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A commensal symbiosis model with Holling type functional response

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Abstract

A two species commensal symbiosis model with Holling type functional response takes the form

$$\frac{dx}{dt} = x\left(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\right),$$
$$\frac{dy}{dt} = y(a_2 - b_2 y)$$

is investigated, where $a_i, b_i, i = 1, 2$, p and c_1 are all positive constants, $p \geq 1$. Local and global stability property of the equilibria is investigated. We also show that depending on the ratio of $\frac{a_2}{b_2}$, the first component of the positive equilibrium $x^*(p)$ may be the increasing or decreasing function of p or independent of p. Our study indicates that the unique positive equilibrium is globally stable and the system always permanent.

Keywords: Commensal symbiosis model, stability.
2010 MSC: 34C25, 92D25, 34D20, 34D40.

1. Introduction

Though much progress has been made on the mutualism model ([2, 4–7, 9–15, 17–19, 23, 24, 27–29]), there are still only few work on the commensal symbiosis model([16, 20, 21, 25, 26]). Sun and

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Wei [21] first time proposed a intraspecific commensal model:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x \left(\frac{k_1 - x + ay}{k_1} \right), \\ \frac{dy}{dt} &= r_2 y \left(\frac{k_2 - y}{k_2} \right). \end{aligned} \tag{1.1}$$

They investigated the local stability of all equilibrium points. However, they did not give any information about the global dynamic behaviors of the system.

Han and Chen [16] incorporated the feedback control variables to the commensal symbiosis model system, and proposed the following model:

$$\begin{aligned} \dot{x} &= x\left(b_1 - a_{11}x + a_{12}y - \alpha_1 u_1\right), \\ \dot{y} &= y\left(b_2 - a_{22}y - \alpha_2 u_2\right), \\ \dot{u}_1 &= -\eta_1 u_1 + a_1 x, \\ \dot{u}_2 &= -\eta_2 u_2 + a_2 y. \end{aligned} \tag{1.2}$$

They showed that system (1.2) admits a unique globally stable positive equilibrium.

Xie et al. [25] proposed the following discrete commensal symbiosis model

$$\begin{aligned} x_1(k+1) &= x_1(k) \exp\left\{a_1(k) - b_1(k)x_1(k) + c_1(k)x_2(k)\right\}, \\ x_2(k+1) &= x_2(k) \exp\left\{a_2(k) - b_2(k)x_2(k)\right\}. \end{aligned} \tag{1.3}$$

They investigated the positive ω-periodic solution of the system (1.3).

Xue et al. [26] further incorporate the delay to system (1.3), and proposed the following discrete commensalism system

$$\begin{aligned} x(n+1) &= x(n) \exp\left[r_1(n)\left(1 - \frac{x(n-\tau_1)}{K_1(n)} + \alpha(n)\frac{y(n-\tau_2)}{K_1(n)}\right)\right], \\ y(n+1) &= y(n) \exp\left[r_2(n)\left(1 - \frac{y(n-\tau_3)}{K_2(n)}\right)\right]. \end{aligned} \tag{1.4}$$

They investigated the almost periodic solution of the system (1.4).

Miao et al. [20] studied the persistent property of the following periodic Lotka-Volterra commensal symbiosis model with impulsive.

$$\begin{aligned} \frac{dx_1}{dt} &= x_1\left(a_1(t) - b_1(t)x_1 + c_1(t)x_2\right), \\ \frac{dx_2}{dt} &= x_2\left(a_2(t) - b_2(t)x_2\right), \qquad t \neq \tau_k, \\ x_i(\tau_k^+) &= (1 + h_{ik})x_i(\tau_k), \qquad t = \tau_k,\ k = 1, 2, \cdots. \end{aligned} \tag{1.5}$$

Their results indicate that impulsive is one of the important reasons that can change the long time behaviors of species.

It brings to our attention that all of the above system are based on the traditional Lotka-Volterra model, which made the assumption that the influence of the second species to the first one is linearize. This may not be suitable. Now, if we assume that the functional response between two species is of

Holling type ([1, 3, 8, 22, 30]), then we could establish the following two species commensal symbiosis model

$$\begin{aligned} \frac{dx}{dt} &= x\left(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\right), \\ \frac{dy}{dt} &= y(a_2 - b_2 y), \end{aligned} \tag{1.6}$$

where $a_i, b_i, i = 1, 2$, p and c_1 are all positive constants, $p \geq 1$.

The aim of this paper is to investigate the local and global stability property of the possible equilibria of system (1.6). We arrange the paper as follows: In the next section, we will investigate the existence and local stability property of the equilibria of system (1.6). In Section 3, by constructing some suitable Dulac function, we will investigate the global stability property of the system. In Section 4, we will investigate the relationship of the x^* and p. In Section 5, two examples together with their numerical simulations are presented to show the feasibility of our main results.

2. The existence and local stability of the equilibria

The equilibria of system (1.6) is determined by the system

$$\begin{aligned} x\left(a_1 - b_1 x + \frac{c_1 y^p}{1+y^p}\right) &= 0, \\ y(a_2 - b_2 y) &= 0. \end{aligned} \tag{2.1}$$

Hence, system (1.6) admits four possible equilibria, $A_0(0,0)$, $A_1\left(\frac{a_1}{b_1}, 0\right)$, $A_2\left(0, \frac{a_2}{b_2}\right)$ and $A_3\left(x^*, y^*\right)$, where

$$x^* = \frac{a_1(\frac{a_2}{b_2})^p + c_1(\frac{a_2}{b_2})^p + a_1}{b_1\left(1 + (\frac{a_2}{b_2})^p\right)}, \quad y^* = \frac{a_2}{b_2}. \tag{2.2}$$

Concerned with the local stability property of the above four equilibria, we have

Theorem 2.1. *$A_0(0,0)$, $A_1(\frac{a_1}{b_1}, 0)$ and $A_2\left(0, \frac{a_2}{b_2}\right)$ are unstable; $A_3\left(x^*, y^*\right)$ is locally stable.*

Proof. The Jacobian matrix of the system (1.6) is calculated as

$$J(x,y) = \begin{bmatrix} a_1 - 2b_1 x + \dfrac{c_1 y^p}{1+y^p} & \dfrac{c_1 p x y^{p-1}}{(1+y^p)^2} \\ 0 & -2b_2 y + a_2 \end{bmatrix}. \tag{2.3}$$

Then the Jacobian matrix of the system (1.6) about the equilibrium $A_0(0,0)$, $A_1(\frac{a_1}{b_1}, 0)$ and $A_2(0, \frac{a_2}{b_2})$ are given by

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \tag{2.4}$$

$$\begin{bmatrix} -a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \tag{2.5}$$

and

$$\begin{bmatrix} a_1 + \dfrac{c_1(\frac{a_2}{b_2})^p}{1 + (\frac{a_2}{b_2})^p} & 0 \\ 0 & -a_2 \end{bmatrix} \tag{2.6}$$

respectively. One could easily see that all of the above three matrix has at least one positive eigenvalues, which means that $A_0(0,0)$, $A_1(\frac{a_1}{b_1},0)$ and $A_2(0,\frac{a_2}{b_2})$ are all unstable.

The Jacobian matrix about the equilibrium A_3 is given by

$$\begin{bmatrix} -\dfrac{a_1(\frac{a_2}{b_2})^p + c_1(\frac{a_2}{b_2})^p + a_1}{1+(\frac{a_2}{b_2})^p} & F_{12} \\ 0 & -a_2 \end{bmatrix} \tag{2.7}$$

where

$$F_{12} = \frac{c_1 p b_2 \left(a_1(\frac{a_2}{b_2})^{2p} + c_1(\frac{a_2}{b_2})^{2p} + a_1(\frac{a_2}{b_2})^p\right)}{b_1 a_2 \left(1+(\frac{a_2}{b_2})^p\right)^3}. \tag{2.8}$$

The eigenvalues of the above matrix are $\lambda_1 = -\dfrac{a_1(\frac{a_2}{b_2})^p + c_1(\frac{a_2}{b_2})^p + a_1}{1+(\frac{a_2}{b_2})^p} < 0, \lambda_2 = -a_2 < 0$. Hence, $A_3(x^*,y^*)$ is locally stable.

This ends the proof of Theorem 2.1. □

3. Global stability of the positive equilibrium

Theorem 2.1 shows that the system always admits a positive equilibrium, and this equilibrium is locally stable. One interesting thing is whether the system (1.6) could admits cycle or not, which means that the two species could be coexistent in a periodic oscillation form. The aim of this section is to show that such phenomenon could not be happened.

Lemma 3.1 ([28]).

$$\frac{dy}{dt} = y(a - by) \tag{3.1}$$

has a unique globally attractive positive equilibrium $y^* = \frac{a}{b}$.

Theorem 3.2. $A_3\big(x^*,y^*\big)$ *is globally stable.*

Proof. Firstly, we prove that every solution of system (1.6) that starts in R^2_+ is uniformly bounded. Noting that the second equation of (1.6) takes the form

$$\frac{dy}{dt} = y(a_2 - b_2 y). \tag{3.2}$$

By applying Lemma 3.1 to system (3.2), we know that system (3.2) has a unique globally attractive positive equilibrium $y^* = \frac{a_2}{b_2}$.

It follows from the first equation of system (1.6) that

$$\frac{dx}{dt} \le x(a_1 - b_1 x + c_1),$$

and so

$$\limsup_{t\to+\infty} x(t) \le \frac{a_1+c_1}{b_1}. \tag{3.3}$$

Hence, there exists a $\varepsilon > 0$ such that for all $t > T$

$$x(t) < \frac{a_1+c_1}{b_1} + \varepsilon, \qquad y(t) < \frac{a_2}{b_2} + \varepsilon. \tag{3.4}$$

Let $D = \left\{(x,y) \in R_+^2 : x < \frac{a_1+c_1}{b_1} + \varepsilon,\ \ y < \frac{a_2}{b_2} + \varepsilon\right\}$. Then every solution of system (1.6) starts in R_+^2 is uniformly bounded on D. Also, from Theorem 2.1, there is a unique local stable positive equilibrium $A_3(x^*, y^*)$. To show that $A_3(x^*, y^*)$ is globally stable, it is enough to show that the system admits no limit cycle in the area D. Let's consider the Dulac function $u(x,y) = x^{-1}y^{-1}$, then

$$\frac{\partial(uP)}{\partial x} + \frac{\partial(uQ)}{\partial y} = -b_1 y^{-1} - b_2 x^{-1} < 0,$$

where $P(x,y) = x\left(a_1 - b_1 x + \dfrac{c_1 y^p}{1+y^p}\right)$, $Q(x,y) = y(a_2 - b_2 y)$. By Dulac Theorem [29], there is no closed orbit in area D. Consequently, $A_3(x^*, y^*)$ is globally asymptotically stable.

This completes the proof of Theorem 3.2. □

4. Relationship of x^* and p

Since we are interesting in the influence of functional response to the dynamic behaviors of the system (1.6). From the previous section, we had showed that the functional response has no influence on the persistent property of the system, since the system always admits a unique global stable positive equilibrium, which means that the system is permanent.

To further investigate the influence of the functional response, bring attention to the expression of x^* and y^*, one could see that y^* is independent of p, which means that the functional response has no influence on the final density of the second species. However, x^* is the function of p, hence it seems interesting to investigate the relationship of x^* and p.

Noting that

$$\frac{dx^*}{dp} = \frac{c_1\left(\frac{a_2}{b_2}\right)^p \ln\frac{a_2}{b_2}}{b_1(1+\left(\frac{a_2}{b_2}\right)^p)^2}. \tag{4.1}$$

Then

(1) if $a_2 > b_2$, then $\ln\frac{a_2}{b_2} > 0$, and so $\frac{dx^*}{dp} > 0$, consequently, $x^*(p)$ is the strict increasing function of p;

(2) if $a_2 = b_2$, then $\ln\frac{a_2}{b_2} = 0$, and so $\frac{dx^*}{dp} = 0$, consequently, $x^*(p)$ will not be changed with p;

(3) if $a_2 < b_2$, then $\ln\frac{a_2}{b_2} < 0$, and so $\frac{dx^*}{dp} < 0$, consequently, $x^*(p)$ is the strict decreasing function of p.

5. Numerical simulations

Now let us consider the following examples.

Example 5.1. Consider the following system

$$\begin{aligned}\frac{dx}{dt} &= x\left(1 - 2x + \frac{y}{1+y}\right),\\ \frac{dy}{dt} &= y(1-2y).\end{aligned} \tag{5.1}$$

In this system, corresponding to system (1.6), we take $a_1 = a_2 = c_1 = 1, b_1 = b_2 = 2$. From Theorem 3.2, the unique positive equilibrium $\left(\frac{2}{3}, \frac{1}{2}\right)$ is globally stable. Numeric simulation (Figure 1) also support this assertion.

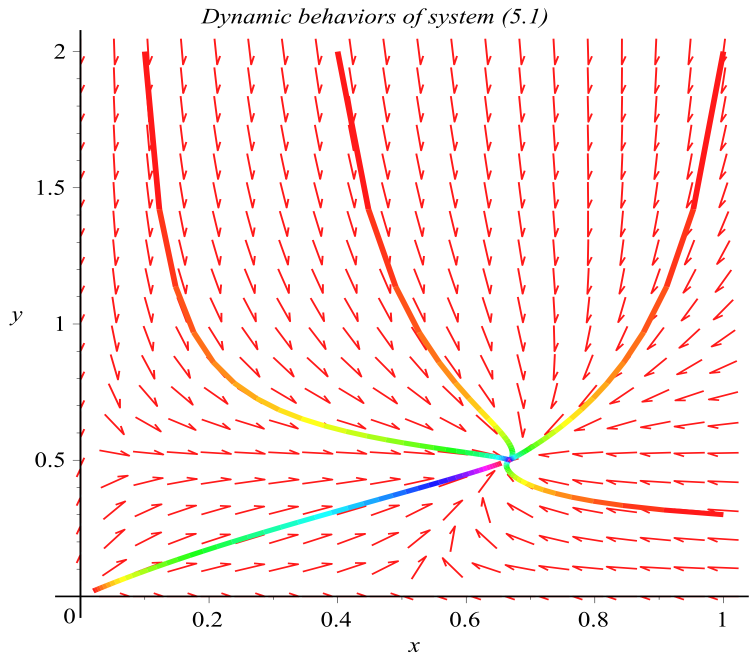

Figure 1: Numerical simulations of system (5.1) with the initial conditions $(x(0), y(0)) = (0.4, 2), (1, 0.3), (0.02, 0.02), (1, 2)$ and $(0.1, 2)$, respectively.

Example 5.2. Consider the following system

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - 2x + \frac{y^p}{1+y^p}\right), \\ \frac{dy}{dt} &= y(a_2 - 2y). \end{aligned} \tag{5.2}$$

(1) Take $a_2 = 1$. In this case, $x^*(p) = \frac{1}{2}\dfrac{2(\frac{1}{2})^p + 1}{1 + (\frac{1}{2})^p}$. Figure 2 shows that $x^*(p)$ is the decreasing function of p.

(2) Take $a_2 = 2$. In this case, $x^*(p) = \frac{3}{4}$. Obviously, $x^*(p)$ is independent of p;

(3) Take $a_2 = 3$. In this case, $x^*(p) = \frac{1}{2}\dfrac{2(\frac{3}{2})^p + 1}{1 + (\frac{3}{2})^p}$. Figure 3 shows that $x^*(p)$ is the increasing function of p.

6. Conclusion

We propose a two species commensal symbiosis model with Holling type functional response, our study shows that the dynamic behaviors of the system is similar to the Lotka-Volterra type commensal symbiosis model, i.e., the system admits a unique globally stable positive equilibrium.

We mentioned here that the analysis method of this paper could not be applied to the case $0 < p < 1$, and we leave this for future investigation.

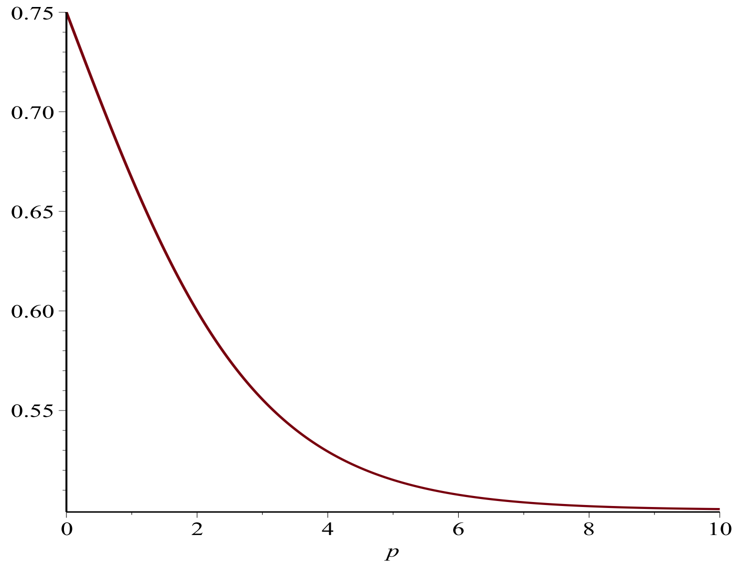

Figure 2: Numerical simulations of $x^*(p)$, where $p \in [0, 10]$.

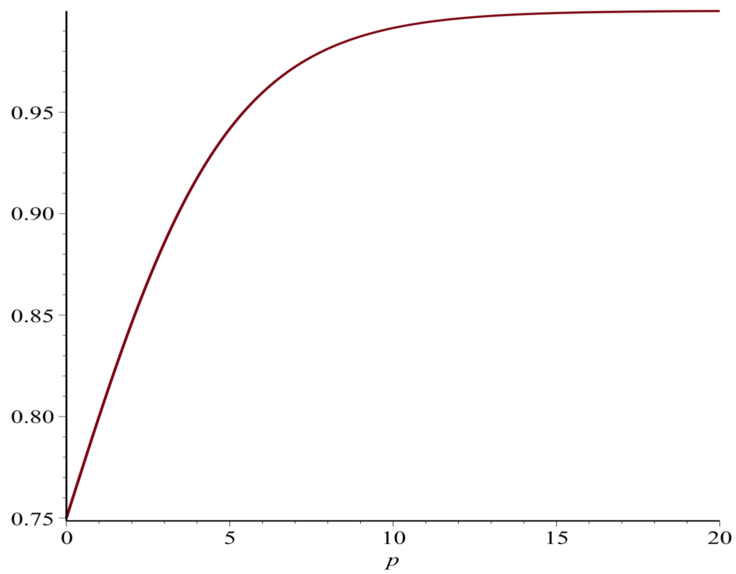

Figure 3: Numerical simulations of $x^*(p)$, where $p \in [0, 20]$.

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