ψ be as given in the proof of Theorem 3.1. Put $r_1 = \frac{1}{2}(\frac{\theta_1}{c})^2$, $r_2 = \frac{1}{2}(\frac{\theta_2}{c})^2$ and $w(t) := x_0$ for all $t\in[0,T]$. The condition

$$\frac{2(\frac{\theta_1}{c})^2}{\delta\int_0^T e^{Q(t)}dt} < |x_0|^2 < \frac{\frac{1}{2}(\frac{\theta_2}{c})^2}{(\sum_{i,j=1}^{N}\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt},$$

in conjunction with (3.2) yields

$$2r_1 < \Phi(w) < \frac{r_2}{2}.$$

Since $\mu < \delta^*_{\lambda,g}$, one has

$$\begin{aligned}\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{r_1} &= \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1])}\int_0^T e^{Q(t)}[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))]dt}{r_1}\\ &\leq \frac{\int_0^T e^{Q(t)}\sup\limits_{|x|\leq\theta_1}F(t,x)dt+\frac{\mu}{\lambda}G^{\theta_1}}{\frac{1}{2}(\frac{\theta_1}{c})^2}\\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_0^T e^{Q(t)}F(t,x_0)dt+\frac{\mu}{\lambda}G_{x_0}}{\frac{1}{2}|x_0|^2(\sum_{i,j=1}^N\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}\\ &\leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},\end{aligned}$$

and

$$\begin{aligned}\frac{2\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2])}\Psi(u)}{r_2} &= \frac{2\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2])}\int_0^T e^{Q(t)}[F(t,u(t))+\frac{\mu}{\lambda}G(t,u(t))]dt}{r_2}\\ &\leq \frac{2\int_0^T e^{Q(t)}\sup\limits_{|x|\leq\theta_2}F(t,x)dt+\frac{\mu}{\lambda}G^{\theta_2}}{\frac{1}{2}(\frac{\theta_2}{c})^2}\\ &< \frac{1}{\lambda} < \frac{2}{3}\frac{\int_0^T e^{Q(t)}F(t,x_0)dt+\frac{\mu}{\lambda}G_{x_0}}{\frac{1}{2}|x_0|^2(\sum_{i,j=1}^N\|a_{ij}\|_\infty)\int_0^T e^{Q(t)}dt}\\ &\leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.\end{aligned}$$

Therefore, (b_1) and (b_2) of Theorem 2.2 are verified. Finally, we show that $\Phi - \lambda\Psi$ satisfies the assumption *2.* of Theorem 2.2. For this purpose let u^1 and u^2 be two local minima for $\Phi - \lambda\Psi$. Thus u^1 and u^2 are critical point for $\Phi - \lambda\Psi$. Since the functions F and G are nonnegative, we have

$$(\lambda F + \mu G)(t, su_1 + (1-s)u_2) \geq 0,$$

and hence, $\Psi(su_1 + (1-s)u_2) \geq 0$, for all $s \in [0,1]$. Then, since the weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$ (see [29, Theorem 2.2]), by Theorem 2.2 the problem (1.1) possesses at least three periodic solutions v^j; $j = 1,2,3$ such that $\|v^j\|_\infty < \theta_2$, $j = 1,2,3$. □

A special case of Theorem 3.1 is the following theorem.

Theorem 3.3. *Let $F : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuously differentiable function such that $F(0, \cdots, 0) = 0$. Assume that*

$$\liminf_{\xi \to 0} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2} = \limsup_{|x| \to +\infty} \frac{F(x)}{|x|^2} = 0.$$

Then, there is $\lambda^* > 0$ such that for each $\lambda > \lambda^*$ and for every continuously differentiable function $G : \mathbb{R}^N \to \mathbb{R}$ such that $G(0, \cdots, 0) = 0$, satisfying the asymptotical condition

$$\limsup_{|x| \to \infty} \frac{G(x)}{|x|^2} < +\infty,$$

there exists $\delta^*_{\lambda,g} > 0$ such that, for each $\mu \in [0, \delta^*_{\lambda,g}[$, the problem

$$\begin{cases} -\ddot{u}(t) - q(t)\dot{u}(t) + A(t)u(t) = \lambda \nabla F(u(t)) + \mu \nabla G(u(t)) & a.e. \ t \in [0, T], \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

admits at least three classical solutions.

Proof. Fix $\lambda > \lambda^* := \frac{|x_0|^2 (\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt}{2F(x_0) \displaystyle\int_0^T e^{Q(t)} dt}$ for some non zero vector $x_0 \in \mathbb{R}^N$. Since

$$\liminf_{\xi \to 0} \frac{\max_{|x| \leq \xi} F(x)}{\xi^2} = 0,$$

there is a sequence $\{\theta_n\} \subset]0, +\infty[$ such that $\lim_{n \to \infty} \theta_n = 0$ and

$$\lim_{n \to \infty} \frac{\max_{|x| \leq \theta_n} F(x)}{\theta_n^2} = 0.$$

Hence, there exists $\overline{\theta} > 0$ such that

$$\frac{\max\limits_{|x| \leq \overline{\theta}} F(x)}{\overline{\theta}^2} < \min \left\{ \frac{1}{c^2 (\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt} \frac{F(x_0)}{|x_0|^2} ; \ \frac{1}{2\lambda c^2 \int_0^T e^{Q(t)} dt} \right\},$$

and $\overline{\theta} < |x_0| c (\delta \int_0^T e^{Q(t)} dt)^{\frac{1}{2}}$. Theorem 3.1 concludes the result. □

Moreover, the following result is a consequence of Theorem 3.2.

Theorem 3.4. *Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a nonnegative continuously differentiable function such that $F(0, 0, 0) = 0$,*

$$\lim_{\xi \to 0^+} \frac{\max_{|(x_1, x_2, x_3)| \leq \xi} F(x_1, x_2, x_3)}{\xi^2} = 0,$$

and

$$\max_{|(x_1, x_2, x_3)| \leq 6} F(x_1, x_2, x_3) < \frac{4}{3c^2(e^3 - 1)} F(1, 1, 1).$$

Then, for every $\lambda \in \left(\frac{27}{4F(1,1,1)}, \frac{9}{c^2(e^3 - 1) \max_{|(x_1,x_2,x_3)| \leq 6} F(\xi)} \right)$ and for every nonnegative continuously differentiable function $G : \mathbb{R}^3 \to \mathbb{R}$ such that $G(0, 0, 0) = 0$, there exists $\delta^*_{\lambda,G} > 0$ such that, *for each* $\mu \in [0, \delta^*_{\lambda,G}[$, *the problem*

$$\begin{cases} -\ddot{u}(t) - \dot{u}(t) + A(t)u(t) = \lambda \nabla F(u(t)) + \mu \nabla G(u(t)) & a.e.\ t \in [0,3], \\ u(0) - u(3) = \dot{u}(0) - \dot{u}(3) = 0, \end{cases}$$

where $A(t)$ *is a third-order identity matrix, admits at least three classical solutions.*

Proof. Choose $N = 3$, $T = 3$, $q(t) = 1$ for all $t \in [0,3]$, $\theta_2 = 6$ and $x_0 = (1,1,1)$. Therefore,

$$\frac{3}{2} \frac{|x_0|^2 (\sum_{i,j=1}^N \|a_{ij}\|_\infty) \int_0^T e^{Q(t)} dt}{\displaystyle\int_0^T e^{Q(t)} F(t, x_0) dt} = \frac{27}{2F(1,1,1)},$$

and

$$\frac{1}{c^2} \frac{\theta_2^2}{2 \displaystyle\int_0^T e^{Q(t)} \sup_{|\xi| \le \theta_2} F(t,\xi) dt} = \frac{1}{c^2} \frac{18}{(e^3 - 1) \displaystyle\max_{|(x_1,x_2,x_3)| \le 6} F(x_1, x_2, x_3)}.$$

Moreover, since

$$\lim_{x \to 0^+} \frac{\max_{|(x_1,x_2,x_3)| \le \xi} F(x_1,x_2,x_3)}{\xi^2} = 0,$$

there exists a positive constant $\theta_1 < c\sqrt{\frac{3(e^3-1)}{2}}$ such that

$$\frac{\displaystyle\max_{|(x_1,x_2,x_3)| \le \theta_1} F(x_1,x_2,x_3)}{\theta_1^2} < \frac{2}{27c^2} F(1,1,1),$$

and

$$\frac{\theta_1^2}{\displaystyle\max_{|(x_1,x_2,x_3)| \le \theta_1} F(x_1,x_2,x_3)} > \frac{18}{\displaystyle\max_{|(x_1,x_2,x_3)| \le 6} F(x_1,x_2,x_3)}.$$

Hence, since the assumptions of Theorem 3.2 are fulfilled, we have the conclusion from Theorem 3.2. □

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