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Computational coupled fixed points for $\mathbb{F}$-contractive mappings in metric spaces endowed with a graph

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Abstract

The purpose of this work is to present some existence theorems for coupled fixed points of F-type contractive operator in metric spaces endowed with a directed graph. Our results generalize the main result obtained by Chifu and Petrusel [C. Chifu, G. Petrusel, Fixed Point Theory Appl., **2014** (2014), 13 pages]. We also present applications to some nonlinear integral system equations to support the results.

Keywords: Coupled fixed point, directed graph, nonlinear integral equations.
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1. Introduction

In metric fixed point theory, the strong motivation for establishing fixed point results of mappings which satisfy Banach type contractive condition is due to their applications to large class of problems. In 2006, Bhaskar and Lakshmikantham [1] introduced the concept of coupled fixed point for mixed monotone operators and used it to solve various existence problems including periodic boundary value

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problem. Most recently, Chifu and Petrusel [2] applied the ideas from Jachymski [3] to introduce the concept of coupled G-contraction and proved some interesting coupled fixed point theorems.

Following this direction of research, in this paper, we present some existence results for coupled fixed points of F-type contractive operator in metric spaces endowed with a directed graph. Our results extend and generalize the main result given in [2]. Applications to some integral systems are also discussed.

2. Preliminaries

Throughout the article $\mathbb{N}, \mathbb{R}^+$, and $\mathbb{R}$ will denote the set of natural numbers, positive real numbers, and real numbers, respectively.

Now, we briefly recall basic notions and notation from the literature. In particular, we refer to [2, 3].

Let (X, d) be a metric space and Δ be the diagonal of $X \times X$. Let G be a directed graph, such that the set $V(G)$ of its vertices coincides with X and $\Delta \subseteq E(G)$, where $E(G)$ is the set of the edges of the graph. Assume also that G has no parallel edges and, thus, one can identify G with the pair $(V(G), E(G))$.

Throughout the paper we shall say that G with the above-mentioned properties satisfies standard conditions.

Let us denote by G^{-1} the graph obtained from G by reversing the direction of edges. Thus,

$$E(G^{-1}) = \{(x, y) \in X \times X : (y, x) \in E(G)\}.$$

Let us consider the function $f : X \times X \to X$.

Definition 2.1 ([2]). An element $(x, y) \in X \times X$ is called coupled fixed point of the mapping f, if $f(x, y) = x$ and $f(y, x) = y$.

We shall denote by $\mathrm{CFix}(f)$ the set of all coupled fixed points of a mapping f, that is,

$$\mathrm{CFix}(f) = \{(x, y) \in X \times X : f(x, y) = x \text{ and } f(y, x) = y\}.$$

Definition 2.2 ([2]). We say that $f : X \times X \to X$ is edge preserving if $[(x, u) \in E(G), (y, v) \in E(G^{-1})]$, then, $(f(x, y), f(u, v)) \in E(G)$ and $(f(y, x), f(v, u)) \in E(G^{-1})$.

Definition 2.3 ([2]). The mapping $f : X \times X \to X$ is called G-continuous if for all $(x, y) \in X \times X, (x^*, y^*) \in X \times X$ and for any sequence $(n_i)_{i\in\mathbb{N}}$ of positive integers, with $f^{ni}(x, y) \to x^*, f^{ni}(y, x) \to y^*$, as $i \to \infty$, and $(f^{ni}(x, y), f^{ni+1}(x, y)) \in E(G), (f^{ni}(y, x), f^{ni+1}(y, x)) \in E(G^{-1})$, we have that

$$f(f^{ni}(x, y), f^{ni}(y, x)) \to f(x^*, y^*),$$

$$f(f^{ni}(y, x), f^{ni}(x, y)) \to f(y^*, x^*),$$

as $i \to \infty$.

Definition 2.4 ([2]). Let (X, d) be a complete metric space and G be a directed graph. We say that the triple (X, d, G) has the property (A_1), if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$, as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G)$, for $n \in \mathbb{N}$, we have $(x_n, x) \in E(G)$.

Definition 2.5 ([2]). Let (X, d) be a complete metric space and G be a directed graph. We say that the triple (X, d, G) has the property (A_2), if for any sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \to x$, as $n \to \infty$, and $(x_n, x_{n+1}) \in E(G^{-1})$, for $n \in \mathbb{N}$, we have $(x_n, x) \in E(G^{-1})$.

Following Wardowski [5], we denote by $\mathcal{F}$ the family of all functions, $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

$(F1)$ F is strictly increasing on $\mathbb{R}^+$;

$(F2)$ for every sequence $\{s_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\to\infty} s_n = 0$ if and only if $\lim_{n\to\infty} F(s_n) = -\infty$;

$(F3)$ there exists a real number $k \in (0,1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

Example 2.6. The following function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$:

(i) $F(s) = \ln s$, with $s > 0$,

(ii) $F(s) = \ln s + s$, with $s > 0$.

Definition 2.7 ([5])**.** Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be an F-contraction on X if there exists $\tau > 0$ such that

$$d(fx, fy) > 0 \Rightarrow \tau + F\left(d(fx, fy)\right) \leq F(d(x, y))$$

for all $x, y \in X$ and $F \in \mathcal{F}$.

2.1. Coupled fixed point theorems

Let (X, d) be a metric space endowed with a directed graph G satisfying the standard conditions. The set denoted by $(X \times X)^f$ is defined as follows:

$$(X \times X)^f = \{(x, y) \in X \times X : (x, f(x, y)) \in E(G) \text{ and } (y, f(y, x)) \in E(G^{-1})\}.$$

Proposition 2.8 ([2])**.** *If $f : X \times X \to X$ is edge preserving, then:*

(i) $(x, u) \in E(G)$ and $(y, v) \in E(G^{-1})$ implies $(f^n(x, y), f^n(u, v)) \in E(G)$ and $(f^n(y, x), f^n(v, u)) \in E(G^{-1})$;

(ii) $(x, y) \in (X \times X)^f$ implies $(f^n(x, y), f^{n+1}(x, y)) \in E(G)$ and $(f^n(y, x), f^{n+1}(y, x)) \in E(G^{-1})$ for all $n \in \mathbb{N}$;

(iii) $(x, y) \in (X \times X)^f$ implies $(f^n(x, y), f^n(y, x)) \in (X \times X)^f$ for all $n \in \mathbb{N}$.

Definition 2.9. The mapping $f : X \times X \to X$ is called an F-G-contraction if:

(i) f is edge preserving;

(ii) there exists a number $\tau > 0$ such that

$$\tau + F(d(f(x, y), f(u, v))) \leq F(\max\{d(x, u), d(y, u)\})$$

$$\text{for all } (x, u) \in E(G), (y, v) \in E(G^{-1}) \text{ with } d(f(x, y), f(u, v)) > 0.$$

Remark 2.10. If F is defined by $F(s) = \ln s$, for all $s > 0$, and the fact that $\frac{(a+b)}{2} \leq \max\{a, b\}$ for all non-negative real numbers a and b, the $F - G$-contraction reduces to G-contraction given in [2].

Lemma 2.11. *Let (X,d) be a metric space endowed with a directed graph G and let $f: X\times X\to X$ be a F-G -contraction. Then, for all $(x,u)\in E(G),(y,v)\in E(G^{-1})$, we have*

$$F(d(f^n(x,y),f^n(u,v)))\leq F(d(x,u))-n\tau$$

or

$$F(d(f^n(x,y),f^n(u,v)))\leq F(d(y,v))-n\tau$$

and

$$F(d(f^n(y,x),f^n(v,u)))\leq F(d(x,u))-n\tau$$

or

$$F(d(f^n(y,x),f^n(v,u)))\leq F(d(y,v))-n\tau.$$

Proof. Let $(x,u)\in E(G)$ and $(y,v)\in E(G^{-1})$. Since f is edge preserving, we have

$$(f(x,y),f(u,v))\in E(G)\text{ and }(f(y,x),f(v,u))\in E(G^{-1}).$$

from Proposition 2.8 (i), it follows that $(f^n(x,y),f^n(u,v))\in E(G)$ and $(f^n(y,x),f^n(v,u))\in E(G^{-1})$. Since f is an F-G-contraction, we obtain

$$\begin{aligned}F(d(f^2(x,y),f^2(u,v))) =&F(d(f(f(x,y),f(y,x)),f(f(u,v),f(v,u))))\\ \leq& F(\max\{d(f(x,y),f(u,v)),d(f(y,x),f(v,u)))\})-\tau.\end{aligned}$$

If $\max\{d(f(x,y),f(u,v)),d(f(y,x),f(v,u))\}=d(f(x,y),f(u,v))$, then, we have

$$\begin{aligned}F(d(f^2(x,y),f^2(u,v))) =&F(d(f(x,y),f(u,v))-\tau\\ \leq& F(\max\{d(x,u),d(y,u)\})-2\tau.\end{aligned}\tag{2.1}$$

If $\max\{d(f(x,y),f(u,v)),d(f(y,x),f(v,u))\}=d(f(y,x),f(v,u))$, then, we have

$$\begin{aligned}F(d(f^2(x,y),f^2(u,v))) =&F(d(f(y,x),f(v,u)))-\tau\\ \leq& F(\max\{d(y,v),d(x,u)\})-2\tau.\end{aligned}\tag{2.2}$$

Thus, from (2.1) and (2.2), we conclude that

$$F(d(f^2(x,y),f^2(u,v)))\leq F(\max\{d(y,v),d(x,u)\})-2\tau.$$

Now, we consider the two cases:
Case I: If $\max\{d(y,v),d(x,u)\}=d(x,u)$, then we get

$$F(d(f^2(x,y),f^2(u,v)))\leq F(d(x,u))-2\tau.$$

Case II: If $\max\{d(y,v),d(x,u)\}=d(y,v)$, then we get

$$F(d(f^2(x,y),f^2(u,v)))\leq F(d(y,v))-2\tau.$$

Therefore, by mathematical induction, we get

$$F(d(f^n(x,y),f^n(u,v)))\leq F(d(x,u))-n\tau$$

or

$$F(d(f^n(x,y),f^n(u,v)))\leq F(d(y,v))-n\tau.$$

Similarly, we can write

$$\begin{aligned} F(d(f^2(y,x),f^2(v,u))) =& F(d(f(f(y,x),f(x,y),(f(f(v,u),f(u,v))))) \\ \le& F(\max\{d(f(y,x),f(v,u),d(f(x,y),f(u,v)))\}) - \tau. \end{aligned}$$

If $\max\{d(f(y,x),f(v,u),d(f(x,y),f(u,v)))\} = d(f(y,x),f(v,u))$, then, we have

$$\begin{aligned} F(d(f^2(y,x),f^2(v,u))) =& F(d(f(y,x),f(v,u)) - \tau \\ \le& F(\max\{d(y,v)d(x,u)\}) - 2\tau. \end{aligned} \tag{2.3}$$

If $\max\{d(f(y,x),f(v,u),d(f(x,y),f(u,v)))\} = d(f(x,y),f(u,v))$, then, we have

$$\begin{aligned} F(d(f^2(y,x),f^2(v,u))) =& F(d(f(x,y),f(u,v)) - \tau \\ \le& F(\max\{d(x,u),d(y,v)\}) - 2\tau. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we conclude that

$$F(d(f^2(y,x),f^2(v,u))) \le F(\max\{d(x,u),d(y,v)\}) - 2\tau.$$

Following the similar reasoning including induction as done above, we get

$$F(d(f^n(y,x),f^n(v,u))) \le F(d(x,u)) - n\tau$$

or

$$F(d(f^n(y,x),f^n(v,u))) \le F(d(y,v)) - n\tau.$$

□

Lemma 2.12. *Let (X,d) be a complete metric space endowed with a directed graph G and let $f : X \times X \to X$ be an F-G-contraction. Then, for each $(x,y) \in (X \times X)^f$, there exist $\hat{x} \in X$ and $\hat{y} \in X$ such that $(f^n(x,y))_{n\in\mathbb{N}}$ converges to $\hat{x}$ and $(f^n(y,x))_{n\in\mathbb{N}}$ converges to $\hat{y}$, as $n \to \infty$.*

Proof. Let $(x,y) \in (X \times X)^f$, that is, $(x,f(x,y)) \in E(G)$ and $(y,f(y,x)) \in E(G^{-1})$. By Lemma 2.11, we consider $u = f(x,y)$ and $v = f(y,x)$, then we obtain

$$F(d(f^n(x,y),f^n(f(x,y))),f(y,x)) \le F(d(x,f(x,y))) - n\tau$$

that is,

$$F(d(f^n(x,y),f^{n+1}(x,y))) \le F(d(x,f(x,y))) - n\tau \tag{2.5}$$

or

$$F(d(f^n(x,y),f^{n+1}(x,y))) \le F(d(y,f(y,x))) - n\tau$$

and

$$F(d(f^n(y,x),f^{n+1}(y,x))) \le F(d(x,f(x,y))) - n\tau$$

or (2.6)

$$F(d(f^n(y,x),f^{n+1}(y,x))) \le F(d(y,f(y,x))) - n\tau.$$

Now, taking limit as $n \to \infty$ in (2.5), we get

$$\lim_{n\to\infty} F(d(f^n(x,y),f^{n+1}(x,y))) = -\infty.$$

Then, by (F_2), we have

$$\lim_{n\to\infty} d(f^n(x,y), f^{n+1}(x,y)) = 0. \tag{2.7}$$

Now, from (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (d(f^n(x,y), f^{n+1}(x,y)))^k (d(f^n(x,y), f^{n+1}(x,y))) = 0. \tag{2.8}$$

Therefore, from (2.5), for all $n \in \mathbb{N}$ we deduce that

$$\begin{aligned}&(d(f^n(x,y), f^{n+1}(x,y)))^k (F(d(f^n(x,y), f^{n+1}(x,y)))) - F(d(x, f(x,y))) \\ &\qquad \le -n\tau(d(f^n(x,y), f^{n+1}(x,y)))^k \le 0,\\ \text{or}&\\ &(d(f^n(x,y), f^{n+1}(x,y)))^k (F(d(f^n(x,y), f^{n+1}(x,y)))) - F(d(y, f(y,x))) \\ &\qquad \le -n\tau(d(f^n(x,y), f^{n+1}(x,y)))^k \le 0.\end{aligned} \tag{2.9}$$

Now, using (2.7), (2.8) and taking limit $n \to \infty$ in (2.9), we get

$$\lim_{n\to\infty} (nd(f^n(x,y), f^{n+1}(x,y)))^k = 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$n(d(f^n(x,y), f^{n+1}(x,y)))^k \le 1 \quad \forall n \ge n_0$$

that is,,

$$d(f^n(x,y), f^{n+1}(x,y)) \le \frac{1}{n^{\frac{1}{k}}} \quad \forall n \ge n_0.$$

Now, for all $m > n > n_0$, we have

$$\begin{aligned} d(f^n(x,y), f^m(x,y)) &\le d(f^n(x,y), f^{n+1}(x,y)) + ... + d(f^{m-1}(x,y), f^m(x,y)) \\ &\le \sum_{n\ge n_0}^{\infty} \frac{1}{n^{\frac{1}{k}}},\end{aligned}$$

which is convergent as $k \in (0,1)$. Therefore as $m, n \to \infty$, we get $d(f^n(x,y), f^m(x,y)) \to 0$.

Similarly, we can obtain

$$d(f^n(y,x), f^m(y,x)) \to 0 \text{ as } m, n \to \infty.$$

Therefore, $(f^n(x,y))_{n\in\mathbb{N}}$ and $(f^n(y,x))_{n\in\mathbb{N}}$ are Cauchy sequences in X. Since (X,d) is a complete metric space, then there exists $\hat{x} \in X$ and $\hat{y} \in X$ such that $(f^n(x,y))_{n\in\mathbb{N}}$ converges to point $\hat{x}$ and $(f^n(y,x))_{n\in\mathbb{N}}$ converges to point $\hat{y}$, as $n \to \infty$. □

Theorem 2.13. *Let (X,d) be a complete metric space endowed with a directed graph G and let $f : X \times X \to X$ be an F-G-contraction. Suppose that:*

(i) *f is G-continuous;*

or

(ii) *The triple (X,d,G) satisfy the properties $(A_1), (A_2)$ and F is continuous. Then,* $\mathrm{CFix}(f) \neq \emptyset$ *if and only if $(X \times X)^f \neq \emptyset$.*

Proof. Suppose that $\mathrm{CFix}(f) \neq \emptyset$, then there exists $(x^*, y^*) \in \mathrm{CFix}(f)$ such that $(x^*, f(x^*, y^*)) = (x^*, x^*) \in \Delta \subset E(G)$ and $(y^*, f(y^*, x^*)) = (y^*, y^*) \in \Delta \subset E(G^{-1})$.

So $(x^*, f(x^*, y^*)) \in E(G)$ and $(y^*, f(y^*, x^*)) \in E(G^{-1})$ that is, $(x^*, y^*) \in (X \times X)^f$ and thus $(X \times X)^f \neq \emptyset$.

Suppose that $(X \times X)^f \neq \emptyset$, then there exists $(x, y) \in (X \times X)^f$, that is, $(x, f(x, y)) \in E(G)$ and $(y, f(y, x)) \in E(G^{-1})$.

Let $(n_i)_{i\in\mathbb{N}}$ be a sequence of positive integers. From Proposition 2.8 (ii), we have

$$\begin{aligned} (f^{ni}(x, y), f^{ni+1}(x, y)) \in& E(G), \\ (f^{ni}(y, x), f^{ni+1}(y, x)) \in& E(G^{-1}). \end{aligned} \tag{2.10}$$

further, from Lemma 2.12, there exists $x^* \in X$ and $y^* \in X$ such that

$$f^{ni}(x, y) \to x^*(x) \text{ and } f^{ni}(y, x) \to y^*(y) \text{ as } i \to \infty.$$

Now, we shall prove that $f(x^*, y^*) = x^*$ and $f(y^*, x^*) = y^*$.

Case I. Suppose f is a G-continuous mapping, then we get

$$f(f^{ni}(x, y), f^{ni}(y, x)) \to f(x^*, y^*) \text{ and } f(f^{ni}(y, x), f^{ni}(x, y)) \to f(y^*, x^*) \text{ as } i \to \infty.$$

Now,

$$d(f(x^*, y^*), x^*) \leq d(f(x^*, y^*), f^{ni+1}(x, y)) + d(f^{ni+1}(x, y), x^*).$$

Since f is G-continuous and $f^{ni}(x, y) \to x^*$, therefore, we get

$$d(f(x^*, y^*), x^*) = 0 \text{ that is, } f(x^*, y^*) = x^*.$$

By the same argument, we can be proved that $f(y^*, x^*) = y^*$.

Thus (x^*, y^*) is a coupled fixed point of the mapping f and hence, $\mathrm{CFix}(f) \neq \emptyset$.

Case II. Suppose that the triple (X, d, G) has the properties (A_1) and (A_2), then we get

$$(f^n(x, y), x^*) \in E(G) \text{ and } (f^n(y, x), y^*) \in E(G^{-1}).$$

Now,

$$\begin{aligned} d(f(x^*, y^*), x^*) &\leq d(f(x^*, y^*), f^{n+1}(x, y)) + d(f^{n+1}(x, y), x^*) \\ &\leq d(f(x^*, y^*), f(f^n(x, y), f^n(y, x))) + d(f^{n+1}(x, y), x^*), \\ F(d(f(x^*, y^*), x^*) - d(f^{n+1}(x, y), x^*)) &\leq F(d(f(x^*, y^*), f(f^n(x, y), f^n(y, x)))) \\ &\leq F(\max\{d(x^*, f^n(x, y)), d(y^*, f^n(y, x))\}) - \tau. \end{aligned}$$

Then, as $n \to \infty$ and continuity of F, we get

$$F(d(f(x^*, y^*), x^*)) \leq -\infty$$

that is, $d(f(x^*, y^*), x^*) = 0$.

Similarly, we can prove that $d(f(y^*, x^*), y^*) = 0$.

Thus $(x^*, y^*) \in \mathrm{CFix}(f)$. □

Theorem 2.14. *Under the condition of Theorem* 2.13*, if* $(x^*, y^*) \in \mathrm{CFix}(f)$ *with* $(x^*, y^*) \in E(G)$ *and* $(y^*, x^*) \in E(G^{-1})$*, then* $x^* = y^*$.

Proof. Suppose $x^* \neq y^*$. Then as $(x^*, y^*) \in E(G), (y^*, x^*) \in E(G^{-1})$ and f is an F-G-contraction mapping, we have

$$\begin{aligned} F(d(x^*, y^*)) =& F(d(f(x^*, y^*)), d(f(y^*, x^*))) \\ \leq& F(\max\{d(x^*, y^*), d(y^*, x^*)\}) - \tau \\ =& F(d(x^*, y^*)) - \tau, \end{aligned}$$

a contradiction. Hence $x^* = y^*$. □

Remark 2.15. From Remark 2.10, we obtain Theorem 2.1 and Theorem 2.3 given in [2].

3. Applications

Integral Equations arise in many scientific and engineering problems. A large class of initial and boundary value problem can be converted to Volterra or Fredholm type integral equations.

In this section, we apply our main theorem to the existence for a solutions of the some integral systems.

At first we considered the following integral systems equation given in [4]

$$\begin{aligned} x(t) &= \int_0^T g(t, s, x(s), y(s))ds + h(t), \\ y(t) &= \int_0^T g(t, s, y(s), x(s))ds + h(t), \end{aligned} \tag{3.1}$$

where $t \in [0, T]$ with $T > 0$.

Let $X := C([0, T], \mathbb{R}^n)$ with the usual supremum norm, that is, $\|x\| = \max_{t\in[0,T]}|x(t)|$ for all $x \in C([0, T], \mathbb{R}^n)$. Consider also the graph G defined by using the partial order relation that is,

$$x, y \in X, x \leq y \Leftrightarrow x(t) \leq y(t) \text{ for any } t \in [0, T].$$

Therefore, $(X, \|\cdot\|)$ is a complete metric space endowed with a directed graph G.

If we consider $E(G) = \{(x, y) \in X \times X : x \leq y\}$, then the diagonal Δ of $X \times X$ is included in $E(G)$. On the other hand $E(G^{-1}) = \{(x, y) \in X \times X : y \leq x\}$.

Moreover, $(X, \|\cdot\|, G)$ has the properties (A_1) and (A_2).

In this case $(X \times X)^f = \{(x, y) \in X \times X : x \leq f(x, y) \text{ and } f(y, x) \leq y\}$.

Theorem 3.1. *Consider system* (3.1). *Suppose that*

(i) $g : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $h : [0, T] \to \mathbb{R}^n$ are continuous;

(ii) for all $x, y, u, v \in \mathbb{R}^n$ with $x \leq u, y \leq v$, we have $g(t, s, x, y) \leq g(t, s, u, v)$ for all $t, s \in [0, T]$;

(iii) there exists a number $\tau > 0$ such that

$$|g(t, s, x, y) - g(t, s, u, v)| \leq \frac{\tau}{T}\max\{|x - u|, |y - v|\}$$

for all $t, s \in [0, T], x, y, u, v \in \mathbb{R}^n, x \leq u, v \leq y$;

(iv) there exists $(x_0, y_0) \in X \times X$ such that

$$x_0(t) = \int_0^T g(t, s, x_0(s), y_0(s))ds + h(t)$$

and

$$y_0(t) = \int_0^T g(t, s, y_0(s), x_0(s))ds + h(t),$$

where $t \in [0, T]$.

Then there exists at least one solution of the integral system (3.1).

Proof. Let $f : X \times X \to X, (x, y) \mapsto f(x, y)$, where

$$f(x, y)(t) = \int_0^T g(t, s, x(s), y(s))ds + h(t); \quad t \in [0, T].$$

Then the system (3.1) can be written as

$$x = f(x, y) \text{ and } y = f(y, x).$$

Now, Let $x, y, u, v \in X$ such that $x \leq u$ and $v \leq y$. Then we have

$$\begin{aligned} f(x, y)(t) &= \int_0^T g(t, s, x(s), y(s))ds + h(t) \\ &\leq \int_0^T g(t, s, u(s), v(s))ds + h(t) \\ &= f(u, v)(t), \text{ for all } t \in [0, T], \end{aligned}$$

and

$$\begin{aligned} f(y, x)(t) &= \int_0^T g(t, s, y(s), x(s))ds + h(t) \\ &\leq \int_0^T g(t, s, v(s), u(s))ds + h(t) \\ &= f(v, u)(t), \text{ for all } t \in [0, T]. \end{aligned}$$

Hence, f is G-edge preserving.

From condition (iv), it follows that

$$(X \times X)^f = \{(x, y) \in X \times X : x \leq f(x, y) \text{ and } y \leq f(y, x)\} \neq \emptyset.$$

Further,

$$\begin{aligned} |f(x, y)(t) - f(u, v)(t)| &\leq \int_0^T |g(t, s, x(s), y(s)) - g(t, s, u(s), v(s))|ds \\ &\leq \frac{\tau}{T} \int_0^T (\max\{|x(s) - u(s)|, |y(s) - v(s)|\})ds \\ &\leq \tau(\max\{\|x - u\|, \|y - v\|\}) \end{aligned}$$

that is,

$$\|f(x,y)-f(u,v)\| \leq \tau(\max\{\|x-u\|,\|y-v\|\})$$

or

$$d(f(x,y),f(u,v)) \leq \tau(\max\{d(x,u),d(y,v)\}.$$

By passing through a logarithm, we can write

$$\ln d(f(x,y),f(u,v)) \leq \ln(e^{-\tau}\max\{d(x,u),d(y,v)\})$$

and hence

$$\tau+\ln d(f(x,y),f(u,v)) \leq \ln(\max\{d(x,u),d(y,v)\}).$$

Now, we observe that the function $F:\mathbb{R}^+\to\mathbb{R}$ defined by $F(\alpha)=\ln\alpha$, for each $\alpha\in C([0,T],\mathbb{R}^n)$, is in $\mathcal{F}$ and so we reduce that the operator f satisfies all condition of Theorem 2.13 and hence there exists a coupled fixed point $(x^*,y^*)\in X\times X$ of the mapping f which is the solution of the integral system 3.1. □

In the next theorem we provide some conditions for existence of solution of nonlinear integral equation of fredholm type. Precisely, we prove the following theorem.

Theorem 3.2. *Consider the non-linear integral equation of Fredholm type:*

$$P(x,y)=h(x,y)+\int_0^a\int_0^b\{k_1(x,y,t,s)+k_2(x,y,t,s)\}(f(t,s,p(t,s))+g(t,s,q(t,s)))dtds, \quad (3.2)$$

where $k_1,k_2\in C(I_a\times I_b\times I_a\times I_b,R)$ and $f,g\in C(I_a\times I_b\times R\times R)$. Assume that there exists $\lambda,\mu>0$ and $\tau>0$ such that

$$\sup_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b k_1(x,y,t,s)dtds\right| \leq \frac{e^{-\tau}}{2(\lambda+\mu)}$$

and

$$\sup_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b k_2(x,y,t,s)dtds\right| \leq \frac{e^{-\tau}}{2(\lambda+\mu)}.$$

For $p_1\geq p_2,(x,y)\in I_a\times I_b$, f and g satisfy

$$0\leq f(x,y,p)-f(x,y,q)\leq\lambda(p-q)$$

and

$$-\mu(p-q)\leq g(x,y,p)-g(x,y,q)\leq 0.$$

If the coupled upper-lower solutions of (3.2) *exist, then there exists a unique solution of the integral equation* (3.2).

Proof. Let $X=C(I_a\times I_b,R)$, where $I_a=[0,a]$ and $I_b=[0,b]$ with the metric defined by:

$$d(w_1,w_2)(x,y)=\sup_{(x,y)\in I_a\times I_b}\left|w_1(x,y)-w_2(x,y)\right|$$

for all $w_1,w_2\in X$.

Also, consider the graph G defined by the partial order relation that is, $w_1(x,y) \leq w_2(x,y)$. Then X is a complete metric space, moreover (X,d,G) have the properties (A_1) and (A_2).

Now, define $H : X \times X \to X$ by

$$\begin{aligned} H(p,q)(x,y) = & \int_0^a \int_0^b k_1(x,y,t,s)(f(t,s,p(t,s)) + g(t,s,q(t,s)))dtds \\ & + \int_0^a \int_0^b k_2(x,y,t,s)(f(t,s,q(t,s)) + g(t,s,p(t,s)))dtds + h(x,y) \end{aligned}$$

for all $(x,y) \in I_a \times I_b$ and $p,q \in X$. Then H is edge preserving mapping.

Now, for $p_1, p_2, q_1, q_2 \in X$ with $p_1 \geq p_2$ and $q_1 \geq q_2$, then

$$\begin{aligned} d(H(p_1,q_1),H(p_2,q_2)) &= \sup_{(x,y)\in I_a\times I_b} \left| H(p_1,q_1)(x,y) - H(p_2,q_2)(x,y) \right| \\ &= \sup_{(x,y)\in I_a\times I_b} \Bigg| \int_0^a \int_0^b k_1(x,y,t,s)(f(t,s,p_1(t,s)) + g(t,s,q_1(t,s)))dtds \\ &\quad + \int_0^a \int_0^b k_2(x,y,t,s)(f(t,s,q_1(t,s)) + g(t,s,p_1(t,s)))dtds \\ &\quad - \int_0^a \int_0^b k_1(x,y,t,s)(f(t,s,p_2(t,s)) + g(t,s,q_2(t,s)))dtds \\ &\quad + \int_0^a \int_0^b k_2(x,y,t,s)(f(t,s,q_2(t,s)) + g(t,s,p_2(t,s)))dtds \Bigg| \\ &= \sup_{(x,y)\in I_a\times I_b} \Bigg| \int_0^a \int_0^b k_1(x,y,t,s)\{f(t,s,p_1(t,s)) - f(t,s,p_2(t,s)) \\ &\quad + g(t,s,q_1(t,s) - g(t,s,q_2(t,s))\}dtds \\ &\quad + \int_0^a \int_0^b k_2(x,y,t,s)\{f(t,s,p_1(t,s)) - f(t,s,p_2(t,s)) \\ &\quad + g(t,s,q_1(t,s) - g(t,s,q_2(t,s))\}dtds \Bigg| \\ &\leq \sup_{(x,y)\in I_a\times I_b} \Bigg| \int_0^a \int_0^b k_1(x,y,t,s)[\lambda\{p_1(t,s) - p_2(t,s)\} \\ &\quad + \mu\{q_1(t,s) - q_2(t,s)\}]dtds \Bigg| \\ &\quad + \sup_{(x,y)\in I_a\times I_b} \Bigg| \int_0^a \int_0^b k_2(x,y,t,s)[\lambda\{q_1(t,s) - q_2(t,s)\} \\ &\quad + \mu\{p_1(t,s) - p_2(t,s)\}]dtds \Bigg| \\ &\leq \sup_{(x,y)\in I_a\times I_b} \left| \int_0^a \int_0^b k_1(x,y,t,s)[\lambda d(p_1,p_2) + \mu d(q_1,q_2)]dtds \right| \end{aligned}$$

$$
\begin{aligned}
&+ \sup_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b k_2(x,y,t,s)[\lambda d(q_1,q_2)+\mu d(p_1,p_2)]dtds\right| \\
&\leq \sup_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b k_1(x,y,t,s)d\tau ds\right|[\lambda d(p_1,p_2)+\mu d(q_1,q_2)] \\
&+ \sup_{(x,y)\in I_a\times I_b}\left|\int_0^a\int_0^b k_2(x,y,t,s)dtds\right|[\lambda d(q_1,q_2)+\mu d(p_1,p_2)] \\
&\leq \frac{e^{-\tau}}{2(\lambda+\mu)}[\lambda d(p_1,p_2)+\lambda d(q_1,q_2)] \\
&+ \frac{e^{-\tau}}{2(r+s)}[\lambda d(q_1,q_2)+\mu d(p_1,p_2)] \\
&= \frac{e^{-\tau}}{2(\lambda+\mu)}[(\lambda+\mu)d(p_1,p_2)+(\lambda+\mu)d(q_1,q_2)] \\
&= e^{-\tau}[\frac{d(p_1,p_2)+d(q_1,q_2)}{2}] \\
&\leq e^{-\tau}\max\{d(p_1,p_2),d(q_1,q_2)\}.
\end{aligned}
$$

By passing though a logarithm we can write

$$\ln(d(H(p_1,q_1),H(p_2,q_2))) \leq \ln(e^{-\tau}\max\{d(p_1,p_2),d(q_1,q_2)\}),$$

that is,

$$\tau+\ln(d(H(p_1,q_1),H(p_2,q_2))) \leq \ln(\max\{d(p_1,p_2),d(q_1,q_2)\}),$$

that is, the operator H is an F - G contraction.

Now, let $\alpha(x,y)$ and $\beta(x,y)$ be the coupled upper - lower solution of (3.2), then we have

$$\alpha(x,y) \leq H(\alpha(x,y),\beta(x,y))$$

and

$$\beta(x,y) \leq H(\beta(x,y),\alpha(x,y))$$

for all $(x,y)\in I_a\times I_b$, i.e $(X\times X)^H \neq \emptyset$.

Thus, all the hypothesis of the Theorem 2.13 are fulfilled and thus, there exists a coupled solution (x^*,y^*) of (3.2). □

Example 3.3. Consider the nonlinear Fredholm integral equation

$$p(x,y)=\frac{1}{(1+x+y)^2}-\frac{x}{6(1+y)}+\int_0^1\int_0^1\frac{x}{(1+y)}(1+t+s)p^2(t,s)dtds.$$

Here, we have $h(x,y)=\frac{1}{(1+x+y)^2}-\frac{x}{6(1+y)}$, $k_1(x,y,t,s)=\frac{x}{1+y}(1+t+s)$, and $f(t,s,p(t,s))=p^2(t,s)$.

Taking initial $p_0(t,s)=\frac{1}{1+t+s}$ and using the iterative scheme

$$p_{n+1}=\frac{1}{(1+x+y)^2}-\frac{x}{6(1+y)}+\int_0^1\int_0^1\frac{x}{(1+y)}(1+t+s)p_n^2dtds,$$

the exact solution is $p(x,y)=\frac{1}{(1+x+y)^2}$.

The Figure 1 is the exact solution while the Figure 2 shows approximate solution given after 25^{th} iteration. Error defined as |exact solution - approximation solution| is plotted in Figure 3.

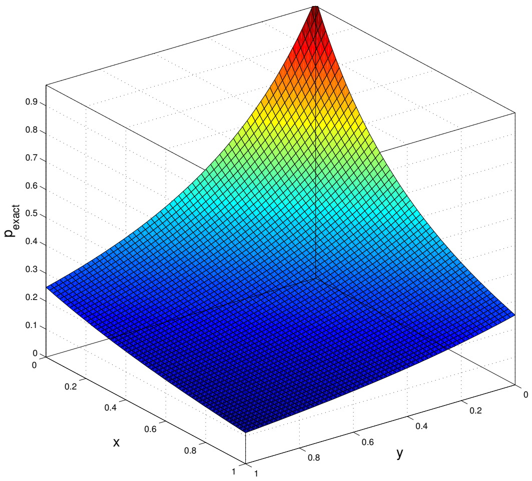

Figure 1: Plot of the exact solution

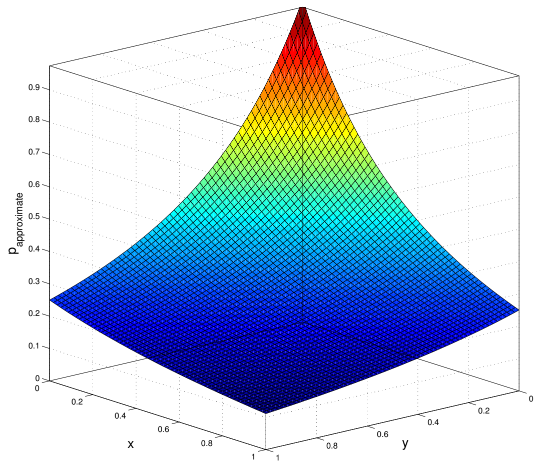

Figure 2: Plot of the approximate solution

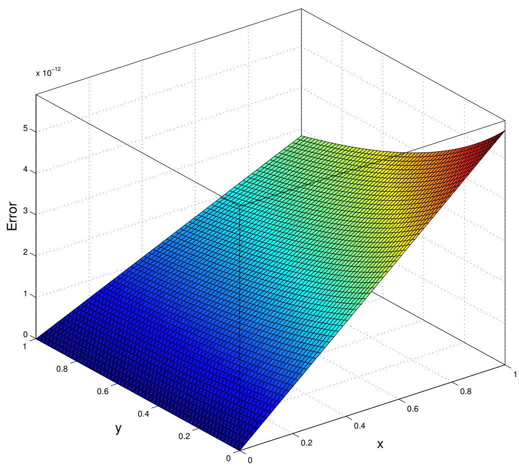

Figure 3: Plot of the error

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References

[1] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393. 1

[2] C. Chifu, G. Petrusel, *New results on coupled fixed point theory in metric spaces endowed with a directed graph*, Fixed Point Theory Appl., **2014** (2014), 13 pages. 1, 2, 2.1, 2.2, 2.3, 2.4, 2.5, 2.8, 2.10, 2.15

[3] J. Jachymski, *The contraction principle for mappings on a metric space with a graph*, Proc. Amer. Math. Soc., **136** (2008), 1359–1373. 1, 2

[4] S. Suantai, P. Charoensawan, T. A. Lampert, *Common coupled fixed point theorems for θ-ψ-contraction mappings endowed with a directed graph*, Fixed Point Theory Appl., **2015** (2015), 11 pages. 3

[5] D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl., **2012** (2012), 6 pages. 2, 2.7