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Construction of a regularized asymptotic solution of an integro-differential equation with a rapidly oscillating cosine

Abdukhafiz Bobodzhanov[a], Burkhan Kalimbetov[b,*], Nilufar Pardaeva[c]

[a]*The National Research University, MPEI, Krasnokazarmennaya 14, Moscow, Russia.*

[b]*M. Auezov South Kazakhstan University, Tauke-xan 5, Shymkent, Kazakhstan.*

[c]*Almalyk branch of the NRTU MISA, Amir Temur 56, Almalyk, Uzbekistan.*

Abstract

In this paper, we consider a singularly perturbed integro-differential equation with a rapidly oscillating right-hand side, which includes an integral operator with a slowly varying kernel. Earlier, singularly perturbed differential and integro-differential equations with rapidly oscillating coefficients were considered. The main goal of this work is to generalize the Lomov's regularization method and to identify the rapidly oscillating right-hand side to the asymptotics of the solution to the original problem.

Keywords: Singular perturbation, integro-differential equation, oscillating right-hand side, solvability of iterative problems, regularization problems.

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1. Introduction

When studying various applied problems related to the properties of media with a periodic structure, it is necessary to study differential equations with rapidly oscillating inhomogeneities. Equations of this type are often found, for example, in electrical systems under the influence of high frequency external forces. The presence of such forces creates serious problems for the numerical integration of the corresponding differential equations. Therefore, asymptotic methods are usually applied to such equations, the most famous of which are the Feshchenko-Shkil-Nikolenko splitting method [16–18, 42] and Lomov's regularization method [38–40]. The splitting method is especially effective when applied to homogeneous equations, and in the case of inhomogeneous differential equations, the Lomov regularization method turned out to be the most effective. However, both of these methods were developed mainly for singularly perturbed equations that do not contain an integral operator. The transition from differential equations to integro-differential equations requires a significant restructuring of the algorithm of the regularization method. The integral term generates new types of singularities in solutions that differ from

*Corresponding author

Email address: bkalimbetov@mail.ru (Burkhan Kalimbetov)

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the previously known ones, which complicates the development of the algorithm for the regularization method. The splitting method, as far as we know, has not been applied to integro-differential equations.

In this study Lomov's regularization method is generalized to previously unexplored classes of problems of the type

$$L_\varepsilon y(t,\varepsilon) \equiv \varepsilon \frac{dy}{dt} - A(t)y - \int_{t_0}^{t} K(t,s)y(s,\varepsilon)ds = h_1(t) + h_2(t)\cos\frac{\beta(t)}{\varepsilon},$$
$$y(t_0,\varepsilon) = y^0, \; t \in [t_0, T], \;\; t_0 > 0 \tag{1.1}$$

where $A(t), h_1(t), h_2(t), \beta(t)$ are scalar functions, $\beta'(t) > 0$ ia the frequency of the rapidly oscillating sine, y^0 is a constant number, $\varepsilon > 0$ is a small parameter. The function $\lambda_1(t) = A(t)$ is the eigenvalue of the limit operator $a(t)$, and the functions $\lambda_2(t) = -i\beta'(t)$ and $\lambda_3(t) = +i\beta'(t)$ are associated with the presence of a rapidly oscillating sine in equation (1.1).

Problem (1.1) will be considered under the following conditions:

1) $A(t), \beta(t) \in C^\infty([t_0,T], \mathbf{R}), h_1(t), h_2(t) \in C^\infty([t_0,T], \mathbf{C}), \;\; K(t,s) \in C^\infty(\{t_0 \leqslant s \leqslant t \leqslant T\}, \mathbf{C});$
2) $A(t) < 0 \;\; \forall t \in [t_0,T].$

The problem is posed of constructing a regularized [38, 39] asymptotic solution to problem (1.1). The problem with parametric amplification from the standpoint of the regularization method was studied in [40], where a regularized asymptotic solution was constructed. A generalization of the idea of the regularization method for integral and integro-differential equations with rapidly oscillating coefficients was studied in [3–5, 26–28, 30], based on our studies for singularly perturbed integro-differential equations with slowly and rapidly varying kernels [6–9, 12, 24, 31–34, 37]. Singularly perturbed integro-differential equations with partial derivatives are studied in the works [10, 11, 23, 25, 35] and singularly perturbed differential, inegro-differential equations with fractional derivatives in the works [2, 20–22]. Based on the algorithm of the regularization method for integro-differential equations with rapidly oscillating coefficients, the time has come to study singularly perturbed integral and integro-differential equations with rapidly oscillating inhomogeneities [1, 13–15, 29, 36].

Thus, we begin to develop an algorithm for constructing a regularized asymptotic solution [38] to problem (1.1).

2. Solution space and regularization of problem (1.1)

Denote by $\sigma_j = \sigma_j(\varepsilon)$, independent on t quantities $\sigma_1 = e^{+\frac{i}{\varepsilon}\beta(t_0)}$, $\sigma_2 = e^{-\frac{i}{\varepsilon}\beta(t_0)}$, and rewrite equation (1.1) as

$$\begin{aligned} L_\varepsilon y(t,\varepsilon) &\equiv \varepsilon \frac{dy}{dt} - A(t)y - \int_{t_0}^{t} K(t,s)y(s,\varepsilon)ds \\ &= h_1(t) + + \left(e^{-\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_1 + e^{+\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta}\sigma_2 \right), \; y(t_0,\varepsilon) = y^0. \end{aligned} \tag{2.1}$$

We introduce regularizing variables (see [38])

$$\tau_j = \frac{1}{\varepsilon}\int_{t_0}^{t} \lambda_j(\theta)\,d\theta \equiv \frac{\psi_j(t)}{\varepsilon}, \;\; j = \overline{1,3}, \tag{2.2}$$

and instead of the problem (2.1) consider the problem

$$\begin{aligned} \tilde{L}_\varepsilon \tilde{y}(t,\tau,\varepsilon) &\equiv \varepsilon\frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{3}\lambda_j(t)\frac{\partial \tilde{y}}{\partial \tau_j} - \lambda_1(t)\tilde{y} - \int_{t_0}^{t} K(t,s)\tilde{y}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds \\ &= h_1(t) + \frac{1}{2}h_2(t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right), \quad \tilde{y}(t_0,0,\varepsilon) = y^0, \end{aligned} \tag{2.3}$$

for the function $\tilde{y} = \tilde{y}(t,\tau,\varepsilon)$, where (according to (2.2)) $\tau = (\tau_1,\tau_2,\tau_3)$, $\psi = (\psi_1,\psi_2,\psi_3)$. It is clear that if $\tilde{y} = \tilde{y}(t,\tau,\varepsilon)$ is the solution of the problem (2.3), then the vector function $\tilde{y} = \tilde{y}\left(t,\frac{\psi(t)}{\varepsilon},\varepsilon\right)$ is an exact solution of the problem (2.1). Thus, problem (2.3) is extended with respect to problem (2.1). However, it cannot be considered fully regularized, since the integral term

$$J\tilde{y} \equiv J\left(\tilde{y}(t,\tau,\varepsilon)|_{t=s,\tau=\psi(s)/\varepsilon}\right) = \int_{t_0}^{t} K(t,s)\,\tilde{y}(s,\frac{\psi(s)}{\varepsilon},\varepsilon)ds$$

has not been regularized in it.

To regularize it, we introduce the class M_ε asymptotically invariant with respect to the operator $J\tilde{y}$ (see [38, p. 62]). Consider first the space U of vector functions $y(t,\tau,\sigma)$, representable by the sums

$$y(t,\tau,\sigma) = y_0(t,\sigma) + \sum_{j=1}^{3} y_j(t,\sigma)e^{\tau_j}, y_j(t,\sigma) \in C^\infty\left([t_0,T],\mathbf{C}^1\right), \quad j = \overline{0,3}. \tag{2.4}$$

Note that in (2.4) the elements of the space U depend on the constants $\sigma_1 = \sigma_1(\varepsilon)$ and $\sigma_2 = \sigma_2(\varepsilon)$ (bounded in $\varepsilon > 0$), which do not affect the development of the algorithm described below, therefore, henceforth, in the recording of element (2.4) of this space U, for the sake of brevity, the dependence on $\sigma = (\sigma_1,\sigma_2)$ is omitted.

Let us show that the class $M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon}$ is asymptotically invariant under the operator J. The image of the operator J on the element (2.4) of the space U has the form

$$Jy(t,\tau) = \int_{t_0}^{t} K(t,s)\,y_0(s)\,ds + \sum_{j=1}^{3}\int_{t_0}^{t} K(t,s)\,y_j(s)\,e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\lambda_j(\theta)d\theta}ds.$$

Integrating by parts, we will have

$$\begin{aligned} J_j(t,\varepsilon) &= \int_{t_0}^{t} K(t,s)y_j(s)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\lambda_j(\theta)d\theta}ds \\ &= \varepsilon\int_{t_0}^{t}\frac{K(t,s)y_j(s)}{\lambda_j(s)}de^{\frac{1}{\varepsilon}\int_{t_0}^{s}\lambda_j(\theta)d\theta} \\ &= \varepsilon\left[\frac{K(t,s)y_j(s)}{\lambda_j(s)}e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\lambda_j(\theta)d\theta}\Bigg|_{s=t_0}^{s=t} - \int_{t_0}^{t}\left(\frac{\partial}{\partial s}\frac{K(t,s)y_j(s)}{\lambda_j(s)}\right)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\lambda_j(\theta)d\theta}ds\right] \\ &= \varepsilon\left[\frac{K(t,t)y_j(t)}{\lambda_j(t)}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_j(\theta)d\theta} - \frac{K(t,t_0)y_j(t_0)}{\lambda_j(t_0)}\right] - \varepsilon\int_{t_0}^{t}\left(\frac{\partial}{\partial s}\frac{K(t,s)y_j(s)}{\lambda_j(s)}\right)e^{\frac{1}{\varepsilon}\int_{t_0}^{s}\lambda_j(\theta)d\theta}ds, j = \overline{1,3}. \end{aligned}$$

Continuing this process further, we obtain the expansion

$$J_j(t,\varepsilon)=\sum_{\nu=0}^{\infty}(-1)^{\nu}\varepsilon^{\nu+1}\left[\left(I_j^{\nu}\left(K(t,s)y_j(s)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_j(\theta)d\theta}-\left(I_j^{\nu}\left(K(t,s)y_j(s)\right)\right)_{s=t_0}\right],$$

where it is indicated as

$$I_j^0=\frac{1}{\lambda_j(s)},\quad I_j^{\nu}=\frac{1}{\lambda_j(s)}\frac{\partial}{\partial s}I_j^{\nu-1},\ j=\overline{1,3},\ (\nu\geqslant 1).$$

Hence, the image of the operator J on the element (2.4) of the space U can be represented as a series

$$\begin{aligned}Jy\left(t,\tau\right)=&\int_{t_0}^{t}K\left(t,s\right)y_0\left(s\right)ds\\&+\sum_{j=1}^{3}\sum_{\nu=0}^{\infty}(-1)^{\nu}\left[\left(I_j^{\nu}\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t}e^{\frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_j(\theta)d\theta}-\left(I_j^{\nu}\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t_0}\right].\end{aligned}$$

It is easy to show (see, for example, [41, pp. 291-294]) that this series converges asymptotically as $\varepsilon\to+0$ (uniformly in $t\in[t_0,T]$). This means that the class M_ε is asymptotically invariant (for $\varepsilon\to+0$) with respect to the operator J.

Let us introduce operators $R_\nu:U\to U$, acting on each element $y\left(t,\tau\right)\in U$ of the form (2.4) according to the law

$$\begin{aligned}R_0y\left(t,\tau\right)&=\int_{t_0}^{t}K\left(t,s\right)y_0\left(s\right)ds,\\R_1y\left(t,\tau\right)&=\sum_{j=1}^{3}\left[\left(I_j^0\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t}e^{\tau_j}-\left(I_j^0\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t_0}\right],\\R_{\nu+1}y\left(t,\tau\right)&=\sum_{j=1}^{3}(-1)^{\nu}\left[\left(I_j^{\nu}\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t}e^{\tau_j}-\left(I_j^{\nu}\left(K\left(t,s\right)y_j\left(s\right)\right)\right)_{s=t_0}\right],\ \nu\geqslant 1.\end{aligned}\tag{2.5}$$

Now let $\tilde{y}\left(t,\tau,\varepsilon\right)$ be arbitrary continuous function in $\left(t,\tau\right)\in[t_0,T]\times\left\{\tau:\mathrm{Re}\tau_j\leqslant 0,\ \ j=\overline{1,3}\right\}$ having an asymptotic expansion

$$\tilde{y}\left(t,\tau,\varepsilon\right)=\sum_{k=0}^{\infty}\varepsilon^k y_k\left(t,\tau\right),\ \ y_k\left(t,\tau\right)\in U\tag{2.6}$$

converging as $\varepsilon\to+0$ (uniformly in $\left(t,\tau\right)\in[t_0,T]\times\left\{\tau:\mathrm{Re}\tau_j\leqslant 0,j=\overline{1,3}\right\}$). Then the image $J\tilde{y}\left(t,\tau,\varepsilon\right)$ of this function expands into an asymptotic series

$$J\tilde{y}\left(t,\tau,\varepsilon\right)=\sum_{k=0}^{\infty}\varepsilon^kJy_k\left(t,\tau\right)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s\left(t,\tau\right)|_{\tau=\psi(t)/\varepsilon}.$$

This equality is the basis for introducing an extension of the operator J on series of the form (2.6),

$$\tilde{J}\tilde{y}\left(t,\tau,\varepsilon\right)\equiv\tilde{J}\left(\sum_{k=0}^{\infty}\varepsilon^k y_k\left(t,\tau\right)\right)=\sum_{r=0}^{\infty}\varepsilon^r\sum_{s=0}^{r}R_{r-s}y_s(t,\tau).\tag{2.7}$$

Although the operator $\tilde{J}$ is defined formally, its usefulness is obvious, since in practice the N-th approximation of the asymptotic solution of problem (2.1) is usually constructed, in which only the N-th partial sums of the series (2.6) that have not formal, but true meaning. Now we can write down a problem that is completely regularized with respect to the original problem (2.1),

$$\tilde{L}_\varepsilon \tilde{y}(t,\tau,\varepsilon) \equiv \varepsilon\frac{\partial \tilde{y}}{\partial t} + \sum_{j=1}^{3}\lambda_j(t)\frac{\partial \tilde{y}}{\partial \tau_j} - \lambda_1(t)\tilde{y} - \tilde{J}\tilde{y} = h_1(t) + \frac{1}{2}h_2(t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right),\ \tilde{y}(t_0,0,\varepsilon) = y^0, \quad (2.8)$$

where the operator $\tilde{J}$ has the form (2.7).

3. Iterative problems and their solvability in the space U

Substituting series (2.6) into (2.8) and equating the coefficients at the same powers ε, we get the following iterative problems

$$Ly_0(t,\tau) \equiv \sum_{j=1}^{3}\lambda_j(t)\frac{\partial y_0}{\partial \tau_j} - \lambda_1(t)y_0 - R_0y_0 = h_1(t) + \frac{1}{2}h_2(t)\left(e^{\tau_2}\sigma_1 + e^{\tau_3}\sigma_2\right), y_0(t_0,0) = y^0, \quad (3.1)$$

$$Ly_1(t,\tau) = -\frac{\partial y_0}{\partial t} + R_1y_0,\ y_1(t_0,0) = 0, \quad (3.2)$$

$$Ly_2(t,\tau) = -\frac{\partial y_1}{\partial t} + R_1y_1 + R_2y_0,\ y_2(t_0,0) = 0,$$

$$\vdots$$

$$Ly_k(t,\tau) = -\frac{\partial y_{k-1}}{\partial t} + R_ky_0 + \cdots + R_1y_{k-1},\ y_k(t_0,0) = 0,\ k \geqslant 1. \quad (3.3)$$

Each of the iterative problems (3.3) can be written as

$$Ly(t,\tau) \equiv \sum_{j=1}^{3}\lambda_j(t)\frac{\partial y}{\partial \tau_j} - \lambda_1(t)y - R_0y = H(t,\tau),\quad y(t_0,0) = y_*, \quad (3.4)$$

where $H(t,\tau) = H_0(t) + \sum_{j=1}^{3}H_j(t)e^{\tau_j}$ is the known function of the space U, $y_* \in \mathbf{C}$ is constant, and the operator R_0 has the form (see (2.5))

$$R_0y \equiv R_0\left(y_0(t) + \sum_{j=1}^{3}y_j(t)e^{\tau_j}\right) = \int_{t_0}^{t}K(t,s)y_0(s)\,ds.$$

We introduce a scalar (for each $t \in [t_0,T]$) product in the space U

$$< z,w > \equiv < z_0(t) + \sum_{j=1}^{3}z_j(t)e^{\tau_j}, w_0(t) + \sum_{j=1}^{3}w_j(t)e^{\tau_j} > \overset{def}{=} \sum_{j=0}^{3}\left(z_j(t), w_j(t)\right),$$

where $(*,*)$ denotes the usual scalar product in the complex space $\mathbf{C}$. Let us prove the following statement.

Theorem 3.1. *Let conditions 1) and 2) be satisfied and the right-hand side* $H(t,\tau) = H_0(t) + \sum_{j=1}^{3}H_j(t)e^{\tau_j}$ *of equation (3.4) belongs to the space* U. *Then for the solvability of equation (3.4) in* U *it is necessary and sufficient that the identity*

$$< H_1(t,\tau), e^{\tau_1} > \equiv 0 \Leftrightarrow H_1(t) \equiv 0,\quad \forall t \in [t_0,T]. \quad (3.5)$$

Proof. We will define the solution of the equation (3.4) as an element (2.4) of the space U:

$$y\left(t,\tau\right)=y_0\left(t\right)+\sum_{j=1}^{3}y_j\left(t\right)e^{\tau_j}. \tag{3.6}$$

Substituting (3.6) into equation (3.4), we will have

$$\sum_{j=1}^{3}\left[\lambda_j\left(t\right)-a\left(t\right)\right]y_j\left(t\right)e^{\tau_j}-\lambda_1\left(t\right)y_0\left(t\right)-\int_{t_0}^{t}K\left(t,s\right)y_0\left(s\right)ds=H_0\left(t\right)+\sum_{j=1}^{3}H_j\left(t\right)e^{\tau_j}.$$

Equating here separately the free terms and coefficients at the same exponents, we obtain the following equations

$$-\lambda_1\left(t\right)y_0\left(t\right)-\int_{t_0}^{t}K\left(t,s\right)y_0\left(s\right)ds=H_0\left(t\right)\quad\Leftrightarrow\quad y_0\left(t\right)=\int_{t_0}^{t}\frac{K\left(t,s\right)}{-\lambda_1\left(t\right)}y_0\left(s\right)ds+\frac{H_0\left(t\right)}{-\lambda_1\left(t\right)}, \tag{3.7}$$

$$\left[\lambda_j\left(t\right)-\lambda_1\left(t\right)\right]y_j\left(t\right)=H_j\left(t\right),\ \ j=\overline{1,3}. \tag{3.8}$$

Due to the smoothness of the kernel $-{\lambda_1}^{-1}(t)K\left(t,s\right)$ and heterogeneity $-{\lambda_1}^{-1}(t)H_0\left(t\right)$ the integral equation (3.7) has a unique solution $y_0\left(t\right)\in C^\infty\left([t_0,T],\mathbf{C}\right)$. As $\lambda_{2,3}\left(t\right)=\pm i\beta'\left(t\right)$ are purely imaginary functions, and the function $a\left(t\right)$ is real, then the equations (3.8) for $j=2,3$ are solvable in the space $C^\infty\left([t_0,T],\mathbf{C}\right)$. The equation (3.1) is solvable in space $C^\infty\left([t_0,T],\mathbf{C}\right)$ if and only if the identity $H_1\left(t\right)\equiv 0\ \forall t\in[t_0,T]$ holds. It is easy to see that this identity coincides with identity (3.5).

Thus, condition (3.5) is necessary and sufficient for the solvability of system (3.4) in the space U. □

Remark 3.2. If identity (3.5) is satisfied, then under conditions 1) and 2) equation (3.4) has the following solution in the space U

$$y\left(t,\tau\right)=y_0\left(t\right)+\alpha_1\left(t\right)e^{\tau_1}+\sum_{j=2}^{3}\frac{H_j(t)}{\lambda_j(t)-\lambda_1(t)}e^{\tau_j}, \tag{3.9}$$

where $\alpha_1\left(t\right)\in C^\infty\left([t_0,T],\mathbf{C}\right)$ is arbitrary function and $y_0\left(t\right)$ is the solution of the integral equation (3.7).

4. Unique solvability of a general iterative problem in the space U: remainder term theorem

As seen from (3.9), the solution to equation (3.4) is determined ambiguously. However, if it is subjected to additional conditions

$$y\left(t_0,0\right)=y_*,\quad -\frac{\partial y}{\partial t}+R_1y+Q\left(t,\tau\right),e^{\tau_1}>\equiv 0,\ \ \forall t\in\left[t_0,T\right], \tag{4.1}$$

where $Q\left(t,\tau\right)=Q_0\left(t\right)+\sum\limits_{j=1}^{3}Q_j\left(t\right)e^{\tau_j}$ is a known function of the space U and y_* is a constant number of the complex space $\mathbf{C}$, then problem (3.4) will be uniquely solvable in the space U. More precisely, the following result holds.

Theorem 4.1. *Let conditions 1) and 2) be satisfied and the right-hand side* $H\left(t,\tau\right)$ *of equation* (3.4) *belongs to the space* U *and satisfies the orthogonality condition* (3.5). *Then equation* (3.4) *under additional conditions* (4.1) *is uniquely solvable in* U.

Proof. Under condition (3.5), equation (3.4) has a solution to (3.3) in the space U, where the function $\alpha_1(t) \in C^\infty([t_0,T], \mathbf{C})$ is so far arbitrary. Subordinate (3.3) to the initial condition $y(t_0,0) = y_*$. We will have

$$y_* = y_0(t_0) + \alpha_1(t_0) + \sum_{j=2}^{3} \frac{H_j(t_0)}{\lambda_j(t_0) - \lambda_1(t_0)} \Leftrightarrow \alpha_1(t_0) = y_* + a^{-1}(t_0) H_0(t_0) - \sum_{j=2}^{3} \frac{H_j(t_0)}{\lambda_j(t_0) - \lambda_1(t_0)}. \tag{4.2}$$

Let us now subordinate the solution (3.9) to the second condition (4.1). The right side of this equation has the form

$$\begin{aligned} &-\frac{\partial y_0}{\partial t} + R_1 y_0 + Q(t,\tau) \\ &= -\dot{y}_0(t) - \dot{\alpha}_1(t) e^{\tau_1} - \sum_{j=2}^{3} \left(\frac{H_j(t)}{\lambda_j(t) - \lambda_1(t)} \right)^{\bullet} e^{\tau_j} \\ &\quad + \frac{K(t,t)\,\alpha_1(t)}{\lambda_1(t)} e^{\tau_1} - \frac{K(t,t_0)\,\alpha_1(t_0)}{\lambda_1(t_0)} + \sum_{j=2}^{3} \left[\frac{K(t,t) y_j(t)}{\lambda_j(t)} e^{\tau_j} - \frac{K(t,t_0) y_j(t_0)}{\lambda_j(t_0)} \right] + Q(t,\tau). \end{aligned} \tag{4.3}$$

Multiplying (4.2) scalarly by e^{τ_1}, we obtain the differential equation

$$-\dot{\alpha}_1(t) + \frac{K(t,t)\alpha_1(t)}{\lambda_1(t)} + Q_1(t) = 0.$$

Adding the initial condition (4.1) to it, we uniquely find the function $\alpha_1(t)$, and, therefore, construct a solution (3.9) to problem (3.4) in the space U uniquely. □

Applying Theorems 3.1 and 4.1 to iterative problems (3.3), we find uniquely their solutions in the space U and construct series (2.6). Let $y_{\varepsilon N}(t) = \sum_{k=0}^{N} \varepsilon^k y_k\left(t, \frac{\psi(t)}{\varepsilon}\right)$ is the restriction of the N^{th} partial sum of series (2.6) for $\tau = \frac{\psi(t)}{\varepsilon}$. Same as in [41], it is easy to prove the following statement.

Lemma 4.2. *Let conditions 1) – 2) be satisfied. Then the partial sum $y_{\varepsilon N}(t)$ satisfies problem (1.1) up to $O(\varepsilon^{N+1})(\varepsilon \to +0)$, i.e.,*

$$\begin{aligned} \varepsilon \frac{dy_{\varepsilon N}(t)}{dt} &\equiv a(t) y_{\varepsilon N}(t) + \int_{t_0}^{t} K(t,s) y_{\varepsilon N}(s) ds \\ &\quad + h_1(t) + h_2(t) \cos\frac{\beta(t)}{\varepsilon} + \varepsilon^{N+1} R_N(t,\varepsilon), y_{\varepsilon N}(t_0) = y^0, \ \forall t \in [t_0,T], \end{aligned} \tag{4.4}$$

where $\|R_N(t,\varepsilon)\|_{C[t_0,T]} \leqslant \bar{R}_N$ *for all* $t \in [t_0,T]$ *and for all* $\varepsilon \in (0, \varepsilon_N]$.

Consider now the following problem

$$\varepsilon \frac{dz(t,\varepsilon)}{dt} = a(t) z(t,\varepsilon) + \int_{t_0}^{t} K(t,s) z(s,\varepsilon) ds + \Phi(t,\varepsilon), z(t_0,\varepsilon) = 0, t \in [t_0,T]. \tag{4.5}$$

Let us show that this problem is solvable in the space $C^1([t_0,T], \mathbf{C})$ (i.e., it has a solution for any right-hand side $\Phi(t,\varepsilon) \in C([t_0,T], \mathbf{C})$) and that in this case there is an estimate

$$\|z(t,\varepsilon)\|_{C[t_0,T]} \leqslant \frac{\nu_0}{\varepsilon} \|\Phi(t,\varepsilon)\|_{C[t_0,T]}.$$

Theorem 4.3. *Let conditions 1)–2) be satisfied. Then, for sufficiently small* $\varepsilon \in (0, \varepsilon_0]$, *problem* (4.4) *for any right-hand side* $\Phi(t, \varepsilon) \in C[t_0, T]$ *has a unique solution* $z(t, \varepsilon)$ *in the space* $C^1([t_0, T], \mathbf{C})$ *and estimate* (4.5) *holds, where* ν_0 *is a constant independent of* $\varepsilon > 0$.

Proof. Introduce an additional unknown function

$$u(t, \varepsilon) = \int_{t_0}^{t} K(t, s) z(s, \varepsilon) ds.$$

Differentiating it with respect to t, we will have

$$\frac{du(t, \varepsilon)}{dt} = K(t, t) z(t, \varepsilon) + \int_{t_0}^{t} \frac{\partial K(t, s)}{\partial t} z(s, \varepsilon) ds.$$

From this and (4.4) it follows that the vector function $w = \{z, u\}$ satisfies the following system

$$\varepsilon \frac{dw(t,\varepsilon)}{dt} = \begin{pmatrix} a(t) & 1 \\ 0 & 0 \end{pmatrix} w(t, \varepsilon) + \varepsilon \begin{pmatrix} 0 \\ K(t, t)z + \int_{t_0}^{t} \frac{\partial K(t,s)}{\partial t} z(s, \varepsilon) ds \end{pmatrix} + \begin{pmatrix} \Phi(t, \varepsilon) \\ 0 \end{pmatrix}, \; w(t_0, \varepsilon) = 0. \qquad (4.6)$$

Denote by $Y(t, s, \varepsilon)$ the normal fundamental matrix of the homogeneous system $\varepsilon \frac{dw}{dt} = \begin{pmatrix} a(t) & 1 \\ 0 & 0 \end{pmatrix} w$, i.e., the matrix satisfying the equation

$$\varepsilon \frac{dY(t, s, \varepsilon)}{dt} = \begin{pmatrix} a(t) & 1 \\ 0 & 0 \end{pmatrix} Y(t, s, \varepsilon), \;\; Y(s, s, \varepsilon) = I, \;\; t_0 \leqslant s \leqslant t \leqslant T.$$

Since the matrix $\begin{pmatrix} a(t) & 1 \\ 0 & 0 \end{pmatrix}$ is a matrix of simple structure and its spectrum $\{a(t), 0\}$ lies in the half-plane $\operatorname{Re} \lambda(t) \leqslant 0$, then the Cauchy matrix $Y(t, s, \varepsilon)$ is uniformly bounded, i.e.,

$$\|Y(t, s, \varepsilon)\| \leqslant c_0, \;\; \forall (t, s, \varepsilon): \;\; t_0 \leqslant s \leqslant t \leqslant T, \;\; \varepsilon > 0,$$

where the constant $c_0 > 0$ does not depend on $\varepsilon > 0$ (see, for example, [38, pp. 119-120]). We now write down an integral system equivalent to system (4.6):

$$w(t, \varepsilon) = \int_{t_0}^{t} Y(t, \zeta, \varepsilon) \begin{pmatrix} 0 \\ K(\zeta, \zeta) z(\zeta, \varepsilon) + \int_{t_0}^{\zeta} \frac{\partial K(\zeta,s)}{\partial t} z(s, \varepsilon) ds \end{pmatrix} d\zeta + \frac{1}{\varepsilon} \int_{t_0}^{t} Y(t, \zeta, \varepsilon) \begin{pmatrix} \Phi(\zeta, \varepsilon) \\ 0 \end{pmatrix} d\zeta. \qquad (4.7)$$

Since for each $\varepsilon > 0$ there exists a solution $w(t, \varepsilon)$ of system (4.6) in the space $C^1([t_0, T], \mathbf{C})$, then substituting it into (4.7), we obtain the identity. Let's move on to the norms

$$\begin{aligned} \|w(t, \varepsilon)\| \leqslant & \int_{t_0}^{t} \|Y(t, \zeta, \varepsilon)\| \cdot \|K(\zeta, \zeta)\| \cdot \|z(\zeta, \varepsilon)\| d\zeta \\ & + \int_{t_0}^{t} \|Y(t, \zeta, \varepsilon)\| \int_{t_0}^{\zeta} \|\frac{\partial K(\zeta, s)}{\partial t}\| \cdot \|z(s, \varepsilon)\| ds d\zeta + \frac{1}{\varepsilon} \int_{t_0}^{t} \|Y(t, \zeta, \varepsilon)\| \cdot \|\Phi(\zeta, \varepsilon)\| d\zeta \end{aligned}$$

$$
\begin{aligned}
&\leqslant c_0k_0\int_{t_0}^{t}\|w(\zeta,\varepsilon)\|d\zeta+c_0k_1\int_{t_0}^{t}\int_{t_0}^{\zeta}\|w(s,\varepsilon)\|dsd\zeta+\frac{T_0}{\varepsilon}c_0\|\Phi(t,\varepsilon)\|_{C[t_0,T]}\\
&\leqslant c_0k_0\int_{t_0}^{t}\|w(s,\varepsilon)\|ds+c_0k_1\int_{t_0}^{t}\int_{t_0}^{t}\|w(s,\varepsilon)\|dsd\zeta+\frac{c_0T_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]}\\
&\leqslant c_0k_0\int_{t_0}^{t}\|w(s,\varepsilon)\|ds+c_0k_1\int_{t_0}^{T}\int_{t_0}^{t}\|w(s,\varepsilon)\|dsd\zeta+\frac{c_0T_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]}\\
&\leqslant (c_0k_0+c_0k_1T_0)\int_{t_0}^{t}\|w(s,\varepsilon)\|ds+\frac{c_0T_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]},
\end{aligned}
$$

where $T_0=T-t_0$, $\|K(t,s)\|_{C([t_0,T]\times[t_0,T])}=k_0$, $\|\partial K(t,s)/\partial t\|_{C([t_0,T]\times[t_0,T])}=k_1$. We got the inequality

$$
\|w(t,\varepsilon)\|\leqslant\frac{c_0T_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]}+(c_0k_0+c_0k_1T_0)\int_{t_0}^{t}\|w(s,\varepsilon)\|ds.
$$

Applying the Gronwall-Bellman lemma [19] to this inequality, we have

$$
\begin{aligned}
\|w(t,\varepsilon)\|&\leqslant\frac{c_0T_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]}e^{(c_0k_0+c_0k_1T_0)\int_{t_0}^{t}ds}\\
&=\frac{c_0T_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]}e^{(c_0k_0+c_0k_1T_0)(t-t_0)}\\
&\leqslant\frac{\nu_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]}.\ \Rightarrow\ \|z(t,\varepsilon)\|_{C[t_0,T]}\leqslant\frac{\nu_0}{\varepsilon}\|\Phi(t,\varepsilon)\|_{C[t_0,T]},
\end{aligned}
$$

where $\nu_0=c_0T_0\cdot t\in[t_0,T]\max e^{(c_0k_0+c_0k_1T_0)(t-t_0)}$. □

Theorem 4.4. *Let conditions 1) and 2) be satisfied. Then for $\varepsilon\in(0,\varepsilon_0]$, where $\varepsilon_0>0$ is small enough, the problem (1.1) has a unique solution $y(t,\varepsilon)\in C^1([t_0,T],\mathbf{C})$; in this case, the estimate*

$$
\|y(t,\varepsilon)-y_{\varepsilon N}(t)\|_{C[t_0,T]}\leqslant C_N\varepsilon^{N+1}\ (N=0,1,2,\dots)
$$

holds true, where the constant $C_N>0$ does not depend on $\varepsilon\in(0,\varepsilon_0]$.

Proof. By the Lemma, the partial sum $y_{\varepsilon N}(t)$ satisfies the problem (4.3), so the remainder $r_N(t,\varepsilon)\equiv y(t,\varepsilon)-y_{\varepsilon N}(t)$ satisfies the following problem

$$
\varepsilon\frac{r_N(t,\varepsilon)}{dt}=a(t)r_N(t,\varepsilon)+\int_{t_0}^{t}K(t,s)r_N(s,\varepsilon)ds-\varepsilon^{N+1}R_N(t,\varepsilon),\quad r_N(t_0,\varepsilon)=0,
$$

where $\Phi(t,\varepsilon)=-\varepsilon^{N+1}\int_{t_0}^{t}R_N(s,\varepsilon)ds$. By Theorem 4.3, we have the estimate

$$
\|r_N(t,\varepsilon)\|_{C[t_0,T]}\leqslant\varepsilon^N\bar{R}_N
$$

for all $N=0,1,2,\dots$ and all $\varepsilon\in(0,\varepsilon_N]$, which means that the partial sum $y_{\varepsilon,N+1}(t)=y_{\varepsilon N}(t)+\varepsilon^{N+1}y_{N+1}(t,\frac{\psi(t)}{\varepsilon})$ satisfies the inequality

$$
\|y(t,\varepsilon)-y_{\varepsilon,N+1}(t)\|_{C[t_0,T]}\equiv\|(y(t,\varepsilon)-y(t))-\varepsilon^{N+1}y_{N+1}(t,\frac{\psi(t)}{\varepsilon})\|_{C[t_0,T]}\leqslant\bar{C}_{N+1}\varepsilon^{N+1}.
$$

Using the inequality $\|a-b\| \geqslant |\|a\|-\|b\||$, valid for any numbers a and b, we will have

$$\|y(t,\varepsilon)-y_{\varepsilon N}(t)\|_{C[t_0,T]} \leqslant \left(\bar{C}_N + \left\|y_{N+1}(t,\frac{\psi(t)}{\varepsilon})\right\|_{C[t_0,T]}\right)\varepsilon^{N+1},$$

whence we derive the estimate

$$\|y(t,\varepsilon)-y_{\varepsilon N}(t)\|_{C[t_0,T]} \leqslant C_N\varepsilon^{N+1},$$

where the constant $C_N > 0$ does not depend on $\varepsilon \in (0,\varepsilon_N]$. □

5. Construction of the solution of the first iteration problem

Using Theorem 3.1, let us try to find a solution to the first iterative problem (3.1). Since the right-hand side $h_1(t)+h_2(t)\cos\frac{\beta(t)}{\varepsilon}$ of the equation (3.1) satisfies condition (3.5), this equation has (according to (3.3)) the solution in the space U in the form

$$y_0(t,\tau) = y_0^{(0)}(t) + \alpha_1^{(0)}(t)e^{\tau_1} + h_{21}(t)\sigma_1 e^{\tau_2} + h_{31}(t)\sigma_2 e^{\tau_3}, \tag{5.1}$$

where $\alpha_1^{(0)}(t) \in C^\infty([t_0,T],C)$ is an arbitrary function, $y_0^{(0)}(t)$ is the solution of the integral equation $-\lambda_1(t)y_0(t) - \int\limits_{t_0}^{t} K(t,s)y_0(s)\,ds = h_1(t)$ and introduced the notation

$$h_{21}(t) = \frac{1}{2}\frac{h_2(t)}{\lambda_2(t)-\lambda_1(t)}, \qquad h_{31}(t) = \frac{1}{2}\frac{h_2(t)}{\lambda_3(t)-\lambda_1(t)}.$$

Subjecting (5.1) to the initial condition $y_0(t_0,0) = y^0$, we will have

$$\begin{aligned} &y_0^{(0)}(t_0) + \alpha_1^{(0)}(t_0) + h_{21}(t_0)\sigma_1 + h_{31}(t_0)\sigma_2 = y^0 \\ &\Leftrightarrow \quad \alpha_1^{(0)}(t_0) = y^0 + \lambda_1^{-1}(t_0)h_1(t_0) - h_{21}(t_0)\sigma_1 - h_{31}(t_0)\sigma_2. \end{aligned} \tag{5.2}$$

For the complete calculation of the function $\alpha_1^{(0)}(t)$, we proceed to the next iterative problem (3.2). Substituting solution (5.1) of the equation (3.1) into it, we arrive at the following equation

$$\begin{aligned} y_1(t,\tau) = &-\frac{d}{dt}\left(y_0^{(0)}(t)\right) - \frac{d}{dt}\left(\alpha_1^{(0)}(t)\right)e^{\tau_1} - \frac{d}{dt}(h_{21}(t))\sigma_1 e^{\tau_2} \\ &-\frac{d}{dt}(h_{31}(t))\sigma_2 e^{\tau_3} + \left[\frac{K(t,t)\alpha_1^{(0)}(t)}{\lambda_1(t)}e^{\tau_1} - \frac{K(t,t_0)\alpha_1^{(0)}(t_0)}{\lambda_1(t_0)}\right] \\ &+\sum_{j=2}^{3}\left[\frac{K(t,t)h_{j1}(t)}{\lambda_j(t)}e^{\tau_j} - \frac{K(t,t_0)h_{j1}(t_0)}{\lambda_j(t_0)}\right]. \end{aligned} \tag{5.3}$$

This equation is solvable in the space U if and only if its right-hand side satisfies condition (3.5). Separating the coefficient of e^{τ_1} on the right-hand side of equation (5.3) and equating it to zero, we obtain the differential equation

$$-\frac{d\alpha_1^{(0)}(t)}{dt} + \frac{K(t,t)}{\lambda_1(t)}\alpha_1^{(0)}(t) = 0.$$

Adding the initial condition (5.2) to this equation, we uniquely find the function $\alpha_1^{(0)}(t)$:

$$\alpha_1^{(0)}(t) = [y^0 + \lambda_1^{-1}(t_0)h_1(t_0) - h_{21}(t_0)\sigma_1 - h_{31}(t_0)\sigma_2]\exp\left\{\int\limits_{t_0}^{t}\frac{K(\theta,\theta)}{\lambda_1(\theta)}d\theta\right\}$$

and hence we will uniquely construct solution (5.1) of problem (3.1) in the space U. The following iterative problems are solved similarly (3.1).

The main term of the asymptotics of the solution to problem (1.1) is the restriction of function (5.1) for $\tau = \frac{\psi(t)}{\varepsilon}$ and has the following form

$$\begin{aligned} y_{\varepsilon 0}(t) = y_0^{(0)}(t) + h_{21}(t)\,\sigma_1 e^{-\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta} + h_{31}(t)\,\sigma_2 e^{+\frac{i}{\varepsilon}\int_{t_0}^{t}\beta'(\theta)d\theta} \\ + \left[y^0 + \lambda_1^{-1}(t_0)\,h_1(t_0) - h_{21}(t_0)\,\sigma_1 - h_{31}(t_0)\,\sigma_2\right] e^{\int_{t_0}^{t}\frac{K(\theta,\theta)}{\lambda_1(\theta)}d\theta + \frac{1}{\varepsilon}\int_{t_0}^{t}\lambda_1(\theta)d\theta}. \end{aligned} \tag{5.4}$$

6. Conclusions

It can be seen from expression (5.4) for $y_{\varepsilon 0}(t)$ that the construction of the leading term of the asymptotics of the solution to problem (1.1) (or the equivalent problem (2.1)) is significantly influenced by both the rapidly oscillating inhomogeneity and the kernel of the integral operator.

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