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Infinitely many high energy solutions for fourth-order elliptic equations with p-Laplacian in bounded domain

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Abstract

In this paper, we study the following fourth-order elliptic equation with p-Laplacian, steep potential well and sublinear perturbation:

$$\begin{cases} \Delta^2 u - \kappa\Delta_p u + \mu V(x)u = f(x,u) + \xi(x)|u|^{q-2}u, & x \in \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $N \geqslant 5$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ with $p > 2$, $\mu, \kappa > 0$ are parameters, $f \in \mathcal{C}(\Omega \times \mathbb{R}, \mathbb{R})$, $\xi \in L^{\frac{2}{2-q}}(\Omega)$ with $1 \leqslant q < 2$, we have the potential $V \in \mathcal{C}(\Omega, \mathbb{R})$. Using variational methods, we establish the existence of infinitely many nontrivial high energy solutions under certain assumptions on V and f.

Keywords: Variational methods, p-Laplacian, fourth-order elliptic equations.

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1. Introduction

In this paper, we are interested in the existence of solutions to the following fourth-order elliptic equations with p-Laplacian:

$$\begin{cases} \Delta^2 u - \kappa\Delta_p u + \mu V(x)u = f(x,u) + \xi(x)|u|^{q-2}u, & x \in \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta^2 := \Delta(\Delta)$ is the biharmonic operator, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ with $p > 2$, $\kappa, \mu > 0$ are parameters, $f \in \mathcal{C}(\Omega \times \mathbb{R}, \mathbb{R})$, $V \in \mathcal{C}(\Omega, \mathbb{R})$, and $\xi \in L^{\frac{2}{2-q}}(\Omega)$ with $1 \leqslant q < 2$.

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When considering problem (1.1) with the conditions $V(x) = 0$, $p = 2$, and $\xi(x) = 0$, we can derive the following modified problem:

$$\begin{cases} \Delta^2 u + c\Delta u = f(x,u) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

The study of fourth-order elliptic equations has concrete applications in many fields, which arises in the study of traveling waves in suspension bridges (see [19, 20] and the references therein), as well as the study of the static deflection of an elastic plate in a fluid, has been extensively investigated in recent years. For the results of infinitely many nontrivial and sign changing solutions of problem (1.2), we refer the readers to [1, 3, 4, 8, 10–14, 16–18, 21, 26, 30] and the references therein.

For the whole space $\mathbb{R}^N$ case, the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. To overcome this difficulty, some authors assumed that the potential V satisfies certain coercive condition, see [15, 24, 27] and the references therein. To reduce our statements, we make the following assumptions for potential V:

(V_1) $V \in \mathcal{C}(\Omega, \mathbb{R})$ and $V \geqslant 0$ on Ω;

(V_2) there exists a constant $c > 0$ such that the set $\{V < c\} = \{x \in \Omega \mid V(x) < c\}$ is nonempty and has finite measure;

(V_3) $\Sigma = \operatorname{int} V^{-1}(0)$ is a nonempty open set and has smooth boundary with $\overline{\Sigma} = V^{-1}(0)$.

From (V_1)-(V_3), we can see μV represents a steep potential well whose depth is controlled by μ. Bartsch and Wang first introduced this problem for the case of a nonlinear Schrödinger equation and the potential μV with V satisfying (V_1)-(V_3) [5, 6]. Later, the authors in [29] considered the case $\xi(x) = 0$ and $p = 2$

$$\begin{cases} \Delta^2 u - \Delta u + \mu V(x) = f(x,u) \text{ in } \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N). \end{cases} \tag{1.3}$$

Using (V_1)-(V_3) they proved the existence and concentration of solutions for problem (1.3). In [28], by the genus properties in critical point theory, Zhang et al. considered the regularity and existence of infinitely many small energy solutions for problem (1.3). In [22], using the Gagliardo-Nirenberg inequality and Mountain Pass Lemma, the existence and multiplicity of nontrivial solutions were obtained of the following biharmonic equation with p-Laplacian problem:

$$\begin{cases} \Delta^2 u - \nu\Delta_p u + \mu V(x) u = f(x,u), \quad x \in \mathbb{R}^N, \\ u \in H^2\left(\mathbb{R}^N\right), \end{cases}$$

where $N \geqslant 1$, $p \geqslant 2$, $\nu \in \mathbb{R}$ and $\mu > 0$ are parameters. In [9], Benhanna and Choutri considered the following biharmonic equation:

$$\begin{cases} \Delta^2 u - \Delta_p u + \mu V(x) u = f(x,u) + \nu\xi(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \\ u \in H^2\left(\mathbb{R}^N\right). \end{cases}$$

By using the Mountain pass theorem, Ekeland's variational principle and Gagliardo-Nirenberg inequality, the existence of at least two nontrivial solutions for this biharmonic equation was obtained. In [23], Fenglong and al studied the following problem

$$\begin{cases} \Delta^2 u - \lambda\Delta_p u = f(x,u) - \frac{\mu}{|\Omega|} \int\limits_\Omega f(y, u(y))\, dy, \quad x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u)}{\partial \nu} = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $|\Omega|$ is the measure of Ω, $\lambda, \mu \in \mathbb{R}$ are parameters. By introducing an appropriate function space with constraint $\int\limits_\Omega u\, dx = 0$, they obtained infinitely many sign-changing solutions for problem (1.4).

Our goal is to obtain the existence of a sequence of infinitely many high-energy solutions to problem (1.1). The strategy of the proof for this assertion is based on applications of the dual-fountain theorem that were primarily introduced by Bartsch [7] with consideration for the variational nature of the problem. The dual-fountain theorem, as a key tool, is a dual version of fountain theorem in [6], which is a generalization of the symmetric mountain-pass theorem in [2] and a powerful technique for ensuring the existence of multiple solutions to elliptic equations of the variational type. This paper is organized as follows. We first briefly review definitions and collect some preliminary results for the Sobolev spaces and Gagliardo-Nirenberg inequalities. Next, we will give the existence criteria of infinitely many nontrivial high energy solutions without the well known Ambrosetti-Rabinowitz condition.

2. Preliminaries

In this section, we briefly recall definitions and some elementary properties of Sobolev spaces. For simplicity, c_i, C, C_i are used to represent different different generic positive constants. Let $s \in [2, 2^*]$, 2^* is the critical Sobolev exponent, that is $2^* = \frac{2N}{N-4}$. We give the definition the following norm:

$$|u|_s = \|u\|_{L^s(\Omega)} = \left(\int_\Omega |u(t)|^s dt \right)^{\frac{1}{s}}.$$

We define Sobolev space H^2 as follows:

$$H^2(\Omega) = \left\{ u \in L^2(\Omega) : \nabla u \in L^2(\Omega), \Delta u \in L^2(\Omega) \right\},$$

with the norm

$$\|u\|_{H^2}^2 = \int_\Omega |\Delta u|^2 + |\nabla u|^2 + u^2 \, dx.$$

Let

$$E = \left\{ u \in H^2(\Omega) \cap H_0^1(\Omega) : \text{ and } \int_\Omega \left(|\Delta u|^2 + V(x) u^2 \right) \, dx < +\infty \right\},$$

be equipped with the inner product

$$(u, v) = \int_\Omega \left(\Delta u \Delta v + V(x) u v \right) \, dx, \quad u, v \in E,$$

and the norm

$$\|u\|^2 = \int_\Omega (|\Delta u|^2 + V(x) u^2) \, dx, \quad u \in E.$$

Then E is a Hilbert space with the inner product defined above. Moreover, by Gagliardo-Nirenberg inequality, there exists $C_1 > 0$ such that:

$$\int_\Omega |\nabla u|^2 \, dx \leqslant C_1^2 \left(\int_\Omega |\Delta u|^2 \, dx \right)^{\frac{1}{2}} \left(\int_\Omega u^2 \, dx \right)^{\frac{1}{2}} \leqslant \frac{C_1^2}{2} \left(\int_\Omega |\Delta u|^2 \, dx + \int_\Omega u^2 \, dx \right),$$

which indicates that:

$$\int_\Omega \left(|\Delta u|^2 + u^2 \right) \, dx \leqslant \|u\|_{H^2}^2 \leqslant \left(1 + \frac{C_1^2}{2} \right) \int_\Omega \left(|\Delta u|^2 + u^2 \right) \, dx. \tag{2.1}$$

It follows from the Hölder, Gagliardo-Nirenberg inequalities, and conditions (V_1)-(V_2) that there exists $C_2 > 0$, such that:

$$\int_\Omega u^2\,dx = \int_{\{V\geqslant c\}} u^2\,dx + \int_{\{V<c\}} u^2\,dx \leqslant \frac{1}{c}\int_{\{V\geqslant c\}} V(x)u^2\,dx + |\{V<c\}|^{\frac{4}{N}}\left(\int_\Omega |u|^{2^*}\,dx\right)^{\frac{2}{2^*}}$$
$$\leqslant \frac{1}{c}\int_\Omega V(x)u^2\,dx + C_2^2|\{V<c\}|^{\frac{4}{N}}\int_\Omega |\Delta u|^2\,dx.$$

Combining the above inequality with (2.1) yields:

$$\|u\|_{H^2}^2 \leqslant \alpha_N\|u\|^2, \tag{2.2}$$

where

$$\alpha_N = \left(1+\frac{C_1^2}{2}\right)\max\left\{1+C_2^2|\{V<c\}|^{\frac{4}{N}}, \frac{1}{c}\right\},$$

which implies that the imbedding $E \hookrightarrow H^2(\mathbb{R}^N)$ is continuous. For $\mu > 0$, we introduce another inner product and normal

$$(u,v)_\mu = \int_\Omega (\Delta u\Delta v + \mu V(x)uv)\,dx, \quad u,v\in E, \qquad \|u\|_\mu^2 = \int_\Omega (|\Delta u|^2 + \mu V(x)u^2)\,dx, \quad u\in E.$$

Let $E_\mu = (E, \|\cdot\|_\mu)$, then E_μ is a Hilbert space and

$$\|u\| \leqslant \|u\|_\mu, \quad \text{for } \mu\geqslant 1. \tag{2.3}$$

By (V_1)-(V_2), the Hölder and Gagliardo-Nirenberg inequalities, we can demonstrate that there exist positive constants β_N, $\bar{\mu}$ (independent of μ) such that

$$\|u\|_{H^2}^2 \leqslant \beta_N\|u\|_\mu^2, \quad \text{for all } u\in E_\mu, \mu\geqslant\bar{\mu}.$$

In fact, similar to the inequality (2.2), for $\mu \geqslant \bar{\mu} := \frac{1}{c}\left[1+C_2^2|\{V<c\}|^{\frac{4}{N}}\right]^{-1}$, we obtain:

$$\|u\|_{H^2}^2 \leqslant \beta_N\|u\|_\mu^2, \tag{2.4}$$

where

$$\beta_N = \left(1+\frac{C_1^2}{2}\right)\left(1+C_2^2|\{V<c\}|^{\frac{4}{N}}\right) > 0.$$

By the Hölder, Gagliardo-Nirenberg inequalities, and conditions (V_1)-(V_2), (2.4), for any $s\in[2,2^*]$, one has:

$$\begin{aligned}\int_\Omega |u|^s\,dx &\leqslant \left(\int_\Omega |u|^2\,dx\right)^{\frac{2N-s(N-4)}{8}}\left(\int_\Omega |u|^{2^*}\,dx\right)^{\frac{(s-2)(N-4)}{8}}\\ &\leqslant C_2^{\frac{N(s-2)}{4}}\left(\int_\Omega |u|^2\,dx\right)^{\frac{2N-s(N-4)}{8}}\left(\int_\Omega |\Delta u|^2\,dx\right)^{\frac{N(s-2)}{8}}\\ &\leqslant C_2^{\frac{N(s-2)}{4}}\left(1+\frac{C_1^2}{2}\right)^{\frac{s}{2}}\left(1+C_2^2|\{V<c\}|^{\frac{4}{N}}\right)^{\frac{s}{2}}\|u\|_\mu^s.\end{aligned} \tag{2.5}$$

From (2.5), for any $s \in [2, 2^*]$ and $\mu \geqslant \bar{\mu}$, we have:

$$\int_\Omega |u|^s \, dx \leqslant \gamma_s \|u\|_\mu^s, \tag{2.6}$$

where

$$\gamma_s = C_2^{\frac{N(s-2)}{4}} \left(1 + \frac{C_1^2}{2}\right)^{\frac{s}{2}} \left(1 + C_2^2 |\{V < c\}|^{\frac{4}{N}}\right)^{\frac{s}{2}}.$$

From (2.6), the Gagliardo-Nirenberg inequality, and Young inequalities, there exists $C_3 > 0$, and $t = \frac{2p}{4-p}$, such that:

$$\begin{aligned} \int_\Omega |\nabla u|^p \, dx &\leqslant C_3^p \left(\int_\Omega |\Delta u|^2 \, dx\right)^{\frac{p}{4}} \left(\int_\Omega |u|^t dx\right)^{\frac{p}{2t}} \\ &\leqslant \frac{C_3^p}{2} \left[\left(\int_\Omega |\Delta u|^2 dx\right)^{\frac{p}{2}} + \left(\int_\Omega |u|^t dx\right)^{\frac{p}{t}}\right] \leqslant \frac{C_3^p (\beta_N^{\frac{p}{2}} + \gamma_t^{\frac{p}{t}})}{2} \|u\|_\mu^p. \end{aligned} \tag{2.7}$$

Definition 2.1. We indicate that $u \in E$ is a weak solution to the problem (1.1) if

$$\int_\Omega (\Delta u \Delta v + \mu V(x) u v) \, dx + \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_\Omega f(x, u) v \, dx + \int_\Omega \xi(x) |u|^{q-2} u v \, dx,$$

for all $v \in E$. Let us define functional $J_{\mu,\lambda}$ by

$$J_{\mu,\lambda}(u) = A(u) - \lambda B(u),$$

where

$$A(u) = \frac{1}{2}\|u\|_\mu^2 + \frac{\kappa}{p} \int_\Omega |\nabla(u)|^p \, dx, \quad B(u) = \int_\Omega F(x, u) dx + \frac{1}{q} \int_\Omega \xi(x) |u|^q \, dx,$$

for all $u \in E_\mu$ and $\lambda \in [1, 2]$. It is easy to verify that $J_{\mu,\lambda}(u) : E_\mu \to \mathbb{R}$ is a $\mathcal{C}^1$-functional for $\lambda \in [1, 2]$ and for any $u, v \in E_\mu$, its Fréchet derivative is given by

$$\begin{aligned} \langle J'_{\mu,\lambda}(u), v \rangle = &\int_\Omega (\Delta u \Delta v + \mu V(x) u v) \, dx + \kappa \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \\ &- \lambda \left[\int_\Omega f(x, u) v \, dx + \int_\Omega \xi(x) |u|^{q-2} u v \, dx\right], \end{aligned} \tag{2.8}$$

for all $u, v \in E_\mu$. Hence the critical points of $J_{\mu,1}$ are solutions of (1.1).

For $2 < p < 2^* = \frac{2N}{N-2}$, we assume that

(f_1) $\xi \in L^{\frac{2}{2-q}}(\Omega)$ and $\xi > 0$ on Ω;

(f_2) $\lim\limits_{|u| \to 0} \frac{f(x,u)}{|u|} = 0$ uniformly in $x \in \Omega$;

(f_3) $f \in \mathcal{C}(\Omega \times \mathbb{R}, \mathbb{R})$, there exist two constants $c_2 > 0, p < r < 2_* = \frac{2N}{N-4}$ such that

$$|f(x, u)| \leqslant c_2 \left(1 + |u|^{r-1}\right), \quad \forall (x, u) \in \Omega \times \mathbb{R};$$

(f_4) $F(x, u) \geqslant 0$ and $\lim\limits_{|u| \to \infty} \frac{F(x,u)}{|u|^p} = \infty$ uniformly in $x \in \Omega$.

Let $\{e_j\}$ be a complete orthonormal basis of E. We define

$$E_j := \operatorname{span}\{e_j\}, \quad Y_k := \bigoplus_{j=1}^{k} E_j, \text{ and } Z_k := \bigoplus_{j=k+1}^{\infty} E_j, \quad k \in \mathbb{N}.$$

Also

$$B_k = \{u \in Y_k : \|u\| \leqslant \rho_k\}, \quad S_k = \{u \in Z_k : \|u\| = r_k\},$$

for $\rho_k > r_k > 0$. Clearly, $E = Y_k \oplus Z_k$ with $\dim Y_k < \infty$.

Theorem 2.2 ([31]). *Let X be a Banach space, suppose that $J_{\mu,\lambda} \in C^1(X, \mathbb{R})$ satisfies:*

(A_1) $J_{\mu,\lambda}(u)$ *maps bounded sets into bounded sets uniformly for $\lambda \in [1,2]$, and*

$$J_{\mu,\lambda}(-u) = J_{\mu,\lambda}(u) \quad \textit{for } (\lambda, u) \in [1,2] \times X;$$

(A_2) $B(u) \geqslant 0$, $\forall u \in X$, *and* $A(u) \to \infty$ *or* $B(u) \to \infty$ *as* $\|u\| \to \infty$*;*

(A_3) *there exist $\rho_k > r_k > 0$ such that*

$$e_k(\lambda) := \inf_{u \in Z_k, \|u\| = r_k} J_{\mu,\lambda}(u) > f_k(\lambda) := \max_{u \in Y_k, \|u\| = \rho_k} J_{\mu,\lambda}(u), \quad \forall \lambda \in [1,2].$$

Then

$$e_k(\lambda) \leqslant g_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} J_{\mu,\lambda}(\gamma(u)), \quad \forall \lambda \in [1,2],$$

where $\Gamma_k = \{\gamma \in \mathcal{C}(B_k, X_\lambda) : \gamma$ is odd, $\gamma|_{\partial B_k} = id\}$ $(k \geqslant 2)$. In addition, for almost every $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\sup_n \|u_n(\lambda)\| < \infty, \quad J'_{\mu,\lambda}\left(u_n^k(\lambda)\right) \to 0 \quad \textit{and} \quad J_{\mu,\lambda}\left(u_n^k(\lambda)\right) \to g_k(\lambda), \quad \textit{as } n \to \infty.$$

Lemma 2.3 ([25], Lemma 3.8). *If $1 \leqslant s < 2^*$ then we have that*

$$\eta_k := \sup_{u \in Z_k, \|u\|_\mu = 1} |u|_s \to 0, \quad \tau_k := \sup_{u \in Z_k, \|u\|_\mu = 1} |\nabla u|_p \to 0 \quad \textit{as} \quad k \to \infty.$$

Proof. It is clear that $0 < \eta_{k+1} \leqslant \eta_k$, so $\eta_k \to \eta \geqslant 0 (k \to \infty)$. For every $k \in \mathbb{N}$ (by the definition of η_k), there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and

$$|u_k|_s > \frac{\eta}{2} > 0. \tag{2.9}$$

For any $v = \sum_{i=1}^{\infty} v_i e_i$, we have, by the Cauchy-Schwartz inequality,

$$|\langle u_k, v\rangle| = \left|\left\langle u_k, \sum_{i=1}^{\infty} v_i e_i \right\rangle\right| = \left|\left\langle u_k, \sum_{i=k+1}^{\infty} v_i e_i \right\rangle\right| \leqslant \|u_k\| \left\|\sum_{i=k+1}^{\infty} v_i e_i\right\| = \left(\sum_{i=k+1}^{\infty} v_i^2\right)^{1/2} \to 0,$$

as $k \to \infty$, which implies that $u_k \rightharpoonup 0$ in E_μ. The compact embedding of $E_\mu \hookrightarrow L^s(\Omega)$ implies that

$$u_k \to 0 \quad \text{in } L^s(\Omega).$$

Hence, letting $k \to \infty$ in (2.9), we get $\eta = 0$, which completes the proof. □

3. Main Result

In this section, we employ the variant Fountain Theorem 2.2 and variational method to establish the existence of a sequence of infinitely many solutions whose energy converges to infinity.

Here is the main result of this work.

Theorem 3.1. *Suppose that* (V_1)-(V_2), (f_1), (F_1)-(F_3) *hold,* $2 < p < \frac{2N}{N-2}$, *and* $F(x,-u) = F(x,u)$ *for all* $(x,u) \in \Omega \times \mathbb{R}$, *then, for* $\mu > \mu_0$ *and* $\kappa \in (0, \kappa_0)$, (1.1) *possesses infinitely many high energy solutions* $u_\mu^{(k)} \in E_\mu$ *for any* $k \in \mathbb{N}$, *that is,*

$$\frac{1}{2}\|u_\mu^{(k)}\|_\mu^2 + \frac{\kappa}{p}\int_\Omega |\nabla u_\mu^{(k)}|^p \, dx - \int_\Omega F(x, u_\mu^{(k)}) \, dx - \frac{1}{q}\int_\Omega \xi(x)|u_\mu^{(k)}|^q \, dx \to \infty, \quad \textit{as } k \to \infty.$$

Lemma 3.2. *Under the assumptions* (V_1)-(V_2), (f_1), (F_1)-(F_2) *, for* $\mu \geqslant \mu_0$ *and* $\kappa \in (0, \kappa_0)$, *there exist* $r_k > 0$ *such that*

$$\inf_{u \in Z_k, \|u\| = r_k} J_{\mu,\lambda}(u) > 0, \quad \forall \lambda \in [1,2],$$

where $\mu_0 = \max\{1, \bar{\mu}\}$.

Proof. Using (F_1) and (F_2), it follows that for any $\varepsilon > 0$, there exists a positive constant c_ε that depends on ε such that

$$F(x,u) \leqslant \varepsilon |u|^2 + c_\varepsilon |u|^r \ \text{ for all } (x,u) \in \bar{\Omega} \times \mathbb{R}. \tag{3.1}$$

From (2.6), (3.1), and the Hölder inequality, for any $\mu \geqslant \mu_0, u \in Z_k$, one has

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{2}\|u\|_\mu^2 + \frac{\kappa}{p}\int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega F(x,u) \, dx - \frac{\lambda}{q}\int_\Omega \xi(x)|u|^q \, dx \\ &\geqslant \frac{1}{2}\|u\|_\mu^2 - \lambda\gamma_2\varepsilon\|u\|_\mu^2 - \lambda c_\varepsilon |u|_r^r - \lambda|\xi|_{\frac{2}{2-q}}|u|_2^q \\ &\geqslant \left(\frac{1}{2} - \lambda\gamma_2\varepsilon\right)\|u\|_\mu^2 - \lambda c_\varepsilon \eta_k^r \|u\|_\mu^r - \lambda|\xi|_{\frac{2}{2-q}}\eta_k^q\|u\|_\mu^q. \end{aligned}$$

Let

$$M' = 2\eta_k^q\left(c_\varepsilon \eta_k^{r-q} + |\xi|_{\frac{2}{2-q}}\right).$$

Set $r_k = (6M')^{\frac{1}{2-r}}$, then

$$r_k \to \infty, \text{ as } k \to \infty, \text{ for } r \in (2, 2_*),$$

which implies there exists a positive constant $k_0 \in \mathbb{N}$ such that $r_k = (6M')^{\frac{1}{2-r}} > 1$, for $k \geqslant k_0, k \in \mathbb{N}$. Take $\|u\|_\mu = r_k$, $u \in Z_k$. Then, for $\lambda \in [1,2], k \geqslant k_0, k \in \mathbb{N}$, we have that

$$\begin{aligned} \inf_{u \in Z_k, \|u\| = r_k} J_{\mu,\lambda}(u) &\geqslant \frac{1}{3}\|u\|_\mu^2 - 2c_\varepsilon \eta_k^r\|u\|_\mu^r - 2|\xi|_{\frac{2}{2-q}}\eta_k^q\|u\|_\mu^q \\ &\geqslant \frac{1}{3}\|u\|_\mu^2 - M'\|u\|_\mu^r = \frac{1}{6}r_k^2 = \frac{1}{6}(6M')^{\frac{2}{2-r}} > 0. \end{aligned} \tag{3.2}$$

□

Lemma 3.3. *Assume that conditions of Theorem* 3.1 *hold. Then, for* $\mu \geqslant \mu_0$ *and* $\kappa \in (0, \kappa_0)$, *there exist* $\rho_k > r_k > 0$ *such that*

$$\max_{u \in Y_k, \|u\| = \rho_k} J_{\mu,\lambda}(u) < 0, \quad \forall \lambda \in [1,2].$$

Proof. By condition (F_3), for any $n \in \mathbb{N}$, there exists $G_n > 0$ such that

$$F(x,t) \geqslant n|t|^p, \quad \text{for any } |t| > G_n \text{ and a.e. } x \in \Omega.$$

Denote $F_n := \max_{|t|\leqslant G_n, x\in\bar{\Omega}} |F(x,t)| + nG_n^p$, by direct calculation, we see that

$$F(x,t) \geqslant n|t|^p - F_n, \quad \text{for any } t \in \mathbb{R}, \text{ a.e. } x \in \Omega. \tag{3.3}$$

From (2.6), (3.3), and the equivalence of the norms in the finite dimensional space Y_k, for $\lambda \in [1,2]$, $u \in Y_k$, we obtain

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{2}\|u\|_\mu^2 + \frac{\kappa}{p}\int_\Omega |\nabla u|^p - \lambda\int_\Omega F(x,u)\,dx - \frac{\lambda}{q}\int_\Omega \xi(x)|u|^q\,dx \\ &\leqslant \frac{1}{2}\|u\|_\mu^2 + \frac{\kappa C_3^p(\beta_N^{\frac{p}{2}} + \gamma_t^{\frac{p}{t}})}{2p}\|u\|_\mu^p - nC_4^p\|u\|_\mu^p + F_n|\Omega|. \end{aligned}$$

Then, Choosing an $\kappa < \frac{nC_4^p 2p}{C_3^p(\beta_N^{\frac{p}{2}} + \gamma_t^{\frac{p}{t}})} = \kappa_0$, for some ρ_k large enough, $\|u\| = \rho_k > r_k > 0$, one has

$$\max_{u\in Y_k, \|u\|=\rho_k} J_{\mu,\lambda}(u) < 0, \quad \forall\lambda \in [1,2].$$

□

Lemma 3.4. *Assume that conditions of Theorem 3.1 hold. Then, for* $\mu \geqslant \mu_0$, $\kappa \in (0,\kappa_0)$, $\lambda_n \in [1,2]$, $\lambda_n \to 1, u(\lambda_n) \in E_\mu$ *with*

$$\sup_n \|u(\lambda_n)\| < \infty, \quad J'_{\mu,\lambda_n}(u(\lambda_n)) \to 0 \quad \textit{and} \quad J_{\mu,\lambda}(u(\lambda_n)) \to g_k(\lambda), \quad \textit{as } n \to \infty,$$

$\{u(\lambda_n)\}$ *has a convergent subsequence in* E_μ *for every* $k \in \mathbb{N}$.

Proof. Assume $u(\lambda_n) \to u$ weakly in E_μ. We can assume that there exist a subsequence $\{u(\lambda_n)\}$ and $u \in E_\mu$, such that:

$$\begin{aligned} &u(\lambda_n) \longrightarrow u \ \text{ weakly in } E_\mu, \\ &u(\lambda_n) \longrightarrow u \ \text{ strongly in } L^s(\Omega) \ \text{ for } s \in [2, 2^*), \\ &u(\lambda_n) \longrightarrow u \ \text{ a.e. in } \Omega. \end{aligned} \tag{3.4}$$

Next we prove that $u(\lambda_n) \to u$ in E_μ. We know that

$$\left\langle J'_{\mu,\lambda_n}(u(\lambda_n)) - J'_{\mu,1}(u), u(\lambda_n) - u\right\rangle \to 0, \quad n \to \infty.$$

By (2.8), we have that

$$\begin{aligned} o_n(1) &= \left\langle J'_{\mu,\lambda_n}(u(\lambda_n)) - J'_{\mu,1}(u), u(\lambda_n) - u\right\rangle \\ &\geqslant \|u(\lambda_n) - u|\,|_\mu^2 + \kappa\int_\Omega |\nabla u(\lambda_n)|^{p-2}\nabla u(\lambda_n)\cdot\nabla(u(\lambda_n) - u) \\ &\quad - \kappa\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla(u(\lambda_n) - u) - \int_\Omega |\lambda_n f(x,u(\lambda_n)) - f(x,u(x))||u(\lambda_n) - u|\,dx \\ &\quad - \int_\Omega \xi(x)\,|\lambda_n u(\lambda_n) - u|^{q-1}\,|u(\lambda_n) - u|\,\,dx. \end{aligned} \tag{3.5}$$

By employing equations (2.6) and (2.7), along with the fact that the embedding $E_\mu \hookrightarrow W^{1,p}(\Omega)$ is continuous, it can be concluded that:

$$u(\lambda_n) \to u \ \text{ weakly in } W^{1,p}(\Omega). \tag{3.6}$$

Then, from (3.4), (3.6), the Gagliardo-Nirenberg inequality, the Hölder inequality, and the boundedness of $\{u(\lambda_n)\}$, we get

$$\begin{aligned}
&\int_\Omega (|\nabla u(\lambda_n)|^{p-2}\nabla u(\lambda_n) - |\nabla u|^{p-2}\nabla u)\cdot\nabla(u(\lambda_n)-u)\,dx\\
&= \int_\Omega (|\nabla u(\lambda_n)|^{p-2} - |\nabla u|^{p-2})\nabla u(\lambda_n)\cdot\nabla(u(\lambda_n)-u)\,dx + \int_\Omega |\nabla u|^{p-2}|\nabla(u(\lambda_n)-u)|^2\,dx\\
&\geqslant \int_\Omega (|\nabla u(\lambda_n)|^{p-2} - |\nabla u|^{p-2})|\nabla u(\lambda_n)||\nabla(u(\lambda_n)-u)|\,dx\\
&\geqslant -|\nabla u(\lambda_n)|_p^{p-1}|\nabla(u(\lambda_n)-u)|_p - |\nabla u|_p^{\frac{p-2}{p}}|\nabla u(\lambda_n)|_p|\nabla(u(\lambda_n)-u)|_p\\
&\geqslant -C_7\left[|\nabla u(\lambda_n)|_p^{p-1} + |\nabla u|_p^{\frac{p-2}{p}}|\nabla u(\lambda_n)|_p\right]|\Delta(u(\lambda_n)-u)|_2^{\frac{p}{2}}|u(\lambda_n)-u|_t^{\frac{p}{2}}\\
&\geqslant -C_8\left[\left(|\Delta u(\lambda_n)|_2^2 + |\Delta u|_2^2\right)^{\frac{p}{4}}\right]|u(\lambda_n)-u|_t^{\frac{p}{2}}\\
&\geqslant -C_9\|u(\lambda_n)-u\|_t^{\frac{p}{2}} \to 0, n\to\infty,
\end{aligned} \tag{3.7}$$

where $1 < \frac{p}{p-2}, 2 < t = \frac{2p}{4-p} \leqslant \frac{2N}{N-4}$. From (f_1), (3.4), $\lambda_n \to 1$, and the Hölder inequality, we have that

$$\int_\Omega \xi(x)\,|\lambda_n u(\lambda_n) - u|^{q-1}\,|u(\lambda_n)-u|\,dx \leqslant |\xi|_{\frac{2}{2-q}}\,|u(\lambda_n)-u|_2^q \to 0, \quad n\to\infty. \tag{3.8}$$

By utilizing equations (3.1), (3.4), the inequality $\lambda_n \leqslant 2$, the Hölder inequality, and the boundedness of the sequence $u(\lambda_n)$, we obtain

$$\begin{aligned}
&\int_\Omega |[\lambda_n f(x,u(\lambda_n)) - f(x,u)](u(\lambda_n)-u)|\,dx\\
&\leqslant \int_\Omega (|2f(x,u(\lambda_n))| + |f(x,u)|)\,|u(\lambda_n)-u|\,dx\\
&\leqslant \int_\Omega C_{10}\left[(|2u(\lambda_n)| + |u|) + \left(|2u(\lambda_n)|^{r-1} + |u|^{r-1}\right)\right]|u(\lambda_n)-u|\,dx\\
&\leqslant C_{10}\,(|2u(\lambda_n)|_2 + |u|_2)\,|u(\lambda_n)-u|_2 + \left(|2u(\lambda_n)|_r^{r-1} + |u|_r^{r-1}\right)|u(\lambda_n)-u|_r \to 0, \quad n\to\infty.
\end{aligned} \tag{3.9}$$

It follows from (3.5)-(3.9) that $\|u(\lambda_n)-u\|_\mu^2 \to 0$, which implies $u(\lambda_n)\to u$ in E_μ. □

Proof of Theorem 3.1. From (f_1) and the property $F(x,-u) = F(x,u)$, it is easy to see that $J_{\mu,\lambda}(u)$ maps a bounded set into bounded sets uniformly for $\lambda\in[1,2]$. Clearly, we have

$$J_{\mu,\lambda}(-u) = J_{\mu,\lambda}(u) \ \text{ for all } (\lambda,u)\in[1,2]\times E,$$

which shows that (A_1) of Theorem 2.2 is satisfied. It follows from (F_3) that

$$B(u) \geqslant 0, \quad \text{ for } u\in E.$$

It follows from (2.3) that

$$A(u) = \frac{1}{2}\|u\|_\mu^2 + \frac{\kappa}{p}\int_\Omega |\nabla u|^p\,dx \geqslant \frac{1}{2}\|u\|^2 \to \infty, \quad \text{ as } \|u\|\to\infty,$$

which implies (A_2) of Theorem 2.2 holds. By Lemmas 3.2 and 3.3, conditions (A_2)-(A_3) of Theorem 3.1 are satisfied. Consequently, according to Theorem 2.2, for any $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ for $k \geqslant k_0, k \in \mathbb{N}$, such that:

$$\sup_n \|u_n^k(\lambda)\|_\mu < \infty, \quad J'_{\mu,\lambda}\left(u_n^k(\lambda)\right) \to 0, \quad J_{\mu,\lambda}\left(u_n^k(\lambda)\right) \to g_k(\lambda). \tag{3.10}$$

We also have $g_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} J_{\mu,\lambda}(\gamma(u_n^k(\lambda))) \geqslant e_k(\lambda)$. Let $\beta_k = \frac{1}{6}(6M')^{\frac{2}{2-r}} > 0$, then $\beta_k \to \infty$, as $k \to \infty$. For $k \geqslant k_0, k \in \mathbb{N}$, it follows from (3.2) that

$$g_k(\lambda) \geqslant e_k(\lambda) \geqslant \beta_k.$$

Therefore

$$g_k(\lambda) \in \left[\beta_k, \beta'_k\right], \tag{3.11}$$

where

$$\beta'_k = \max_{u \in B_k} J_{\mu,\lambda}(\gamma(u_n^k(\lambda))), \quad \Gamma_k = \{\gamma \in C\left(B_k, E_\mu\right) : \gamma \text{ is odd }, \gamma|_{\partial B_k} = id\} \ (k \geqslant 2),$$

with

$$B_k = \{u \in Y_k : \|u\| \leqslant \rho_k\}.$$

Choose $\lambda_m \to 1$ as $m \to \infty$, for $\lambda_m \in [1,2]$. Owing to (3.10), we can get the boundedness of $\left\{u_n^k\left(\lambda_m\right)\right\}$, which implies $\left\{u_n^k\left(\lambda_m\right)\right\}$ has a weakly convergent subsequence. Following a similar approach as the proof of Lemma 3.4, it is possible to establish that $\left\{u_n^k\left(\lambda_m\right)\right\}$ possesses a strongly convergent subsequence in E_μ as $n \to \infty$. Let us assume that $\lim_{n\to\infty} u_n^k\left(\lambda_m\right) = u^k\left(\lambda_m\right)$ for $m \in \mathbb{N}$. Subsequently, by using equations (3.10) and (3.11), it follows that for $k \geqslant k_0$, $k \in \mathbb{N}$:

$$J'_{\mu,\lambda_m}\left(u^k\left(\lambda_m\right)\right) = 0, \quad J_{\mu,\lambda_m}\left(u^k\left(\lambda_m\right)\right) \in \left[\beta_k, \beta'_k\right]. \tag{3.12}$$

Next we show $\left\{u^k\left(\lambda_m\right)\right\}$ is bounded in E_μ. We argue by contradiction. In fact, if it is not the case, then

$$\|u^k\left(\lambda_m\right)\|_\mu \to \infty, \quad \text{as } m \to \infty. \tag{3.13}$$

Consider the sequence $v_m = \dfrac{u^k(\lambda_m)}{\|u^k(\lambda_m)\|_\mu}$, which satisfies $\|v_m\|_\mu = 1$. Then v_m has a weakly convergent subsequence in E_μ. By (2.6), we can see that v_m has a strong convergent subsequence in $L^s(\Omega)$, for $s \in [2, 2^*]$. Without loss of generality, we suppose $v_m \to v$ a.e. in $L^s(\Omega)$, then $v_m(x) \to v(x)$ for a.e. $x \in \Omega$. For $q \in [1,2)$, it follows from (f_1), (3.13), and the Hölder inequality that

$$0 \leqslant \lim_{m\to\infty} \frac{\int_\Omega \xi(x)\left|u^k\left(\lambda_m\right)\right|^q dx}{\|u^k\left(\lambda_m\right)\|_\mu^2} \leqslant \lim_{m\to\infty} \frac{|\xi|_{\frac{2}{2-q}} \gamma_2^{q/2} \|u^k\left(\lambda_m\right)\|_\mu^q}{\|u^k\left(\lambda_m\right)\|_\mu^2} = 0.$$

Let $A = \{x \in \Omega : v(x) \neq 0\}$. It's easy to see that A is nonempty. For any $x \in A$, we have $|u_k(x)| \to +\infty$. Then, by (F_3) and the Fatou's Lemma, we have

$$\begin{aligned}\liminf_{m\to\infty} \int_\Omega \frac{F\left(x, u^k(\lambda_m)\right)}{\|u^k(\lambda_m)\|_\mu^p} dx &= \liminf_{m\to\infty} \int_\Omega \frac{F\left(x, u^k(\lambda_m)\right)}{|u^k(\lambda_m)|^p} |v_m(x)|^p \, dx \\ &\geqslant \liminf_{m\to\infty} \int_A \frac{F\left(x, u^k(\lambda_m)\right)}{|u^k(\lambda_m)|^p} |v_m(x)|^p \, dx = +\infty.\end{aligned}$$

For $x \in A$, $\lambda_m \in [1,2]$, combining (F_3), (3.10)-(3.13), and Fatou's lemma, we get

$$\begin{aligned}
1+\frac{\kappa C_3^p\left(\beta_N^{\frac{p}{2}}+\gamma_t^{\frac{p}{t}}\right)}{2p} &\geqslant \frac{1}{\|u^k(\lambda_m)\|_\mu^p}\left(\frac{1}{2}\|u^k(\lambda_m)\|_\mu^2-J\left(u^k(\lambda_m)\right)\right)+\frac{\kappa}{p}\frac{\left|\nabla u^k(\lambda_m)\right|_p^p}{\|u^k(\lambda_m)\|_\mu^p}\\
&=\frac{\lambda_m\left[\int\limits_\Omega F\left(x,u^k\left(\lambda_m\right)\right)\,dx+\frac{1}{q}\int\limits_\Omega \xi(x)\left|u^k\left(\lambda_m\right)\right|^q\,dx\right]}{\|u^k\left(\lambda_m\right)\|_\mu^p}\\
&\geqslant \int\limits_A |\nu_m(x)|^p\frac{F\left(x,u^k\left(\lambda_m\right)\right)}{|u^k\left(\lambda_m\right)|^p}\,dx+\frac{1}{q}\int\limits_\Omega\frac{\xi(x)\left|u^k\left(\lambda_m\right)\right|^q}{\|u^k\left(\lambda_m\right)\|_\mu^p}\,dx\to+\infty.
\end{aligned}$$

This is a contradiction. Hence $\{u^k(\lambda_m)\}$ is bounded in E_μ, which shows $\{u^k(\lambda_m)\}$ has a weakly convergent subsequence. By Lemma 3.4, we know that $\{u^k(\lambda_m)\}$ has a strongly convergent subsequence in E_μ. Suppose

$$\lim_{m\to\infty} u^k(\lambda_m)=u^k(1)=u_\mu^k\in E_\mu.$$

Then, for $k\geqslant k_0, k\in\mathbb{N}$, from (3.12) and $\beta_k\to\infty$, as $k\to\infty$, we have

$$J'_{\mu,1}\left(u_\mu^k\right)=0,\quad J_{\mu,1}\left(u_\mu^k\right)\in\left[\beta_k,\beta'_k\right]\to\infty,\quad \text{as } k\to\infty,$$

which shows u_μ^k is a nontrivial critical point of $J_{\mu,1}$. Consequently, for $k\geqslant k_0, k\in\mathbb{N}$ is arbitrary, we obtain infinitely many nontrivial critical points $u_\mu^{(k)}$ of $J_{\mu,1}$, which are also the nontrivial solutions of (1.1) with high energy, that is,

$$J_{\mu,1}(u_\mu^k)\to+\infty,\quad \text{as } k\to\infty.$$

□

Example 3.5.

$$(\Delta)^2u-\kappa\Delta_pu+\mu Vu=\frac{8\left(\sin^2 x\right)u^{\frac{5}{3}}}{9}+\frac{\ln\left(1.3+\left|\sin x^2\right|\right)}{e^{|x|}\sqrt{(1+x^2)}}|u|^{\frac{3}{2}-2}u,\quad x\in\Omega, \tag{3.14}$$

where $q=\frac{3}{2}$. Obviously, from (3.14), we have

$$f(x,u)=\frac{8\left(\sin^2 x\right)u^{\frac{5}{3}}}{9}\in\mathcal{C}\left(\Omega\times\mathbb{R},\mathbb{R}\right),$$

and

$$\xi(x)=\frac{\ln\left(1.3+\left|\sin x^2\right|\right)}{e^{|x|}\sqrt{(1+x^2)}}.$$

Then $0<\xi(x)=\frac{\ln\left(1.3+\left|\sin x^2\right|\right)}{e^{|x|}\sqrt{(1+x^2)}}\in L^2(\Omega)$, which implies (f_1) is satisfied. From (3.14), we know $F(x,u)=\frac{(\sin^2 x)u^{\frac{8}{3}}}{3}\geqslant 0$, and we have that

$$\begin{aligned}
|f(x,u)|&=\left|\frac{8\left(\sin^2 x\right)u^{5/3}}{9}\right|<\left(1+|u|^{\frac{8}{3}-1}\right),\\
\lim_{|u|\to 0}\frac{|f(x,u)|}{|u|}&=\lim_{|u|\to 0}\frac{8\left(\sin^2 x\right)u^{2/3}}{9}=0,\\
\lim_{|u|\to\infty}\frac{F(x,u)}{|u|^p}&=\lim_{|u|\to\infty}\frac{\left(\sin^2 x\right)u^{\frac{8}{3}-p}}{3}=\infty,
\end{aligned}$$

with $p < r = \frac{8}{3} \in [2, 2_*)$, $c_2 = 1$, which shows that (F_1)-(F_3) of Theorem 3.1 hold. Let

$$V(x) = \begin{cases} 0, & |x| \leqslant 1, \\ 2(|x| - 1), & 1 < |x| \leqslant 2, \\ |x|, & |x| > 2. \end{cases}$$

Thus, it becomes straightforward to verify the satisfaction of (V_1)-(V_3). Moreover, it is evident that $F(x, -u) = F(x, u)$. In view of Theorem 3.1, for $\kappa \in (0, \kappa_0)$, $\mu > \mu_0, k \geqslant k_0$, $k \in \mathbb{N}$, (3.14) has infinitely many high energy solutions $u_\mu^{(k)} \in E_\mu$.

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