J. Math. Computer Sci., 32 (2024), 122–136

Online: ISSN 2008-949X

Journal of Mathematics and Computer Science

Journal Homepage: www.isr-publications.com/jmcs

Resonant fractional order differential equation with two-dimensional kernel on the half-line

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Abstract

This paper derives existence results for a resonant fractional order differential equation with two-dimensional kernel on the half-line using coincidence degree theory. Fractional calculus of Riemann-Liouville type is adopted in the study. The results obtained are illustrated with an example.

Keywords: Half-line, coincidence degree, multipoint, Riemann-Stieltjes integral, two-dimensional kernel, resonance.

2020 MSC: 34A08, 34B15.

1. Introduction

Fractional order boundary value problems have attracted significant attention of researchers recently, because of its wide applications in sciences, engineering, and commerce. Unlike integer order derivatives, fractional derivatives are non local operators. This attribute makes it a powerful tool for modelling complex phenomena with historical attributes, as typical of; viscoelastic media, electromagnetics, acoustics, control theory, electrochemistry, finance and materials science (see [5, 6, 16, 20, 23, 27]). Useful results have been obtained by researchers, using coincidence degree theory to establish the existence of solutions of fractional order boundary value problems (BVPs) (see [7–10, 25, 26]) and the references therein. A fractional order boundary value problem is at resonance if the corresponding homogeneous problem has non-trivial solution.

Different researchers have studied resonant fractional order BVPs under different boundary conditions, intervals, and dimensions of the kernel. Some are on finite interval $[0, 1]$ with finite point or integral boundary conditions in which the $\dim \operatorname{Ker}(L) = 1$ and $1 < \alpha \leqslant 2$ (see [1, 7, 12, 15, 23, 28]).

For other studies in which the $\dim \operatorname{Ker}(L) = 2$ on finite interval $(0, 1)$ see [25, 26]. Recently, Djebali and Aoun [4] studied the following class of fractional multipoint boundary value problem at resonance with $\dim \operatorname{Ker}(L) = 1$ on $(0, +\infty)$,

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f\big(t, u(t), D_{0^+}^{\alpha-1} u(t)\big), & t \in (0, +\infty), \\ I_{0^+}^{2-\alpha} u(0) = 0, \ \ \lim_{t\to\infty} D_{0^+}^{\alpha-1} u(t) = \sum_{i=1}^{m-2} \beta_i D_{0^+}^{\alpha-1} u(\xi_i), \end{cases}$$

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doi: 10.22436/jmcs.032.02.03

Received: 2022-08-05 *Revised:* 2023-01-22 *Accepted:* 2023-05-10

where $1 < \alpha \leqslant 2,\ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < +\infty,\ \beta_i > 0,\ \ i = 1, 2, \ldots, m-2$, with

$$\sum_{i=1}^{m-1} \beta_i = 1.$$

They established existence of solution using coincidence degree argument.

The existence of solutions for resonant fractional order boundary value problems with $\dim\ker L = 2$ on the half-line with multipoint and Riemann-Stieltjes integral boundary conditions have not been widely reported in the literature. Motivated by this, this study focuses on investigating the existence of solution for the following resonant fractional order boundary value problem with $\dim\ker L = 2$, using coincidence degree theory:

$$D_{0^+}^{\alpha} x(t) = f\Big(t, x(t), D_{0^+}^{\alpha-2} x(t), D_{0^+}^{\alpha-1} x(t)\Big), \quad t \in (0, +\infty), \tag{1.1}$$

subject to:

$$x(0) = 0,\ \ D_{0^+}^{\alpha-2} x(0) = \int_0^1 D_{0^+}^{\alpha-2} x(t) dA(t), \quad D_{0^+}^{\alpha-1} x(+\infty) = \sum_{i=1}^{q} \kappa_i D_{0^+}^{\alpha-1} x(\xi_i), \tag{1.2}$$

where $2 < \alpha \leqslant 3,\ \dim\ker L = 2,\ \kappa_i \in \mathbb{R}, i = 1, 2, \ldots, q,\ 0 < \xi_1 < \xi_2 < \xi_3 < \cdots < \xi_q < \infty$, $A(t)$ is a function of bounded variation on $(0, +\infty)$, and $f : (0, +\infty) \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies Caratheodory's conditions namely:

(i) $f(u, g, l)$ is Lebesgue measurable in t for all $(u,g,l) \in \mathbb{R}^3$; and
(ii) f is continuous in (u, g, l) for a.e. $t \in (0, +\infty)$.

Throughout this study, the following assumptions are made.

(H_1) $\sum_{i=1}^{q} \kappa_i = 1,\ \sum_{i=1}^{q} \kappa_i \xi_i^{-1} = 0,\ \int_0^1 dA(t) = 1$, and $\int_0^1 t dA(t) = 0$.

(H_2) $\Delta = \Big(\int_0^1 (e^{-t} + t - 1) dA(t)\Big)\Big(\sum_{i=1}^{q} \kappa_i (\xi_i + 1) e^{-\xi_i}\Big) - \Big(\sum_{i=1}^{q} \kappa_i e^{-\xi_i}\Big)\Big(\int_0^1 ((t+1)e^{-t} + t - 1) dA(t)\Big) \neq 0$.

(H_3) There exist nonnegative functions $d_1(t), d_2(t), d_3(t), d_4(t) \in L^1(0, +\infty)$ such that for all $t \in (0, +\infty)$ and $(u,\ g,\ l) \in \mathbb{R}^3$,

$$|f(t, u, g, l)| \leqslant d_1(t) e^{-mt}|u| + d_2(t) e^{-mt}|g| + d_3(t)|l| + d_4(t),\ m > 0,$$

$\zeta := \|d_1\|_{L^1} + \|d_2\|_{L^1} + \|d_3\|_{L^1}$, and $\|d_i\|_{L^1} = \int_0^\infty |d_i| dt, i = 1, 2, 3$.

(H_4) There exists a non-negative constant M_1 such that, if one of the following properties hold:
  (i) $|D_{0^+}^{\alpha-2} x(t)| > M_1$, for $t \in (0, T]$;
  (ii) $|D_{0^+}^{\alpha-1} x(t)| > M_1$, for all $t \in (T, +\infty)$,

then either $Q_1 N x(t) \neq 0$ or $Q_2 N x(t) \neq 0$, for any $x \in \mathrm{dom} L \setminus \mathrm{Ker} L$.

(H_5) There exists a constant $M > 0$ such that, for any $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2}$, $b_1,\ b_2 \in \mathbb{R}$, satisfying $|b_1|$ or $|b_2| > M$, then either

$$b_1 Q_1 N x(t) + b_2 Q_2 N x(t) < 0 \tag{1.3}$$

or

$$b_1 Q_1 N x(t) + b_2 Q_2 N x(t) > 0, \tag{1.4}$$

then, BVP (1.1)-(1.2) has at least one solution provided $W\zeta < 1$, where $W = \Big(\frac{1}{\Gamma(\alpha)} + \frac{T}{\Gamma(\alpha-1)} + T + 2\Big)$.

The rest of the paper is organized as follows. Section 2 presents some lemmas and definitions which are relevant to the study. Section 3 focuses on the main existence results. Section 4 illustrates the results with an example while Section 5 focuses on conclusion.

2. Preliminaries

In this section, we present some results from fractional calculus of Riemann-Liouville type, Mawhin's coincidence degree theory, and some definitions, lemmas, and theorems that will be useful in the research work.

Definition 2.1 ([4]). The *Riemann-Liouville fractional derivative* of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{d^n}{dt^n} I_{0^+}^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} f(s) ds,$$

where $n = [\alpha] + 1$, provided that the right-hand side integral is point-wise defined on $(0, +\infty)$.

Definition 2.2 ([4]). The *Riemann-Liouville fractional integral* of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds$$

provided the right-hand side integral is point-wise defined on $(0, +\infty)$.

Lemma 2.3 ([14]). *If* $\alpha > 0$ *and* f, $D_{0^+}^{\alpha} f \in L^1(0,1)$, *then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} f(t) = f(t) + b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + \cdots + b_n t^{\alpha-n},$$

where $n = [\alpha] + 1$, $b_i \in \mathbb{R}$ $(i = 1, 2, \cdots, n)$ *are arbitrary constants.*

Lemma 2.4 ([4]). *Given that* $\alpha > 0$, $n \in \mathbb{N}$ *and* $D = \frac{d}{dx}$, *if the fractional derivatives* $(D_{0^+}^{\alpha} f)(t)$ *and* $(D_{0^+}^{\alpha+n} f)(t)$ *exist, then*

$$(D^n D_{0^+}^{\alpha} f)(t) = (D_{0^+}^{\alpha+n} f)(t).$$

Lemma 2.5 ([3]). *Given that* $\alpha > \beta > 0$, *suppose that* $f(t) \in L^1(0,1)$, *then*

$$I_{0^+}^{\alpha} I_{0^+}^{\beta} f(t) = I_{0^+}^{\alpha+\beta} f(t), \quad D_{0^+}^{\beta} I_{0^+}^{\alpha} f(t) = I_{0^+}^{-\beta} I_{0^+}^{\alpha} f(t) = I_{0^+}^{\alpha-\beta} f(t).$$

In particular,

$$D_{0^+}^{\alpha} I_{0^+}^{\alpha} f(t) = I_{0^+}^{-\alpha} I_{0^+}^{\alpha} f(t) = f(t).$$

Lemma 2.6 ([4]). *Suppose that* $\alpha > 0$, $\mu > -1$, $t > 0$, *then*

$$I_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \quad D_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha}.$$

In particular, $D_{0^+}^{\alpha} t^{\alpha-m} = 0$, *for* $m = 1, 2, 3, \ldots, n$, *where* $n = [\alpha] + 1$.

Definition 2.7 ([14]). Let $n \in \mathbb{R}+$ and $m = [n]$. The operator

$$D_{0^+}^{n} f = D_{0^+}^{m} I_{0^+}^{m-n} f$$

is called Riemann-Liouville fractional differential operator of order n. If $n = 0$, then $D_{0^+}^{0} = I$, the identity operator.

Lemma 2.8 ([3]). *Let* $n \in \mathbb{R}+$ *and* $m \in \mathbb{N}$ *such that* $m > n$. *Then*

$$D_{0^+}^{n} = D_{0^+}^{m} I_{0^+}^{m-n}.$$

Let $(G, \|\cdot\|)$ and $(H, \|\cdot\|)$ be real Banach spaces. Suppose $G = \{x(t) : x(t), D_{0+}^{\alpha-2}x(t),\ D_{0+}^{\alpha-1}x(t)\} \in C(0, +\infty)$ such that $\lim_{t\to\infty} e^{-mt}|x(t)|$, $\lim_{t\to\infty} e^{-mt}|D_{+0}^{\alpha-2}x(t)|$, and $\lim_{t\to\infty} |D_{+0}^{\alpha-1}x(t)|$ exist, for $m > 0$, with the norm

$$\|x(t)\|_G = \max\{\|x(t)\|_\infty, \|D_{0+}^{\alpha-2}x(t)\|_\infty, \|D_{0+}^{\alpha-1}x(t)\|_\infty\},$$

where

$$\|x(t)\|_\infty = \sup_{t\geqslant 0} e^{-mt}|x(t)|,\ \|D_{0+}^{\alpha-2}x(t)\|_\infty = \sup_{t\geqslant 0} e^{-mt}|D_{+0}^{\alpha-2}x(t)|,$$

and

$$\|D_{0+}^{\alpha-1}x(t)\|_\infty = \sup_{t\geqslant 0} |D_{+0}^{\alpha-1}x(t)|.$$

Let $H = L^1(0, +\infty)$ be the space of Lebesgue integrable functions with the norm

$$\|h\|_1 = \int_0^{+\infty} |h(t)| dt.$$

Define a linear operator $L : \mathrm{dom}L \subset G \to H$ such that,

$$Lx(t) = D_{0+}^{\alpha}x(t),$$

where

$$\mathrm{dom}L = x \in G : \begin{cases} x(0) = 0,\ \ D_{0+}^{\alpha-2}x(0) = \int_0^1 D_{0+}^{\alpha-2}x(t)dA(t), \\ D_{0+}^{\alpha-1}x(+\infty) = \sum_{i=1}^{q} \kappa_i D_{0+}^{\alpha-1}x(\xi_i). \end{cases}$$

Let $N : G \to H$ be a nonlinear operator such that,

$$Nx(t) = f(t, x(t), D_{+0}^{\alpha-2}x(t), D_{+0}^{\alpha-1}x(t)),\ x \in G.$$

Then the BVP (1.1) can be written as an operator equation

$$Lx(t) = Nx(t).$$

Definition 2.9 ([13]). A linear operator $L : \mathrm{dom}L \subset G \to H$ is called a Fredholm operator of index zero if

(i) $\mathrm{Im}L$ is a closed subset of H; and
(ii) $\dim \mathrm{Ker}L = \mathrm{codim}\ \mathrm{Im}L < +\infty$.

Let $L : \mathrm{dom}L \subset G \to H$ be a Fredholm operator, then, there exist continuous projectors $P : G \to G$ and $Q : H \to H$ such that $\mathrm{Im}\,P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L$, $G = \mathrm{Ker}L \oplus \mathrm{Ker}P$, $H = \mathrm{Im}L \oplus \mathrm{Im}Q$, and the mapping $L|_{\mathrm{dom}L\cap\ \mathrm{Ker}P} : \mathrm{dom}L \cap \mathrm{Ker}P \to \mathrm{Im}L$ is invertible. We denote the inverse of $L|_{\mathrm{dom}L\cap\mathrm{Ker}P}$ by $K_p : \mathrm{Im}L \to \mathrm{dom}L \cap \mathrm{Ker}P$ and the generalized inverse of L is denoted by $K_{P,Q} : H \to \mathrm{dom}L \cap\ \mathrm{Ker}P$ where $K_{P,Q} = K_P(I - Q)$ (see [17, 26]).

Theorem 2.10 ([18]). *Let $L : \mathrm{dom}L \subset G \to H$ be a Fredholm operator of index zero and $N : G \to H$ is L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied.*

(i) $Lx \neq \lambda Nx$ *for any* $x \in (\mathrm{dom}L \setminus \mathrm{Ker}L) \cap \partial\Omega$, $\lambda \in (0, 1)$;
(ii) $Nx \notin \mathrm{Im}L$ *for any* $x \in \mathrm{Ker}L \cap \partial\Omega$;
(iii) $\deg\left(QN\big|_{\mathrm{Ker}L}, \mathrm{Ker}L \cap \Omega, 0\right) \neq 0$,

then, the equation $Lx(t) = Nx(t)$ *has at least one solution in* $\mathrm{dom}L \cap \overline{\Omega}$.

3. Main results

Lemma 3.1. *Suppose* (H_1) *holds. Then*

(i) $\operatorname{Ker}L = \{x(t) \in \operatorname{dom}L : x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2}$ *for all* $t \in (0, +\infty),\ b_1, b_2 \in \mathbb{R}\}$;

(ii) $\operatorname{Im}L = \{h \in H : Q_1 h = 0 = Q_2 h\}$, *where* $Q_1 h = \int_0^1 \int_0^t (t-s)h(s)dsdA(t),\ Q_2 h = \sum_{i=1}^q \kappa_i \int_{\xi_i}^\infty h(s)ds$.

Proof.

(i) The homogeneous boundary value problem $D_{0+}^\alpha x(t) = 0$ has the solution $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + b_3 t^{\alpha-3}$, with $2 < \alpha \leqslant 3$. By using the initial condition $x(0) = 0 \Rightarrow b_3 = 0$, then, $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2}$. Apply the boundary condition (1.2) to $x(t)$ to obtain

$$\int_0^1 dA(t) = 1, \quad \int_0^1 t dA(t) = 0,$$

and

$$\sum_{i=1}^q \kappa_i = 1, \quad \sum_{i=1}^q \kappa_i \xi_i^{-1} = 0.$$

Thus, $\operatorname{Ker}L = \{x(t) \in \operatorname{dom}L : x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2}$ for all $t \in (0, +\infty),\ b_1, b_2 \in \mathbb{R}\}$, provided $\int_0^1 dA(t) = 1,\ \int_0^1 t dA(t) = 0,\ \sum_{i=1}^q \kappa_i = 1$, and $\sum_{i=1}^q \kappa_i \xi_i^{-1} = 0$.

(ii) Assume $h \in \operatorname{Im}L$, then there exists $x(t) \in \operatorname{dom}L$ such that

$$Lx(t) = h(t). \tag{3.1}$$

Solving equation (3.1) yields

$$x(t) = I_{0+}^\alpha h(t) + b_1 t^{\alpha-1} + b_2 t^{\alpha-2} + b_3 t^{\alpha-3}.$$

Applying the initial condition $x(0) = 0$, gives $b_3 = 0$, hence,

$$x(t) = I_{0+}^\alpha h(t) + b_1 t^{\alpha-1} + b_2 t^{\alpha-2}. \tag{3.2}$$

Applying the boundary condition (1.2) to (3.2) leads to

$$D_{0+}^{\alpha-2} x(t) = I_{0+}^2 h(t) + b_1 \Gamma(\alpha) t + + b_2 \Gamma(\alpha-1), \qquad D_{0+}^{\alpha-2} x(0) = \int_0^1 D_{0+}^{\alpha-2} x(t) dA(t).$$

Then,

$$b_2 \Gamma(\alpha-1) = \int_0^1 \int_0^t (t-s)h(s)dsdA(t) + b_2 \Gamma(\alpha-1),$$

which implies that

$$\int_0^1 \int_0^t (t-s)h(s)dsdA(t) = 0 = Q_1 h. \tag{3.3}$$

To obtain $Q_2 h$, we apply the boundary condition $D_{0+}^{\alpha-1} x(+\infty) = \sum_{i=1}^q \kappa_i D_{0+}^{\alpha-1} x(\xi_i)$ to (3.2) and simplify to get

$$\sum_{i=1}^q \kappa_i \int_{\xi_i}^\infty h(s)ds = 0 = Q_2 h. \tag{3.4}$$

□

Definition 3.2. Let $\Delta = a_{11}a_{22} - a_{12}a_{21}$, where $a_{11} = Q_1 e^{-t} = \int_0^1 (e^{-t} + t - 1) dA(t)$, $a_{12} = Q_2 e^{-t} = \sum_{i=1}^q \kappa_i e^{-\xi_i}$, $a_{21} = Q_1 t e^{-t} = \int_0^1 ((t+1)e^{-t} + t - 1) dA(t)$, $a_{22} = Q_2 t e^{-t} = \sum_{i=1}^q \kappa_i (\xi_i + 1) e^{-\xi_i}$.

Definition 3.3. Let $\psi_1, \psi_2 : H \to H$ such that $\psi_1 h = \frac{1}{\Delta}(a_{22}Q_1 h - a_{21}Q_2 h)e^{-t}$, $\psi_2 h = \frac{1}{\Delta}(a_{11}Q_2 h - a_{12}Q_1 h)e^{-t}$. It can be easily shown that: $\psi_1(\psi_1 h) = \psi_1 h$, $\psi_1(\psi_2 h) = 0$, $\psi_2(\psi_1 h) = 0$, and $\psi_2(\psi_2 h = \psi_2 h$.

Lemma 3.4. *Assume* (H_1)*, then* $L : \text{dom } L \subset G \to H$ *is a Fredholm operator of index zero.*

Proof. Let

$$Px(t) = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1}x(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)} D_{0+}^{\alpha-2}x(0)t^{\alpha-2} \text{ and } Qh = \psi_1 h + (\psi_2 h)t.$$

It is easy to show that

$$P^2x(t) = P\big(Px(t)\big) = Px(t)$$

and

$$Q^2h = Q(\psi_1 h + (\psi_2 h)t = \psi_1(Qh) + \psi_2((Qh))t = Qh.$$

Thus, P and Q are projection operators.

Next, let $h \in H$ such that $h = Qh + (h - Qh)$, then $Qh \in \text{Im}Q$ and $Q(h - Qh) = Qh - Q^2h = Qh - Qh = 0$. Similarly, $Q_1(h - Qh) = Q_2(h - Qh) = 0$. Therefore, $(h - Qh) \in \text{Im}L = \text{Ker}Q$. If $h \in \text{Im}L \cap \text{Im}Q$, then $h = Qh = 0$. Then, $H = \text{Im}Q \oplus \text{Im}L$ and $\dim \text{Ker}L = \text{codim Im}L = 2$. Thus, L is a Fredholm operator of index zero. □

Lemma 3.5. *Let* $L_p = L|_{\text{dom}L \cap \text{Ker}P} : \text{dom}L \cap \text{Ker}P \to \text{Im}L$ *and* $K_P : \text{Im}L \to \text{dom}L \cap \text{Ker}P$ *such that*

$$K_P h = I_{0+}^{\alpha} h = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds, \ h \in \text{Im}L.$$

Then, K_P *is the inverse of* L_P *and* $\|K_P h\|_G \leqslant \|h\|_{L^1}$.

Proof. To prove that $K_P = L_P^{-1}$, let $h \in \text{Im}L \subset H$, suppose $K_P h = I_{0+}^{\alpha} h$. Then, $(L_p K_P)h(t) = D_{0+}^{\alpha}(K_P h(t) = D_{0+}^{\alpha} I_{0+}^{\alpha} h(t) = h(t)$. Thus, for any $x(t) \in \text{dom}L \cap \text{Ker}P$,

$$K_P L_P x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} D_{0+}^{\alpha} x(s)ds = x(t) + b_1 t^{\alpha-1} + b_2 t^{\alpha-2}.$$

$P(K_P L_P x(t)) = 0$, since $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2} \in \text{Ker}L = \text{Im}P$. $Px(t) = x(t)$ and $K_P L_P x(t) = x(t) - Px(t)$. Thus, $x \in \text{dom}L \cap \text{Ker}P$, hence $K_P L_P x(t) = x(t)$. Hence, $K_P = (L|_{\text{dom}L \cap \text{Ker}P})^{-1} = {L_P}^{-1}$.

$$e^{-mt}|K_P h| \leqslant \sup_{t \geqslant 0} \frac{e^{-mt}}{\Gamma(\alpha)} \left| \int_0^t (t-s)^{\alpha-1} h(s)ds \right| \leqslant \frac{1}{\Gamma(\alpha)} \|h\|_{L^1},$$

$$e^{-mt}|D_{0+}^{\alpha-2} K_P h| \leqslant \sup_{t \geqslant 0} \frac{e^{-mt}}{\Gamma(2)} \left| \int_0^t (t-s)h(s)ds \right| \leqslant \|h\|_{L^1},$$

and

$$e^{-mt}|D_{0+}^{\alpha-1} K_P h| \leqslant \sup_{t \geqslant 0} \left| \int_0^t h(s)ds \right| \leqslant \|h\|_{L^1}.$$

We infer that $\|K_P h\|_G \leqslant \|h\|_{L^1}$ for any $h \in \text{Im}L$. □

Definition 3.6. $e^{-mt}x(t)$, $e^{-mt}x(t)D_{0+}^{\alpha-2}x(t)$, and $D_{0+}^{\alpha-1}x(t)$ are said to be equiconvergent at infinity, if given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $|e^{-mt_2}x(t_2) - e^{-mt_1}x(t_1| < \epsilon$, $|e^{-mt_2}D_{0+}^{\alpha-2}x(t_2) - e^{-mt_1}D_{0+}^{\alpha-2}x(t_1)| < \epsilon$, and $|D_{0+}^{\alpha-1}x(t_2) - D_{0+}^{\alpha-1}x(t_1)| < \epsilon$ for any $t_1, t_2 > \delta$ and $x \in G$.

Lemma 3.7. *Suppose that* (H_3) *holds and* $\Omega \subset G$ *is an open bounded subset such that* $\text{dom } L \cap \overline{\Omega} \neq \emptyset$*, then* N *is* L*-compact on* $\overline{\Omega}$*, where* $N : \overline{\Omega} \to H$.

Proof.

(1) We show that $QN(\overline{\Omega})$ is bounded. Since Ω is bounded in G, there exists $K > 0$ such that $\|x\|_G \leqslant K$ for any $x \in \overline{\Omega}$. By using assumption (H_3), (3.3), and (3.4) of Lemma 3.1, we get

$$\begin{aligned}
\big|Q_1Nx(t)\big| &= \Big|\int_0^1\int_0^t (t-s)Nx(s)dsdA(t)\Big| \\
&= \Big|\int_0^1\int_0^t (t-s)f\big(s,x(s),D_{0^+}^{\alpha-2}x(s),D_{0^+}^{\alpha-1}x(s)\big)dsdA(t)\Big| \\
&\leqslant \int_0^1\left(\int_0^{+\infty}\Big|d_1(s)e^{-ms}|u|+d_2(s)e^{-ms}|g|+d_3(s)|l|+d_4(s)\Big|ds\right)dA(t) \\
&\leqslant \big((\|d_1\|_{L^1},\|d_2\|_{L^1},\|d_3\|_{L^1})(\max\{\|x(t)\|_\infty,\|D_{0^+}^{\alpha-2}x(t)\|_\infty,\|D_{0^+}^{\alpha-1}x(t)\|_\infty\})+\|d_4\|_{L^1}\big)\int_0^1 dA(t) \\
&\leqslant \zeta\|x(t)\|_G+\|d_4\|_{L^1}\leqslant \zeta K+\|d_4\|_{L^1}:=K_1, \\
\big|Q_2Nx(t)\big| &= \Big|\sum_{i=1}^q \kappa_i\int_{\xi_i}^{+\infty}Nx(s)\Big|ds \\
&= \Big|\sum_{i=1}^q \kappa_i\int_{\xi_i}^{+\infty}f\big(s,x(s),D_{0^+}^{\alpha-2}x(s),D_{0^+}^{\alpha-1}x(s)\big)\Big|ds \\
&\leqslant \sum_{i=1}^q \kappa_i\int_{\xi_i}^{+\infty}\Big|d_1(s)e^{-ms}|u|+d_2(s)e^{-ms}|g|+d_3(s)|l|+d_4(s)\Big|ds \\
&\leqslant \sum_{i=1}^q \kappa_i\int_0^{+\infty}\Big|d_1(s)e^{-ms}|u|+d_2(s)e^{-ms}|g|+d_3(s)|l|+d_4(s)\Big|ds \\
&\leqslant \zeta\|x(t)\|_G+\|d_4\|_{L^1},\quad \text{where } \sum_{i=1}^q\kappa_i=1 \\
&\leqslant \zeta K+\|d_4\|_{L^1}:=K_1.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|QNx(t)\|_{L^1} &= \int_0^{+\infty}|QNx(s)|ds \\
&\leqslant \int_0^{+\infty}|\psi_1Nx(s)|ds+\int_0^{+\infty}|\psi_2Nx(s)s|ds \\
&\leqslant \int_0^{+\infty}\frac{1}{|\Delta|}(|Q_1Nx(s)a_{22}-Q_2Nx(s)a_{21})e^{-s}|ds \\
&\quad+\int_0^{+\infty}\frac{1}{|\Delta|}(|Q_2Nx(s)a_{11}-Q_1Nx(s)a_{12})se^{-s}|ds \\
&\leqslant \frac{1}{|\Delta|}\int_0^{+\infty}(|Q_1Nx(s)||a_{22}|-|Q_2Nx(s)||a_{21}|)e^{-s}ds \\
&\quad+\frac{1}{|\Delta|}\int_0^{+\infty}(|Q_2Nx(s)|a_{11}|-|Q_1Nx(s)||a_{12}|)se^{-s}ds \\
&\leqslant \frac{1}{|\Delta|}\Big(K_1|a_{22}|+K_1|a_{21}|+K_1|a_{11}|+K_1|a_{12}|\Big) \\
&\leqslant \frac{1}{|\Delta|}\Big(a_{11}|+|a_{12}|+|a_{21}|+|a_{22}|\Big)K_1:=K_2.
\end{aligned}$$

Therefore, $QN(\overline{\Omega})$ is bounded.

(2) We prove that $K_{P,Q}N(\overline{\Omega})$ is compact on $(0,+\infty)$, by establishing that $K_{P,Q}N(\overline{\Omega})$ is bounded, equicontinuous on any subcompact interval of $(0,+\infty)$ and equiconvergent at infinity.

(i) First, given any $x \in \overline{\Omega}$, $Nx = f(s, x(s), D_{0+}^{\alpha-2}x(s), D_{0+}^{\alpha-1}x(s))$ and

$$\begin{aligned}\|Nx(t)\|_{L^1} &= \int_0^{+\infty} |f(s, x(s), D_{0+}^{\alpha-2}x(s), D_{0+}^{\alpha-1}x(s))|ds\\ &\leqslant \int_0^{+\infty} \left|d_1(s)e^{-ms}|u| + d_2(s)e^{-ms}|g| + d_3(s)|l| + d_4(s)\right|ds\\ &\leqslant \zeta\|x(t)\|_G + \|d_4\|_{L^1} \leqslant \zeta K + \|d_4\|_{L^1} := K_1.\end{aligned}$$

Also,

$$\begin{aligned}e^{-mt}|K_{P,Q}Nx(t)| &= \left|\frac{e^{-mt}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(I-Q)Nx(s))ds\right|\\ &\leqslant \frac{e^{-mt}}{\Gamma(\alpha)}\int_0^t (|Nx(s)| + |QNx(s)|)ds\\ &\leqslant \frac{1}{\Gamma(\alpha)}\int_0^\infty (|Nx(s)| + |QNx(s)|)ds \leqslant \frac{1}{\Gamma(\alpha)}(\|Nx\|_{L^1} + \|QNx\|_{L^1} \leqslant \frac{1}{\Gamma(\alpha)}(K_1+K_2),\end{aligned}$$

$$e^{-mt}|D_{0+}^{\alpha-2}K_{P,Q}Nx(t)| = \left|\frac{e^{-mt}}{\Gamma(2)}\int_0^t (t-s)(I-Q)Nx(s)ds\right| \leqslant \int_0^{+\infty}(|Nx(s)| + |QNx(s)|)ds \leqslant K_1 + K_2,$$

and

$$\begin{aligned}|D_{0+}^{\alpha-1}K_{P,Q}Nx(t)| = \left|\int_0^t (I-Q)Nx(s)ds\right| &\leqslant \int_0^t |(I-Q)Nx(s)|ds\\ &\leqslant \int_0^\infty |Nx(s) + QNx(s)|ds \leqslant \|Nx(t)\|_{L^1} + \|QNx(t)\|_{L^1} \leqslant K_1 + K_2.\end{aligned}$$

We infer that, $K_{P,Q}N(\overline{\Omega})$ is bounded.

(ii) We establish that $K_{P,Q}N$ is equicontinuous on any subcompact interval of $(0,+\infty)$. Suppose $x \in \overline{\Omega}$, by the assumption (H_3),

$$|Nx(s)| = |f(s,u,g,l)| \leqslant \left|d_1(s)e^{-ms}|u| + d_2(s)e^{-ms}|g| + d_3(s)|l| + d_4(s)\right|.$$

Let $t_1, t_2 \in (0,+\infty)$ such that $t_1 < t_2$, then

$$\begin{aligned}&|e^{-mt_2}K_{P,Q}Nx(t_2) - e^{-mt_1}K_{P,Q}Nx(t_1)|\\ &\quad= \left|\frac{e^{-mt_2}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q)Nx(s)ds - \frac{e^{-mt_1}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q)Nx(s)ds\right|\\ &\quad\leqslant \frac{e^{-mt_2}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|(I-Q)Nx(s)|ds\\ &\qquad+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|e^{-mt_2}(t_2-s)^{\alpha-1} - e^{-mt_1}(t_1-s)^{\alpha-1}\right||(I-Q)Nx(s)|ds \quad \to 0 \text{ as } t_1 \to t_2,\\ &|e^{-mt_2}D_{0+}^{\alpha-2}K_{P,Q}Nx(t_2) - e^{-mt_1}D_{0+}^{\alpha-2}K_{P,Q}Nx(t_1)|\\ &\quad= \left|\frac{1}{\Gamma(2)}\Big(\int_0^{t_2} e^{-mt_2}(t_2-s)(I-Q)Nx(s)ds - \int_0^{t_1} e^{-mt_1}(t_1-s)(I-Q)Nx(s)ds\Big)\right|\\ &\quad\leqslant \int_{t_1}^{t_2}|e^{-mt_2}(t_2-s)||(I-Q)Nx(s)|ds\end{aligned}$$

$$+\int_0^{t_1}\left|e^{-mt_2}(t_2-s)-e^{-mt_1}(t_1-s)\right||(I-Q)Nx(s)|ds \quad \to 0 \text{ as } t_1\to t_2,$$

$$\begin{aligned}&|D_{0+}^{\alpha-1}K_{P,Q}Nx(t_2)-D_{0+}^{\alpha-1}K_{P,Q}Nx(t_1)|\\ &\quad=\left|\int_0^{t_2}(I-Q)Nx(s)ds-\int_0^{t_1}(I-Q)Nx(s)ds\right|\\ &\quad\leqslant\int_0^{t_2}|(I-Q)Nx(s)|ds+\int_{t_1}^{0}|(I-Q)Nx(s)|ds\leqslant\int_{t_1}^{t_2}|(I-Q)Nx(s)|ds \quad \to 0 \text{ as } t_1\to t_2.\end{aligned}$$

Therefore, $K_{P,Q}N(\overline{\Omega})$ is equicontinuous.

(iii) Next, we prove that $K_{P,Q}N(\overline{\Omega})$ is equiconvergent at infinity. Given any $x\in\overline{\Omega}$, we have $\int_0^{+\infty}|(I-Q)Nx(t)|ds\leqslant\|Nx(t)\|_{L^1}+\|QNx(t)\|_{L^1}\leqslant K_1+K_2$. Thus, given $\epsilon>0$, there exists a positive number k_0 and $\delta>0$ such that for any $t_1,t_2>\delta$ and $0\leqslant s\leqslant k_0$,

$$\int_{k_0}^{+\infty}|(I-Q)Nx(s)|ds<\epsilon,\quad \lim_{t\to\infty}e^{-mt}(t-k_0)^{\alpha-1}=0,\quad \lim_{t\to\infty}e^{-mt}(t-k_0)=0, \text{ and } \lim_{t\to\infty}e^{-mt}=0.$$

Thus,

$$\left|e^{-mt_2}(t_2-k_0)^{\alpha-1}-e^{-mt_1}(t_1-k_0)^{\alpha-1}\right|<\epsilon,\ \left|e^{-mt_2}(t_2-k_0)-e^{-mt_1}(t_1-k_0)\right|<\epsilon,$$

and $\left|e^{-mt_2}-e^{-mt_1}\right|<\epsilon$, where $m>0$. Thus for any $t_1,t_2>\delta>k_0$,

$$\begin{aligned}&\left|e^{-mt_2}K_{P,Q}Nx(t_2)-e^{-mt_1}K_{P,Q}Nx(t_1)\right|\\ &\quad=\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}e^{-mt_2}(t_2-s)^{\alpha-1}(I-Q)Nx(s)ds-\int_0^{t_1}e^{-mt_1}(t_1-s)^{\alpha-1}(I-Q)Nx(s)ds\right|\\ &\quad\leqslant\frac{1}{\Gamma(\alpha)}\int_0^{k_0}\left|e^{-mt_2}(t_2-s)^{\alpha-1}-e^{-mt_1}(t_1-s)^{\alpha-1}\right||(I-Q)Nx(s)|ds\\ &\qquad+\frac{1}{\Gamma(\alpha)}\int_{k_0}^{t_1}e^{-mt_1}(t_1-s)^{\alpha-1}|(I-Q)Nx(s)|ds+\frac{1}{\Gamma(\alpha)}\int_{k_0}^{t_2}e^{-mt_2}(t_2-s)^{\alpha-1}|(I-Q)Nx(s)|ds\\ &\quad\leqslant\frac{\epsilon}{\Gamma(\alpha)}\int_0^{k_0}|(I-Q)Nx(s)|ds+\frac{2\epsilon}{\Gamma(\alpha)}\int_{k_0}^{+\infty}|(I-Q)Nx(s)|ds<\frac{1}{\Gamma(\alpha)}(K_1+K_2)\epsilon.\end{aligned}$$

By similar argument,

$$\left|e^{-mt_2}D_{+0}^{\alpha-2}K_{P,Q}Nx(t_2)-e^{-mt_1}D_{+0}^{\alpha-2}K_{P,Q}Nx(t_1)\right|\leqslant(k_1+k)\epsilon$$

and

$$\begin{aligned}\left|D_{+0}^{\alpha-1}K_{P,Q}Nx(t_2)-D_{+0}^{\alpha-1}K_{P,Q}Nx(t_1)\right|&=\left|\int_0^{t_2}(I-Q)Nx(s)ds-\int_0^{t_1}(I-Q)Nx(s)ds\right|\\ &\leqslant\int_0^{t_2}|(I-Q)Nx(s)|ds+\int_{t_1}^{0}|(I-Q)Nx(s)|ds\\ &\leqslant\int_{t_1}^{t_2}|(I-Q)Nx(s)|ds\leqslant\int_0^{+\infty}|(I-Q)Nx(s)|ds<(K_1+K_2)\epsilon.\end{aligned}$$

Thus, $K_{P,Q}N(\overline{\Omega})$ is equiconvergent at infinity. It follows that $K_{P,Q}N(\overline{\Omega})$ is relatively compact. Hence, N is L-compact on $\overline{\Omega}$. □

Lemma 3.8. *Assume that* (H_3) *and* (H_4) *hold, then the set* $\Omega_1=\{x\in \mathrm{dom}L\setminus \mathrm{Ker}L : Lx(t)=\lambda Nx(t),\ \lambda\in(0,1)\}$ *is bounded in* G.

Proof. Suppose $x \in \Omega_1$ and $Nx \in \mathrm{Im}L = \mathrm{Ker}Q$, then

$$Q_1Nx(t) = 0 = Q_2Nx(t).$$

Thus, from the condition (H_4), there exist $t_1 \in (0,T]$ such that $|D^{\alpha-2}_{0+}x(t_1)| \leqslant M_1$, and $t_2 \in (T,+\infty)$ with $|D^{\alpha-1}_{0+}x(t_2)| \leqslant M_1$. Combinning this with Lemma 2.6, we obtain the following results

$$\begin{aligned}
|D^{\alpha-1}_{0+}x(t)| &= \left|D^{\alpha-1}_{0+}x(t_2) + \int_{t_2}^{t} D^{\alpha}_{0+}x(s)ds\right| \\
&\leqslant |D^{\alpha-1}_{0+}x(t_2)| + \int_{t_2}^{t} |D^{\alpha}_{0+}x(s)|ds \leqslant M_1 + \int_0^{+\infty} |Nx(s)|ds = M_1 + \|Nx\|_{L^1}, \\
|D^{\alpha-2}_{0+}x(0)| &= |D^{\alpha-2}_{0+}x(t_1) - \int_0^{t_1} D^{\alpha-1}_{+0}x(s)ds| \\
&\leqslant |D^{\alpha-2}_{0+}x(t_1)| + \int_0^{t_1} |D^{\alpha-1}_{+0}x(s)|ds \leqslant M_1 + \|D^{\alpha-1}_{0+}x(t)\|_\infty T \leqslant M_1(1+T) + T\|Nx(t)\|_{L^1}.
\end{aligned}$$

Using definition of P and Lemma 2.6, we obtain

$$\begin{aligned}
Px(t) &= \frac{1}{\Gamma(\alpha)}D^{\alpha-1}_{0+}x(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D^{\alpha-2}_{0+}x(0)t^{\alpha-2}, \\
\|Px(t)\|_\infty &= \sup_{t\geqslant 0} e^{-mt}\left|\frac{1}{\Gamma(\alpha)}D^{\alpha-1}_{0+}x(0)t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}D^{\alpha-2}_{0+}x(0)t^{\alpha-2}\right| \\
&\leqslant \frac{1}{\Gamma(\alpha)}|D^{\alpha-1}_{0+}x(0)| \sup_{t\geqslant 0} e^{-mt}t^{\alpha-1} + \frac{1}{\Gamma(\alpha-1)}|D^{\alpha-2}_{0+}x(0)| \sup_{t\geqslant 0} e^{-mt}t^{\alpha-2} \\
&\leqslant \frac{1}{\Gamma(\alpha)}|D^{\alpha-1}_{0+}x(0)| + \frac{1}{\Gamma(\alpha-1)}|D^{\alpha-2}_{0+}x(0)| \\
&\leqslant \frac{M_1 + \|Nx(t)\|_{L^1}}{\Gamma(\alpha)} + \frac{M_1(1+T) + T\|Nx(t)\|_{L^1}}{\Gamma(\alpha-1)}, \\
\|D^{\alpha-2}_{0+}Px(t)\|_\infty &= \sup_{t\geqslant 0} e^{-mt}|D^{\alpha-2}_{0+}Px(t)| \\
&\leqslant \sup_{t\geqslant 0} e^{-mt}\left|\frac{|D^{\alpha-1}_{0+}x(0)D^{\alpha-2}_{0+}t^{\alpha-1}}{\Gamma(\alpha)} + \frac{D^{\alpha-2}_{0+}x(0)D^{\alpha-2}_{0+}t^{\alpha-2}}{\Gamma(\alpha-1)}\right| \\
&\leqslant |D^{\alpha-1}_{0+}x(0)| \sup_{t\geqslant 0} e^{-mt}t + |D^{\alpha-2}_{0+}x(0)| \sup_{t\geqslant 0} e^{-mt} \\
&\leqslant |D^{\alpha-1}_{0+}x(0)| + |D^{\alpha-2}_{0+}x(0)| \\
&\leqslant M_1 + \|Nx(t)\|_{L^1} + M_1(1+T) + T\|Nx(t)\|_{L^1} \\
&= M_1(2+T) + (1+T)\|Nx(t)\|_{L^1}, \\
\|D^{\alpha-1}_{0+}Px(t)\|_\infty &= \sup_{t\geqslant 0} |D^{\alpha-1}_{0+}Px(t)| \\
&= \sup_{t\geqslant 0}\left|\frac{D^{\alpha-1}_{0+}x(o)D^{\alpha-1}_{0+}t^{\alpha-1}}{\Gamma(\alpha)} + \frac{D^{\alpha-2}_{0+}x(o)D^{\alpha-1}_{0+}t^{\alpha-2}}{\Gamma(\alpha-1)}\right| \\
&\leqslant |D^{\alpha-1}_{0+}x(o)| + |D^{\alpha-2}_{0+}x(o)| \sup_{t>0}\frac{1}{t} \leqslant |D^{\alpha-1}_{0+}x(o)| \leqslant M_1 + \|Nx(t)\|_{L^1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\|Px(t)\|_G &= \max\{\|Px(t)\|_\infty, \|D_{0+}^{\alpha-2}Px(t)\|_\infty, \|D_{0+}^{\alpha-1}Px(t)\|_\infty\}\\ &\leqslant \|Px(t)\|_\infty + \|D_{0+}^{\alpha-2}Px(t)\|_\infty + \|D_{0+}^{\alpha-1}Px(t)\|_\infty\\ &\leqslant \frac{1}{\Gamma(\alpha)}(M_1 + \|Nx(t)\|_{L^1}) + \frac{1}{\Gamma\alpha-1}(M_1(1+T) + T\|Nx(t)\|_{L^1}) + M_1(2+T)\\ &\quad + (1+T)\|Nx(t)\|_{L^1} + M_1 + \|Nx(t)\|_{L^1}.\end{aligned} \tag{3.5}$$

It is observed that $(I-P)x \in \operatorname{dom}L \cap \operatorname{Ker}P$ and $LPx = 0$. From the definition, the operator $K_P : \operatorname{Im}L \to \operatorname{dom}L \cap \operatorname{Ker}P$ is such that for any $h \in \operatorname{Im}L$, $K_P h = I_{0+}^{\alpha} h$. Therefore,

$$\|(I-P)x\|_G = \|K_P L(I-P)x\|_G \leqslant \|L(I-P)x\|_{L^1} = \|Lx\|_{L^1} \leqslant \|Nx\|_{L^1}. \tag{3.6}$$

From (3.5) and (3.6), we obtain

$$\|x\|_G = \|Px\|_G + \|(I-P)x\|_G \leqslant \left(\frac{1}{\Gamma(\alpha)} + \frac{1+T}{\Gamma(\alpha-1)} + T + 3\right)M_1 + \left(\frac{1}{\Gamma(\alpha)} + \frac{T}{\Gamma(\alpha-1)} + T + 2\right)\|Nx\|_{L^1}.$$

Let $\left(\frac{1}{\Gamma(\alpha)} + \frac{1+T}{\Gamma(\alpha-1)} + T + 3\right) = V$ and $\left(\frac{1}{\Gamma(\alpha)} + \frac{T}{\Gamma(\alpha-1)} + T + 2\right) = W$, then,

$$\begin{aligned}\|x\|_G &= VM_1 + W\|Nx(t)\|_{L^1} \leqslant VM_1 + W(\zeta\|x\|_G + \|d_4\|_{L^1}),\\ (1-W\zeta)\|x\|_G &\leqslant VM_1 + W\|d_4\|_{L^1},\\ \|x\|_G &\leqslant \frac{VM_1 + W\|d_4\|_{L^1}}{1-W\zeta}.\end{aligned}$$

Thus, Ω_1 is bounded, provided $W\zeta < 1$. □

Lemma 3.9. *Suppose that* (H_4)) *and* (H_5) *hold, then the set* $\Omega_2 = \{x \in \operatorname{Ker}L : Nx \in \operatorname{Im}L\}$ *is bounded in* G.

Proof. Let $x \in \Omega_2$, where $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2}$, $b_1,\ b_2 \in \mathbb{R}$, and $Q_1Nx(t) = 0 = Q_2Nx(t)$. By condition (H_5), it follows that,

$$\begin{aligned}\sup_{t\geqslant 0} e^{-mt}|x(t)| &= \sup_{t\geqslant 0}\frac{|b_1 t^{\alpha-1} + b_2 t^{\alpha-2}|}{e^{mt}} \leqslant |b_1|\sup_{t\geqslant 0}\frac{t^{\alpha-1}}{e^{mt}} + |b_2|\sup_{t\geqslant 0}\frac{t^{\alpha-2}}{e^{mt}} \leqslant |b_1| + |b_2| \leqslant 2M,\\ \sup_{t\geqslant 0}|D_{+0}^{\alpha-1}x(t)| &= \sup_{t\geqslant 0}|b_1 D_{+0}^{\alpha-1}t^{\alpha-1} + b_2 D_{+0}^{\alpha-1}t^{\alpha-2}|\\ &= |b_1\Gamma(\alpha)| + |b_2\Gamma(\alpha-1)|\sup_{t>0}\frac{1}{t} \leqslant |b_1\Gamma(\alpha)| \leqslant M\Gamma(\alpha),\\ \sup_{t\geqslant 0} e^{-mt}|D_{+0}^{\alpha-2}x(t)| &= \sup_{t\geqslant 0} e^{-mt}|b_1 D_{+0}^{\alpha-2}t^{\alpha-1} + b_2 D_{+0}^{\alpha-2}t^{\alpha-2}|\\ &= \sup_{t\geqslant 0} e^{-mt}|b_1\Gamma(\alpha)t + b_2\Gamma(\alpha-1)|\\ &\leqslant |b_1|\Gamma(\alpha)\sup_{t\geqslant 0}\frac{t}{e^{mt}} + |b_2|\Gamma(\alpha-1) \leqslant |b_1|\Gamma(\alpha) + |b_2|\Gamma(\alpha-1 \leqslant \big(\Gamma(\alpha) + \Gamma(\alpha-1)\big)M.\end{aligned}$$

Thus, we conclude that Ω_2 is also bounded in G. □

Lemma 3.10. *Suppose that the condition* (H_5) *holds, then the set*

$$\Omega_3 = \{x \in \operatorname{Ker}L : \nu\lambda Jx(t) + (1-\lambda)QNx(t) = 0,\ \lambda \in [0,1]\}$$

is bounded in G, *where* $\nu = \begin{cases} -1, & \text{if (1.3) holds,}\\ +1, & \text{if (1.4) holds,}\end{cases}$ *and* $J : \operatorname{Ker}L \to \operatorname{Im}Q$ *is a linear isomorphism defined by*

$$J(b_1 t^{\alpha-1} + b_2 t^{\alpha-2}) = \frac{1}{\Delta}(a_{22}|b_1| - a_{21}|b_2|)e^{-t} + \frac{1}{\Delta}(-a_{12}|b_1| + a_{11}|b_2|)te^{-t}, b_1, b_2 \in \mathbb{R}.$$

Proof. If (H_5) holds, $x \in \Omega_3$ is such that $x(t) = b_1 t^{\alpha-1} + b_2 t^{\alpha-2}$, with $b_1, b_2 \in \mathbb{R}$. If $\nu = -1$, $\lambda Jx(t) = (1-\lambda)QNx(t)$, $\lambda \in [0,1]$. By the condition (H_5), we show that there exists a positive number M such that $|b_1| \leqslant M$ and $|b_2| \leqslant M$. If $\lambda = 0$, then $QNx(t) = 0$, which implies that $Q_1Nx(t) = 0 = Q_2Nx(t)$. If $\lambda = 1$, then $Jx(t) = 0$, which implies that $b_1 = 0 = b_2$. Thus, $b_1 \leqslant M$ and $b_2 \leqslant M$. For $\lambda \in (0,1)$, with $\lambda Jx(t) = (1-\lambda)QNx(t)$, we have

$$\begin{cases} \lambda(a_{22}b_1 - a_{12}b_2 = (1-\lambda)(a_{22}Q_1Nx(t) - a_{12}Q_2Nx(t)), \\ \lambda(a_{11}b_2 - a_{21}b_1 = (1-\lambda)(a_{11}Q_2Nx(t) - a_{21}Q_1Nx(t)). \end{cases}$$

Since $\Delta \neq 0$, we have

$$\begin{cases} \lambda|b_1| = |(1-\lambda)Q_1Nx(t)|, \\ \lambda|b_2| = |(1-\lambda)Q_2Nx(t)|. \end{cases}$$

Assuming $|b_1| > M$ or $|b_2| > M$, then

$$\begin{cases} \lambda {b_1}^2 = (1-\lambda)b_1Q_1Nx(t), \\ \lambda {b_2}^2 = (1-\lambda)b_2Q_2Nx(t), \end{cases} \qquad 0 < \lambda({b_1}^2 + {b_2}^2) = (1-\lambda)\Big(b_1Q_1Nx(t) + b_2Q_2Nx(t)\Big) < 0$$

leads to contradiction by (1.3). Hence,

$$|b_1| \leqslant M, \quad |b_2| \leqslant M.$$

Thus, Ω_3 is bounded. If (1.4) holds, by similar argument we can see that Ω_3 is bounded. □

Theorem 3.11. *Suppose that* (H_1)-(H_5) *hold, then, the BVP* (1.1)-(1.2) *has at least one solution in* G *provided* $W\zeta < 1$, *where* $W = \left(\frac{1}{\Gamma(\alpha)} + \frac{T}{\Gamma(\alpha-1)} + T + 2\right)$ *and* $\zeta := \|d_1\|_{L^1} + \|d_2\|_{L^1} + \|d_3\|_{L^1}$.

Proof. Let Ω be a bounded open subset of G such that $\cup_{i=1}^{3}\Omega_i \subset \Omega$, $i = 1,2,3$. We have shown that N is L-compact on $\overline{\Omega}$ by Lemma 3.8. Applying Lemmas 3.9 and 3.10, we have established that

(i) $Lx(t) \neq \lambda Nx(t)$ for any $x \in (\operatorname{dom}L|_{\operatorname{Ker}L}) \cap \partial\Omega$, $\lambda \in (0,1)$;

(ii) $Nx(t) \notin \operatorname{Im}L$ for any $x \in \operatorname{Ker}L \cap \partial\Omega$, where $\partial\Omega$ is the boundary of Ω.

Lastly, we prove that $\deg\{QN|_{\operatorname{Ker}L}, \Omega \cap \operatorname{Ker}L, 0\} \neq 0$. Define

$$H(x,\lambda) = \nu\lambda Jx(t) + (1-\lambda)QNx(t).$$

By Lemma 3.9, we infer that $H(x,\lambda) \neq 0$ for any $x \in \operatorname{Ker}L \cap \Omega$, $\lambda \in [0,1]$. Thus, by the homotopy of degree, we have the

$$\begin{aligned} \deg\{QN|_{\operatorname{Ker}L}, \Omega \cap \operatorname{Ker}L, 0\} &= \deg\{H(\cdot,0), \Omega \cap \operatorname{Ker}L, 0\} \\ &= \deg\{H(\cdot,1), \Omega \cap \operatorname{Ker}L, 0\} = \deg\{\pm J, \Omega \cap \operatorname{Ker}L, 0\} \neq 0. \end{aligned}$$

Accordingly, it follows that the BVP (1.1)-(1.2) has at least one solution in G. □

4. Example

Consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{5}{2}}x(t) = f\Big(t, x(t), D_{0+}^{\frac{1}{2}}x(t), D_{0+}^{\frac{3}{2}}x(t)\Big), \quad t \in (0,+\infty), \\ x(0) = 0, \ D_{0+}^{\frac{1}{2}}x(0) = \int_0^1 D_{0+}^{\frac{1}{2}}x(t)dA(t), \\ D_{0+}^{\frac{3}{2}}x(+\infty) = -\frac{1}{2}D_{0+}^{\frac{3}{2}}x(\frac{1}{4}) + \frac{3}{2}D_{0+}^{\frac{3}{2}}x(\frac{3}{4}). \end{cases} \tag{4.1}$$

$A(t) = 4t - 3t^2$ and

$$f\Big(t, x(t), D^{\frac{1}{2}}_{0+}x(t), D^{\frac{3}{2}}_{0+}x(t)\Big) = \begin{cases} \frac{e^{-2t}}{15}\cos x(t) + \frac{e^{-3t}}{10}\sin(D^{\frac{1}{2}}_{0+}x(t)), & t \in (0,1], \\ \frac{e^{-5t}}{25}\cos(D^{\frac{3}{2}}_{0+}x(t)), & t \in (1, +\infty). \end{cases}$$

Corresponding to BVP (1.1)-(1.2), $\alpha = \frac{5}{2}, m = 2, \kappa_1 = -\frac{1}{2}, \kappa_2 = \frac{3}{2}, \xi_1 = \frac{1}{4}, \xi_2 = \frac{3}{4}$.

For the existence of at least one solution for the BVP, we check if the assumptions (H_1)-(H_5) are satisfied.

(H_1): We see that

$$\sum_{i=1}^{2}\kappa_i = 1, \quad \sum_{i=1}^{2}\kappa_i\xi_i^{-1} = 0, \int_0^1 dA(t) = 1, \text{ and } \int_0^1 t\,dA(t) = 0.$$

(H_2): Determinant $\Delta = a_{11}a_{22} - a_{12}a_{21} \neq 0$, where $a_{11} = -0.5570$, $a_{12} = 0.3191$, $a_{21} = 0.0364$, and $a_{22} = 0.7532$,

$$\Delta = (-0.5570)(0.7532) - (0.3191)(0.0364) = -0.4311 \neq 0.$$

So (H_2) holds.

(H_3): We show that $W\zeta < 1$, where $W = \Big(\frac{1}{\Gamma(\frac{5}{2})} + \frac{1}{\Gamma(\frac{3}{2})} + 3\Big) = 4.8807$ and $d_1(t) = \frac{e^{-2t}}{15}$, $d_2(t) = \frac{e^{-3t}}{10}$, $d_3(t) = \frac{e^{-5t}}{25}$, $d_4(t) = 0$,

$$\zeta = \|d_1\|_{L^1} + \|d_2\|_{L^1} + \|d_3\|_{L^1} = 0.0746.$$

Therefore

$$W\zeta = 4.8807 \times 0.0746 = 0.3641 < 1.$$

Hence, (H_3) holds.

(H_4): Set $M_1 = 5$, if $D^{\frac{1}{2}}_{+0}x(s) > M_1$, then

$$Q_1Nx(t) > \int_0^1\int_0^t (t-s)(\frac{e^{-3s}}{10}M_1 - \frac{e^{-2s}}{15})ds\,dA(t) > -0.003M_1 + 0.0027 \neq 0.$$

Hence, $Q_1Nx(t) \neq 0$. Also,

$$\begin{aligned} Q_2Nx(t) &> \int_1^{+\infty}\frac{e^{-5s}}{25}M_1 + \frac{1}{2}\int_0^{\frac{1}{4}}\Big(\frac{e^{-3s}}{10}M_1 - \frac{e^{-2s}}{15}\Big)ds - \frac{3}{2}\int_0^{\frac{3}{4}}\Big(\frac{e^{-2s}}{15} - \frac{e^{-3s}}{10}M_1\Big)ds \\ &> 0.05358M_1 - 0.0454 \neq 0. \end{aligned}$$

So, $Q_2Nx(t) \neq 0$. Thus, (H_4) holds.

(H_5): Take $M = 10$, if $|b_1|$ or $|b_2| > M$,

$$\begin{aligned} b_1Q_1Nx(t) + b_2Q_2Nx(t) &= b_1\int_0^1\int_0^t (t-s)\left(\frac{e^{-3s}}{10}(b_1\Gamma(\frac{5}{2})s + b_2\Gamma(\frac{3}{2})) + \frac{e^{-2s}}{15}\right)ds\,dA(t) \\ &\quad + b_1b_2\Gamma(\frac{5}{2})\int_1^{\infty}\frac{e^{-5s}}{25}ds + \frac{b_2}{2}\int_0^{\frac{1}{4}}\left(\frac{e^{-3s}}{10}(b_1\Gamma(\frac{5}{2})s + b_2\Gamma(\frac{3}{2})) + \frac{e^{-2s}}{15}\right)ds \\ &\quad - \frac{3b_2}{2}\int_0^{\frac{3}{4}}\left(\frac{e^{-3s}}{10}\left(b_1\Gamma(\frac{5}{2})s + b_2\Gamma(\frac{3}{2})\right) + \frac{e^{-2s}}{15}\right)ds \\ &= 0.0087b_1^2 + 0.0111b_2^2 - 0.0027b_1 + 0.0322b_2 + 0.01583b_1b_2 \\ &> 0.00397M^2 - 0.0349M > 0. \end{aligned}$$

Thus, (H_5) holds. Since the assumptions (H_1)-(H_5) of Theorem 3.11 are satisfied, the boundary value problem (4.1) has at least one solution.

5. Conclusion

The study has shown the existence of solutions for the resonant fractional order boundary value problem with the $\dim \text{KerL} = 2$ on half-line using coincidence degree theory. An example was given to validate the results. It is anticipated that the outcome of the research will further expand the knowledge of fractional order BVP and enrich the existing literature. Further work can still be carried out on this problem by using a nonlinear p-Laplacian operator.

Acknowledgment

The authors would like to show their gratitude to Covenant University Centre for Research, Innovation and Discovery (CUCRID) for the support received and the reviewers for their useful suggestions.

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